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Citation: Chin. Phys. B, 2022, 31 (**4**): 040307. DOI: 10.1088/1674-1056/ac2806
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Quantum walk search algorithm for multi-objective searching with iteration auto-controlling on hypercube

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(Received 22 August 2021; revised manuscript received 4 September 2021; accepted manuscript online 18 September 2021)

Shenvi *et al.* have proposed a quantum algorithm based on quantum walking called Shenvi–Kempe–Whaley (SKW) algorithm, but this search algorithm can only search one target state and use a specific search target state vector. Therefore, when there are more than two target nodes in the search space, the algorithm has certain limitations. Even though a multi-objective SKW search algorithm was proposed later, when the number of target nodes is more than two, the SKW search algorithm cannot be mapped to the same quotient graph. In addition, the calculation of the optimal target state depends on the number of target states m. In previous studies, quantum computing and testing algorithms were used to solve this problem. But these solutions require more Oracle calls and cannot get a high accuracy rate. Therefore, to solve the above problems, we improve the multi-target quantum walk search algorithm, and construct a controllable quantum walk search algorithm under the condition of unknown number of target states. By dividing the Hilbert space into multiple subspaces, the accuracy of the search algorithm is improved from $p_c = (1/2) - O(1/n)$ to $p_c = 1 - O(1/n)$. And by adding detection gate phase, the algorithm can stop when the amplitude of the target state becomes the maximum for the first time, and the algorithm can always maintain the optimal number of iterations, so as to reduce the number of unnecessary iterations in the algorithm process and make the number of iterations reach $t_f = (\pi/2)\sqrt{2^{n-2}}$.

Keywords: multi-objective, quantum walk, search algorithm, accurate probability

PACS: 03.67.Ac, 02.70.–c, 02.20.–a **DOI:** 10.1088/1674-1056/ac2806

1. Introduction

Quantum computing is an emerging computer science and technology in recent years. In the past two decades, this subject has developed rapidly both on theoretically and experimentally, and has become an international research hot spot, with a series of breakthrough research results.[1] Moreover, quantum computing can process the eigenstates of superposition state in parallel. So quantum algorithm has quadratic and exponential advantages over classical algorithm. The research of quantum algorithm has profound theoretical significance and broad application prospect.[2] In 1996, Grover proposed a high performance quantum search algorithm for unstructured databases, which has quadratic property than any possible classical algorithm.[3] Grover search algorithm is formed by repeated application of U operator. The number of iterations in the whole process directly affects the probability of output target state. Since then, quantum search algorithms have be continued to study.[4–6] Furthermore quantum walk is a branch of quantum computing, and it is the counterpart of the classical random walk in the field of quantum mechanics. Quantum walk has the generality and the advantage of parallel computation of quantum algorithm. Therefore, scholars have carried out a series of researches and applications using quantum walk,[7–10] and a lot of search algorithms based on quantum walk are proposed. In 2003, Shenvi, Kempe, and Whaley proposed a quantum algorithm based on quantum walk (This algorithm is called the SKW algorithm). The SKW search algorithm can be used to find hypercube vertices marked by Oracle.[11] In 2009, by examining n neighbors of the measured results, Potoek *et al.* greatly optimized the probability of finding a solution based on search results, and increased the probability of finding a solution to close to 1.[12] Besides, in the last part of the literature, an improved scheme based on the original SKW algorithm is proposed, but its correctness is not proved. In 2014, Zhu *et al.* designed a new algorithm to automatically control the iterative process by adding a phase detection. In the case that the number of target states is unknown, when the operator just iterates to the maximum amplitude of the target state, the algorithm can be stopped automatically.[13] Since search algorithms and algorithms that can be transformed into search problems have a wide range of applicability, after these pioneering articles, scholars have conducted in-depth research. In China, Long[14] constantly improve Grover search algorithm; Sheng,[15–17] Zhou *et al.*[18–21] also in the field of quantum search algorithm and quantum computing have made efforts.

Although the scholar gate has done a lot of work on the quantum walk search algorithm, when searching for multi-target state, the quantum search algorithm still has the problem of many iterations and low precision, which leads to the ineffi-

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cient search algorithm in the application. Therefore, the design of the multi-target search algorithm for quantum wandering is still a problem, especially when the number of unknown target states is unknown. Based on this problem, an unknown number of target states search algorithm based on quantum walk is designed. In this paper, by modifying the search algorithm of multi-objective states in the following two aspects, an automatic search algorithm based on quantum walking is proposed in the case of multi-objective state positions. Firstly, the iterative process is automatically controlled by adding phase detection gate (the operation P for short), in order to find the optimal number of iterations of quantum walk. Because in the quantum walk process, iterative operators have a cyclical effect on any system state, that is, the role of iterative operators is not convergent. If the number of iterations is less or more than the optimal number, the probability of eventually reaching the target state is reduced, so we need to stop the algorithm the first time the magnitude of the target state becomes maximum. The optimal number of iterations at this point reduces unnecessary iterations while ensuring that the target state is measured, in other words, the algorithm effectively reduces time complexity. Secondly, by decomposing the Hilbert space where the search space is located into the direct sum of several disjoint subsets, the goal is to increase the accuracy of the search to the target state. Since the accuracy of searching for a single target state has reached 1, the purpose of decomposing the Hilbert space into several dissection subsets of straight sum is to convert the projection measurement of the target state of the multipart into the measurement of the single target state, so that the accuracy of the search for the multi-target state is also greater than 90%. The controllable quantum walk search algorithm has the characteristics of less iteration times, high efficiency and high accuracy in searching the target state.

The paper is organized as follows. In Section 2, the SKW search algorithm and the multi-objective coin operator are introduced. In Section 3, the composition of the new algorithm and the process of searching the algorithm is described. In Section 4, the SKW search algorithm is compared with the controllable quantum walk search algorithm in terms of time complexity and accurate probability of searching the target node. In Section 5, the advantages of the improved search algorithm are obtained and the results are summarized.

2. Background

The SKW algorithm uses Grover walk as the basic algorithm framework and will be based on quantum walk on the n cube. The hypercube is a graph with $N = 2^n$ nodes with n dimensions, in which each point can be marked with an n-bit binary string. Two nodes described as $\boldsymbol{x}$ and $\boldsymbol{y}$ on the hypercube. If $|\boldsymbol{x}-\boldsymbol{y}| = 1$, two nodes are connected by an edge, where $|\boldsymbol{x}|$ is the hamming weight of $\boldsymbol{x}$. So the Hilbert space of the search algorithm can be written $\mathcal{H} = \mathcal{H}^n \otimes \mathcal{H}^{2n}$. Each state of $\mathcal{H}$ can be defined as a bit string $\boldsymbol{x}$ representing the position on the hypercube, and each state of $\mathcal{H}$ can be defined as a direction d representing the state of the coin space. The shift operator S maps a state $|d,\boldsymbol{x}\rangle$ to the state $|d,\boldsymbol{x}\oplus\boldsymbol{e}_d\rangle$, where $\boldsymbol{e}_d$ is the d-th basis vector on the hypercube. Then S can be expressed as

$$S = \sum_{d=0}^{n-1}\sum_{\boldsymbol{x}} |d,\boldsymbol{x}\oplus\boldsymbol{e}_d\rangle\langle d,\boldsymbol{x}|, \tag{1}$$

where $\boldsymbol{e}_d = 2^d$ corresponds to the edges originating which are from the given vertex. The target vertex that is marked by the oracle $\mathcal{O}$ is defined as $\boldsymbol{x}_{\rm tg}$, the coin operator C' can be expressed as

$$C' = C_0 \otimes \mathcal{I} + (C_1 - C_0) \otimes |\boldsymbol{x}_{\rm tg}\rangle\langle\boldsymbol{x}_{\rm tg}|. \tag{2}$$

In the SKW quantum search algorithm, C_0 is selected as an N-dimensional Grover operator, and C_1 is selected as I. Because of the symmetry of hypercube graphs, vertices can be relabeled in such a way that the labeled vertices become $\boldsymbol{x}_{\rm tg} = 0$. Then the initial state is

$$|\psi_0\rangle = \frac{1}{\sqrt{n2^n}}\sum_{\boldsymbol{x}}\sum_{d=1}^{n-1}|d,\boldsymbol{x}\rangle. \tag{3}$$

Finally, the coin operator C' of the multi-target of the SKW quantum walk algorithm is expressed as

$$C'_m = C_0 \otimes \mathcal{I} + (C_1 - C_0) \otimes \sum_{j=1}^{m} \left|\boldsymbol{x}_{\rm tg}^{(j)}\right\rangle \left\langle \boldsymbol{x}_{\rm tg}^{(j)}\right|, \tag{4}$$

where m represents the number of labeled elements $\boldsymbol{x}_{\rm tg}^{(j)}$, and $j = 1,...,m$.

3. Improvement of algorithm

3.1. Composition of the algorithm

Now we consider the optimization of the quantum walk search algorithm when the number of target states is greater than 1. To formalize the task of finding multiple marked vertices. Let $\mathcal{H}_i$ denote the subspace spanned by states $|d,\boldsymbol{x}\rangle$ such that $|\boldsymbol{x}|$ meet the condition: $|\boldsymbol{x}| = |\boldsymbol{x}| \oplus i$, $i = 1,2,\ldots,n$. The $\mathcal{H}_i$ is i-th subspace of $\mathcal{H}$, and meets

$$\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_l = \mathcal{H}. \tag{5}$$

The subspace $\mathcal{H}_i$ refer to a sign where the target vertex is denoted by $\boldsymbol{x}_{\rm tg} = 0$.

In addition, it was found that when the phase of the superposition state became negative for the first time, the amplitude value of the search target reached its maximum, and the positivity and negativity of the phase in front of the $|++\cdots+\rangle$ component based on X determined the positivity and negativity of the phase of the superposition state in the paper.[13]

Therefore, the phase detection gate can be used to estimate the positive and negative phase. The position where the detection gate moves in should be behind the Oracle operator, that is, when the Oracle operator is applied, the amplitude value of the target component is the largest when the phase and the first negative change. The Oracle operator of quantum walk search algorithm based on complete graph is in the coin operator. So, intuitively, the Phase operator should be embedded into $C'_{\rm m}$, that is, the new quantum walk coin operator should be set as

$$C''_{\rm m} = \text{Oracle} \otimes C'_{\rm m} = (I \otimes C'_{\rm m})(\text{Oracle} \otimes I), \tag{6}$$

$$\widetilde{C''_{\rm m}} = (I \otimes C'_{\rm m}) P (\text{Oracle} \otimes I). \tag{7}$$

Then S can be expressed as

$$S = \sum_{d=0}^{n-1} \sum_{\boldsymbol{x}} \sum_{j=1}^{m} |d, \boldsymbol{x} \oplus \boldsymbol{x}^j_{\rm tg} \oplus \boldsymbol{e}_d\rangle\langle d, \boldsymbol{x} \oplus \boldsymbol{x}^j_{\rm tg}|. \tag{8}$$

Reduce to walking in a straight line with i fork roads. The basis states of the collapsed quantum walk are defined as

$$|L^i, x\rangle = \sqrt{\frac{1}{x\binom{n}{x}}} \sum_{|\boldsymbol{x}|=x} \sum_{x_d=1} \left| d, \boldsymbol{x} \oplus \boldsymbol{x}^j_{\rm tg} \right\rangle, \tag{9}$$

$$|R, x\rangle = \sum_i \sqrt{\frac{1}{(n-x)\binom{n}{x}}} \sum_{|\boldsymbol{x}|=x} \sum_{x_d=0} \left| d, \boldsymbol{x} \oplus \boldsymbol{x}^j_{\rm tg} \right\rangle, \tag{10}$$

and the propagator becomes

$$S = \sum_i \left(\sum_{x=0}^{n-1} |R, x\rangle\langle L^i, x+1| + |L^i, x+1\rangle\langle R, x| \right). \tag{11}$$

When C_0 becomes the Grover coin, it expresses

$$C_0 = \sum_{x=0}^{n} \begin{pmatrix} \cos\omega_x & \sin\omega_x \\ \sin\omega_x & -\cos\omega_x \end{pmatrix} \otimes \sum_{j=1}^{m} |\boldsymbol{x} \oplus \boldsymbol{x}^j_{\rm tg}\rangle\langle \boldsymbol{x} \oplus \boldsymbol{x}^j_{\rm tg}|, \tag{12}$$

where $\sin\omega_x = (2/n)\sqrt{x(n-x)}$, $\cos\omega_x = 1 - 2x/n$.

And we get that the initial superposition state is

$$|\psi_0\rangle = \frac{1}{\sqrt{2^n}} \sum_i |R^i, 0\rangle + \frac{1}{\sqrt{2^n}} |L, n\rangle + \sum_{x=0}^{n-1} \left(\sqrt{\frac{\binom{n-1}{x-1}}{2^n}} |L, x\rangle + \sum_i \sqrt{\frac{\binom{n-1}{x}}{2^n}} |R^i, x\rangle \right). \tag{13}$$

In addition, the unitary operator $U''_{\rm m}$ acting on the Hilbert space becomes

$$U''_{\rm m} = \sum_i \sum_{x=0}^{n-1} |R^i, x\rangle \left(-\cos\omega_{x+1} \langle L, x+1| + \sin\omega_{x+1} \langle R^i, x+1| \right) + \sum_i \sum_{x=1}^{n} |L, x\rangle (\sin\omega_{x-1} \langle L, x-1| + \cos\omega_{x-1} \langle R^i, x-1|), \tag{14}$$

where

$$\begin{aligned} &\sin\omega_{x+1} = (2/n)\sqrt{x+1(n-1-x)}, \\ &\cos\omega_{x+1} = 1 - (2x+2)/n, \\ &\sin\omega_{x-1} = (2/n)\sqrt{x-1(n+1-x)}, \\ &\cos\omega_{x-1} = 1 - (2x-2)/n. \end{aligned}$$

Finally this unitary operator as $\widetilde{U''_{\rm m}} = S\widetilde{C''_{\rm m}}$ is to obtain a final state that is composed the marked states corresponding to the marked vertices when the iterated given number of times.

Next we can define the orthogonal projectors V_i that project to $\mathcal{H}_i$ respectively. The $|\boldsymbol{x}|$ meet the condition:$|\boldsymbol{x}| = |\boldsymbol{x}| \oplus i$. Since $[V_i, C_0] = 0$ and $[V_i, C'] = 0$. It does not change for the move operator S. So we get the following equation:

$$V_i U''_{\rm m} = U''_{\rm m} V_i. \tag{15}$$

Let us introduce the states

$$|\psi_0^{(i)}\rangle = \sqrt{\rm i} V_i |\psi_0^{(i)}\rangle. \tag{16}$$

And the initial state meets $|\psi_0\rangle = (1/\sqrt{\rm i}) \sum_i |\psi_0^{(i)}\rangle$. We can get the action of $U''_{\rm m}$ on $|\psi_0^{(i)}\rangle\rangle$ to

$$U''_{\rm m} |\psi_0^{(i)}\rangle = U'_{\rm m} |\psi_0^{(i)}\rangle. \tag{17}$$

By Eqs. (15) and (17), it can be shown that $U''_{\rm m}$ meets

$$V_{(i)} (U''_{\rm m})^{2r} |\psi_0\rangle = V_{(i)} (U''_{\rm m})^{2r+1} |\psi_0\rangle, \quad (r = 0, 1, 2, \ldots). \tag{18}$$

The iterative operator $(U''_{\rm m})^t$ can be expressed after t steps as

$$(U''_{\rm m})^t |\psi_0\rangle = \sum_i \sum_{x=0}^{n-1} \alpha^t_{i,x} |i, x\rangle. \tag{19}$$

Then we define $V^t_x = \sum_i |\alpha^t_{i,x}|^2$.

3.2. The decision process of *P* operator

The determination of P operator when the amplitude of element connection of groups i are about to reach the maximum value is analyzed in detail as follows:

(i) When the amplitude value of the result of the group $i = 1$ elements $|\boldsymbol{x}|$ does not reach the maximum value through the next mean inversion operator, the groups $i = 2, 3, \ldots$ elements $|\boldsymbol{x}|$ is acted on by the Oracle operator. In this case, the P operator does not stop the algorithm because elements can make the phase sum negative.

(ii) It is assumed that the amplitude of the result of the group $i = 2$ elements $|\boldsymbol{x}|$ has reached the maximum value, and the Oracle operator is applied, but the groups $i = 1, 3, \ldots$ elements $|\boldsymbol{x}|$ has not passed the mean inversion operator and does

not reach the maximum value. But the phase sum of the group $i=2$ elements $|\boldsymbol{x}|$ is already negative, and the phase sum of the groups $i=1,3,\ldots$ elements $|\boldsymbol{x}|$ is positive and close to 0. Therefore, the joint phase sum of groups $i=1,2,3,\ldots$ elements may be positive or negative, that is, there is a certain probability that the algorithm will stop in this iteration.

(iii) In the next iteration, when the solution of the group $i=1$ elements $|\boldsymbol{x}|$ has reached its maximum value, the Oracle operator is applied. At this time, the phase sum of the groups $i=2,3,\ldots$ elements and the group $i=1$ elements is negative, so the algorithm must be stopped in this iteration.

In summary, when the P operator stops the algorithm, the results of group i elements can be measured with a certain probability. However, no matter which group is measured, the maximum amplitude of the element solution can be guaranteed. The circuit diagram is shown below in Fig. 1.

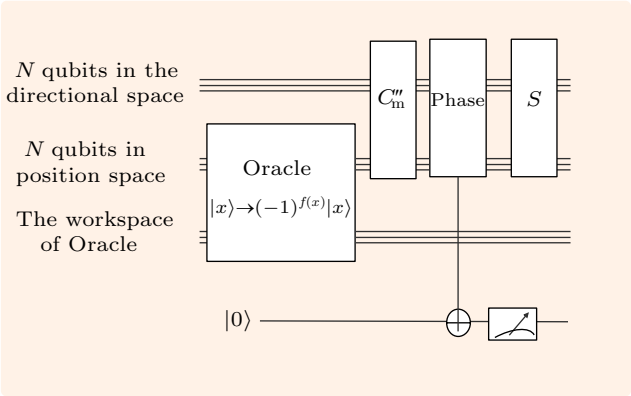


Fig. 1. Quantum circuit diagram of search algorithm. As figure 1 shows that in every iteration of the $U''_{\rm m}$ operator, measure the constant bit after performing phase detecting gate. If the projection measurement of the state is $|0\rangle$, then the algorithm continues. If the projection measurement of the state is $|1\rangle$, the algorithm stops.

3.3. The algorithm process

(I) The initial state is effectively implemented in node space by applying n bit Hadamard operations to state $|0\rangle$. A similar process applies to directional space.

(II) Cycle the following process. The operator is applied to the system state, and the auxiliary bit of measurement is: if the measurement result is $|0\rangle$, the cycle is continued; if the measurement result is $|1\rangle$, the cycle is ended.

(III) Measure the directional space state.

(IV) End of the algorithm.

4. Analysis of algorithm

4.1. Analysis of success rate of algorithm

The partition of $\mathcal{H}$ is the same for all $\boldsymbol{x}_{\rm tg}$ values. We have defined $V_x^t=\sum_i|\alpha_{i,x}^t|^2$. The state after t steps becomes

$$(U''_{\rm m})^t|\psi_0\rangle=\sum_i\sum_{d=0}^{n}\sum_{x=0}^{n-1}\alpha_{i,x}^t|d,x\rangle, \tag{20}$$

and with setting $\alpha_{1,0}^t=\alpha_{i,n}^t=0$ for convenience. The interpretation of V_x^t is from the definitions: V_x^t is the probability of target vertex after t iterations, and V_0^t is the total probability of the nearest neighbors of the target node. We can obtain the following inequality:

$$V_0^{t+1}\le V_1^t,\ \ V_0^{t-1}\le V_1^t. \tag{21}$$

So for any $t>0$, this is the probability

$$p_0=\sum_{d=0}^{n}|\langle d,0|\psi_t\rangle|^2,\ \ p_1=\sum_{|\boldsymbol{x}|=1}\sum_{d}^{n}|\langle d,\boldsymbol{x}|\psi_t\rangle|^2, \tag{22}$$

and, satisfies the inequality

$$p_0\le p_1. \tag{23}$$

The property (23) can also be tested on the numerical results presented in Fig. 2.

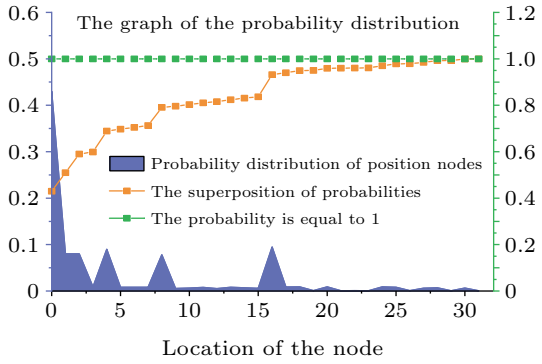


Fig. 2. The graph of the probability distribution. The plot shows the numerical probability distribution of node position after the optimal number of iterations. In accordance with the analytic results, the probability distribution has its maximum for the marked vertex $\boldsymbol{x}_{\rm tg}=0$ (the blue area) reaching a value close to 0.5. In addition, it can also be observed from the graph that nodes near the target node also have high probabilities and the sum of these probabilities (yellow curve) can be compared with the probability of the target node (refer to the axis on the right).

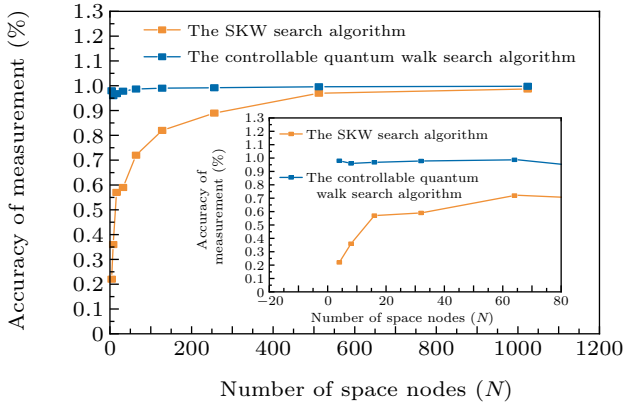


Fig. 3. Comparison of measurement accuracy of two search algorithms. Figure 3 shows the variation trend of the accuracy of the two search algorithms to search the target state with the increase of the number of space nodes. For the controllable quantum walk search (blue curve), the accuracy of searching the target node is basically stable at about 90% as the number of space nodes increases. For the SKW search algorithm (yellow curve), with the increase of the number of space nodes, the accuracy of searching the target node is constantly increasing. This shows that only when the number of spatial nodes is large enough, can the SKW search algorithm search the target state more accurately. This also illustrates a limitation of the SKW search algorithm.

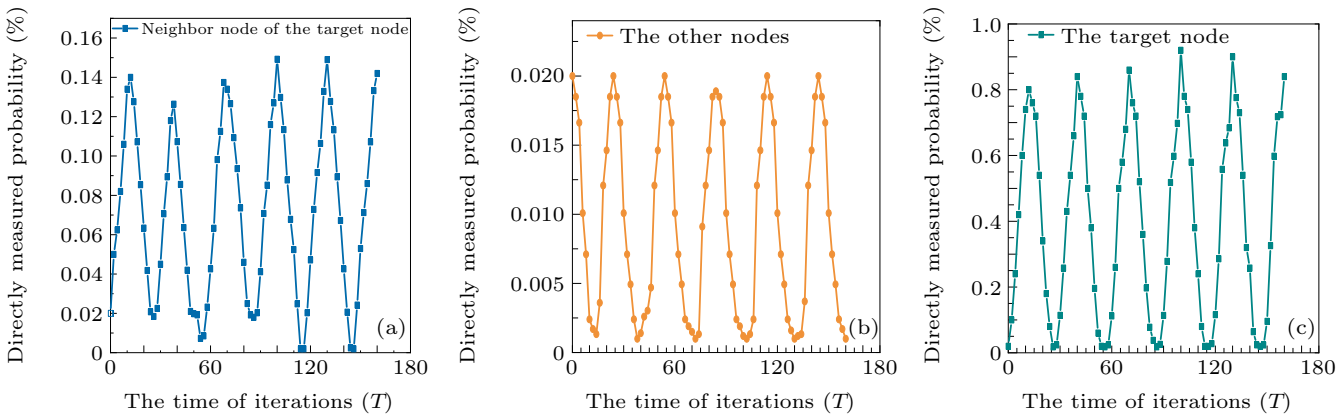


Fig. 4. The probability distribution of different nodes. Figure 4 shows us the variation trend chart of the probability of directly measuring nodes of different classes in space as the number of iterations increases, where the blue line represents the neighbor nodes of the target node, the yellow line represents other nodes, and the green line represents the target node. The following results can be obtained from the figure: The increase of the number of iterations does not simply increase or decrease the measurement probability of different types of nodes, but there is an optimal number of iterations. If the optimal number of iterations can be calculated, the target node can be measured with a probability of 90%.

Therefore, since we have $p_0 = (1/2) - O(1/n)$, the total probability of measuring the target nodes or any of its direct neighbors is

$$p_{\rm c} = p_0 + p_1 \geq 1 - O\left(\frac{1}{n}\right). \tag{24}$$

Since the upper bounded of $p_{\rm c}$ is 1, the total probability must be approaching $p_{\rm c} = 1$ when n is large. Figure 3 is obtained by comparing the accuracy of the quantum walk search algorithm and the SKW search algorithm.

Therefore, if a complete projection measurement of the coin's state is made, the marker element can be determined with the $1 - O(1/n)$ probability after performing a quantum walk search. The projection measurement of the system state after substituting the operator method for an appropriate number of times will make the final probability of getting the target node higher than 90%.

Finally, we analyze the influence of the operator P added on the controllable quantum walk search algorithm in this paper. Figure 4 is the probability distribution of different nodes in the search space is obtained by using the controllable quantum walk search algorithm.

In other words, the operator P added in this paper can stop the algorithm directly when the optimal probability is reached for the first time, that is, the optimal iteration. In this way, the target state can be measured with an absolutely high probability (90%), and the optimal iteration number can be achieved, which can improve the search efficiency.

4.2. Analysis of the time complexity of the algorithm

Now we analyze the time complexity of the search algorithm. We consider the state $U''_{\rm m}$ after t steps with initial state $|\psi_0\rangle$. Then we can unfold $|\psi_0\rangle$ as

$$|\psi_0\rangle = \sqrt{p_0}\left(|\omega'_0\rangle + |-\omega'_0\rangle\right) + \delta|r\rangle, \tag{25}$$

where $|r\rangle$ is a residual normalized vector orthogonal to $|\omega'_0\rangle$ and $|-\omega'_0\rangle$; $\delta = \sqrt{1-2p_0} = O(\sqrt{n/2^n})$; $|\omega'_0\rangle$ and $|-\omega'_0\rangle$ are the two eigenvectors with eigenvalues close to 1. We can get

$$\begin{aligned} U''^t_{\rm m}|\psi_0\rangle &= \sum_j \left(\sqrt{p_0}(\mathrm{e}^{\mathrm{i}\omega'_j t}|\omega'_j\rangle + \mathrm{e}^{-\mathrm{i}\omega'_j t}|-\omega'_j\rangle) + \delta|r'\rangle\right) \\ &= \sum_j \Big(2p_0\cos\omega'_j t|\psi_0\rangle - 2\sqrt{p_0 p_1}\Big(\sin\omega'_j t \\ &\quad + \mathrm{Re}\,\mathrm{e}^{\mathrm{i}\omega'_j t}\Delta\Big)|\psi_1\rangle + \sqrt{1-p_0-p_1}\Big(\mathrm{e}^{\mathrm{i}\omega'_j t}|r_0\rangle \\ &\quad + \mathrm{e}^{-\mathrm{i}\omega'_j t}|r_0^*\rangle\Big) + \delta|r\rangle\Big) \\ &= \sum_j \Bigg(\cos\omega'_j t|\psi_0\rangle - \sin\omega'_j t|\psi_1\rangle \\ &\quad + O\left(\frac{n^{3/4}}{\sqrt{2^n}}\right)|\widetilde{r}\rangle\Bigg), \end{aligned} \tag{26}$$

where $|\widetilde{r}\rangle$ is residual normalized vector.

In the literature,[11] when the number of target states is one, the number of iterations has been derived, so it is directly applied here to get

$$t_{\rm f} = \frac{\pi c}{2m}\sqrt{2^{n-1}}\left[1 \pm O\left(\frac{n^{3/2}}{\sqrt{2^n}}\right)\right]. \tag{27}$$

Using the inequality $1 + 1/(2n) \leq c^2 \leq 1 + 2/n$ to obtain $1 + 1/(4n) \leq c \leq 1 + 1/n$, we get

$$\begin{aligned} t_{\rm f} &= \frac{\pi}{2m}\sqrt{2^{n-1}}\left[1 + O\left(\frac{1}{n}\right)\right] \\ &< \frac{\pi}{2}\sqrt{2^{n-2}}\left[1 + O\left(\frac{1}{n}\right)\right]. \end{aligned} \tag{28}$$

So after $t_{\rm f} = (\pi/2)\sqrt{2^{n-2}}$ steps the probability about measuring $|x_{\rm target}\rangle$ is $p_{\rm c} = 1 - O(1/n)$. Figure 5 is obtained by comparing the the time complexity of the quantum walk search algorithm and the SKW search algorithm.

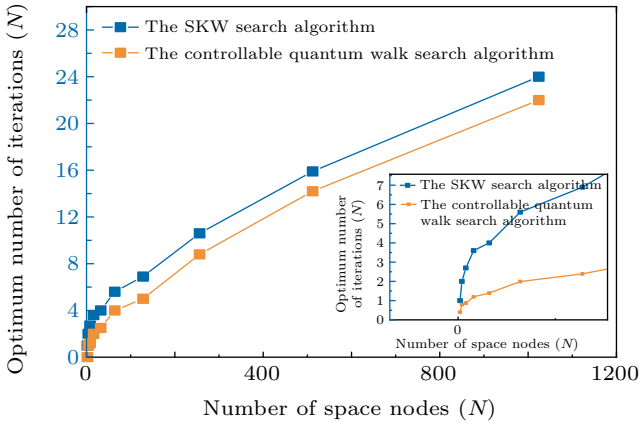


Fig. 5. The comparing the the time complexity of two search algorithms. The figure shows the trend that the time complexity of the two search algorithms changes as the number of space nodes increases. It can be seen from the figure that the time complexity of SKW search algorithm is greater than that of the controllable quantum walk search algorithm. In other words, the controlled quantum walk search algorithm can search the target state at a high rate.

5. Conclusion

In this paper, a search algorithm based on quantum walk is proposed. When the number of target states is unknown, the algorithm can control the iteration by judging the phase sign automatically and will stop when the probability of target states reach the maximum, and the number of iterations reached the optimal value at this time. And the Hilbert space $\mathcal{H}$ is decomposed into multiple subspaces $\mathcal{H}_i$, which can increase the probability of finding the target node. By adding the above two operations, we analyzed the time complexity of the algorithm and the accuracy of searching the target state respectively. The analysis results are as follows: Firstly, by dividing the search space into multiple subspaces, it is calculated that the accuracy of the search algorithm is increased from $(1/2)-O(1/n)$ to $1-O(1/n)$. As shown in Fig. 3, with the increase of the number of space nodes, the accuracy of searching the target state remains above 90%. Secondly, it can be seen from Fig. 4 that even if the accuracy of searching the target state is increased to more than 90%, there is a great difference in the number of iterations. We hope to find the number of iterations that can achieve the accuracy of more than 90% for the first time. Therefore, by increasing the P operator, we can stop when the probability amplitude of the first time is maximum, and the iteration number at this time is exactly the best iteration number. On the premise of ensuring accuracy, finding the optimal number of iterations means the minimum number of iterations. In other words, on the premise of ensuring the accuracy, adding P operator improves the search rate of the target state. Thirdly, by analyzing the time complexity of the controllable quantum walk search algorithm, the number of iterations is calculated as $(\pi/2)\sqrt{2^{n-2}}[1+O(1/n)]$. Compared with the iteration times $(\pi/2)\sqrt{2^{n-1}}[1+O(1/n)]$ of the SKW search algorithm, the controllable quantum walk search algorithm reduces the iteration times of searching to the target node. Therefore, the search algorithm proposed in this paper improves the search accuracy and reduces the number of unnecessary iterations.

Acknowledgments

Project supported by the National Natural Science Foundation of China (Grant Nos. 11975132 and 61772295), the Natural Science Foundation of Shandong Province, China (Grant No. ZR2019YQ01), and the Project of Shandong Provincial Higher Educational Science and Technology Program, China (Grant No. J18KZ012). Thanks to Chen Zhenyu, Yan Dandan, and Ma Yulin for their help in the numerical simulation experiment. Special thanks to the scholars who have made important contributions in quantum computing and quantum search algorithm.

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