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Citation: Chin. Phys. B, 2022, 31 (**3**): 030303. DOI: 10.1088/1674-1056/ac2299
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Tetrapartite entanglement measures of generalized GHZ state in the noninertial frames

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(Received 28 June 2021; revised manuscript received 14 August 2021; accepted manuscript online 1 September 2021)

Using a single-mode approximation, we carry out the entanglement measures, e.g., the negativity and von Neumann entropy when a tetrapartite generalized GHZ state is treated in a noninertial frame, but only uniform acceleration is considered for simplicity. In terms of explicit negativity calculated, we notice that the difference between the algebraic average π_4 and geometric average Π_4 is very small with the increasing accelerated observers and they are totally equal when all four qubits are accelerated simultaneously. The entanglement properties are discussed from one accelerated observer to all four accelerated observers. It is shown that the entanglement still exists even if the acceleration parameter r goes to infinity. It is interesting to discover that all 1-1 tangles are equal to zero, but 1-3 and 2-2 tangles always decrease when the acceleration parameter r increases. We also study the von Neumann entropy and find that it increases with the number of the accelerated observers. In addition, we find that the von Neumann entropy S_{ABCDI}, S_{ABCIDI}, S_{ABICIDI} and S_{AIBICIDI} always decrease with the controllable angle θ, while the entropies $S_{3-3\ \mathrm{non}}$, $S_{3-2\ \mathrm{non}}$, $S_{3-1\ \mathrm{non}}$ and $S_{3-0\ \mathrm{non}}$ first increase with the angle θ and then decrease with it.

Keywords: tetrapartite, generalized GHZ state, entanglement measures, dirac field, noninertial frames

PACS: 03.67.–a, 03.67.Mn, 03.65.Ud, 04.70.Dy **DOI:** 10.1088/1674-1056/ac2299

1. Introduction

The concept of entanglement introduced by Einstein, Podolsky, Rosen[1] and Schrödinger[2,3] in 1930s has become an important topic in quantum information since the shared entangled qubits support us to perform some quantum mechanical forms of communication, e.g., quantum dense coding and quantum teleportation.[4–10] This is because the non-locality is one of its most important characteristics. Among the different kinds of description for the degree of the entanglement, the negativity[11,12] and its relevant whole entanglement measures including the algebraic and geometric averages[13] are often used to study the degree of the entanglement. The von Neumann entropy[14–16] is also applied to study the entanglement property of the entangled quantum system. Up to now, a number of authors have worked out the bipartite and tripartite systems except for many multipartite systems.[10,17–24] In recent years, many endeavors were devoted to the inertial quantum properties, and a lot of interesting contributions in the noninertial frames have been made,[21,25–39] in which the general relativity, quantum field theory and quantum information theory have been combined.[13,21] This is because the non-negligible effects of relative motion have occurred in noninertial frame on quantum dynamics. Moreover, the relativistic field theory provides not only a complete theoretical framework but also an essential theory for many potential experimental setups. From these works mentioned above, it is recognized that the entanglement is degraded when the observer is moving in an acceleration. Since the tripartite entangled state was studied,[30] a lot of papers focused on some certain states, e.g., the GHZ state, the W-state and other relevant entangled states. Along with this line, we have carried out the entanglement measures of the tripartite and tetrapartite systems initialed in the pure or mixed states such as the GHZ, the W-class and the Werner states in noninertial frame.[40–47] Among them, the tripartite entanglement measures for a generalized GHZ state including a polar angle θ have been explored in the noninertial frame.[44] In this work, we are going to investigate the tetrapartite entanglement measures of a generalized GHZ state when the observers are moving in a uniform acceleration. Four different cases, i.e., the observer accelerated from one to four will be studied. The corresponding negativity, the algebraic and geometric averages as well as the von Neumann entropy will be calculated accordingly.

This work is organized as follows. We present in Section 2 the corresponding density matrix of the tetrapartite system initialed in a generalized GHZ state. In Section 3, starting with the density matrix, the negativity (1-1, 2-2 and 1-3 tangles), the algebraic and geometric averages π_4 and Π_4 are

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calculated. The von Neumann entropy is studied in Section 4. We finally present the conclusions in Section 5.

2. Tetrapartite entanglement from one to four accelerated observers

The initial generalized entangled GHZ state to be considered here has the form

$$|\psi_{\rm g}\rangle_{\rm ABCD} = \cos\theta|0000\rangle + \sin\theta|1111\rangle, \quad \theta \in [0,\pi/2]. \quad (1)$$

The subscripts A, B, C and D denote Alice, Bob, Charlie and David, respectively. This state becomes maximally entangled GHZ case when $\theta = \pi/4$. Its degree of entanglement depends on the controllable angle θ, but it is not entangled any more when $\theta = 0$ and $\theta = \pi/2$.

To study its entanglement property in the noninertial frames, as shown in Refs. [48–50], we use the Rindler coordinates to describe a family of observers in a uniform acceleration. Dividing Minkowski space-time into inaccessible regions I and II makes the accelerating observers in region I naturally disconnect from their analogous counterparts located in region II. The Minkowski vacuum state can be expressed by a product of two-mode squeezed states of the Rindler vacuum. In this work, using a single-mode approximation[21,38,40] rather than the general non-single-mode approximation,[51,52] in which Shi and his co-authors have first constructed the vacuum and one-particle structure states in the Kruskal modes and then studied the quantum coherence in a curved space time and the Bell correlation via quantum partially collapsing measurement, the transformations between both the coordinated systems for the fermion field can be expressed as the following form:[21,27,38,40]

$$\begin{aligned} |0_{w_i}\rangle_{\rm M} &= \cos r_i\,|0_{w_i}\rangle_{\rm I}|0_{w_i}\rangle_{\rm II} + \sin r_i\,|1_{w_i}\rangle_{\rm I}|1_{w_i}\rangle_{\rm II}, \\ |1_{w_i}\rangle_{\rm M} &= |1_{w_i}\rangle_{\rm I}|0_{w_i}\rangle_{\rm II}, \end{aligned} \quad (2)$$

where $\cos r_i = (1+{\rm e}^{-2\pi\omega_i c/a_i})^{-1/2}$. Here, a_i denotes the acceleration of the i-th accelerated observer and w_i is its respective frequency. For the acceleration $a_i \in [0,\infty)$, one has $r_i \in [0,\pi/4)$. As treated in our recent work about the tripartite case,[44] only monochromatic mode case $w_{\rm A,\,B,\,C,\,D} = w$ and uniform acceleration $a_{\rm A,\,B,\,C,\,D} = a$ are considered for simplicity. Thus, we have $r_{\rm A} = r_{\rm B} = r_{\rm C} = r_{\rm D} \equiv r$. To show how to expand the $|\psi_{\rm g}\rangle_{\rm ABCD}$ in the Rindler coordinates, let us give an explicit expression for the accelerated observer, say David, i.e.,

$$\begin{aligned} |\psi_{\rm g}\rangle_{\rm ABCDIDII} &= \cos\theta\sin(r_c)\,|0_{\rm A}0_{\rm B}0_{\rm C}1_{\rm DI}1_{\rm DII}\rangle \\ &\quad + \cos\theta\cos(r_c)\,|0_{\rm A}0_{\rm B}0_{\rm C}0_{\rm DI}0_{\rm DII}\rangle \\ &\quad + \sin\theta\,|1_{\rm A}1_{\rm B}1_{\rm C}1_{\rm DI}0_{\rm DII}\rangle. \end{aligned} \quad (3)$$

As mentioned above, the observers are confined to move in the region I so that it is natural to trace out the part of the antiparticle state in region II. We shall study this entanglement system in four different cases.

1) When David is accelerated and its density matrix $\rho_{\rm ABCDI} = |\psi_{\rm g}\rangle_{\rm ABCDI}\cdot{}_{\rm ABCDI}\langle\psi_{\rm g}|$ can be calculated. For simplicity, we only write out explicitly its nonzero elements

$$\begin{aligned} &a_{11} = \beta^2\gamma^2, \quad a_{116} = a_{161} = \alpha\beta\gamma, \\ &a_{22} = \beta^2\delta^2, \quad a_{1616} = \alpha^2, \end{aligned} \quad (4)$$

where and hereafter $\alpha = \sin(\theta)$, $\beta = \cos(\theta)$, $\delta = \sin(r)$ and $\gamma = \cos(r)$.

2) When Charlie and David are accelerated, the corresponding nonzero elements are found as

$$\begin{aligned} &a_{11} = \beta^2\gamma^4, \quad a_{116} = a_{161} = \alpha\beta\gamma^2, \\ &a_{22} = a_{33} = \beta^2\gamma^2\delta^2, \quad a_{44} = \beta^2\delta^4, \quad a_{1616} = \alpha^2. \end{aligned} \quad (5)$$

3) When the observers Bob, Charlie and David are accelerated, the nonzero matrix elements are given by

$$\begin{aligned} &a_{11} = \beta^6\gamma^2, \quad a_{116} = a_{161} = \beta^3\gamma\delta, \\ &a_{22} = a_{33} = a_{55} = \alpha^2\beta^4\gamma^2, \\ &a_{44} = a_{66} = a_{77} = \alpha^4\beta^2\gamma^2, \\ &a_{88} = \alpha^6\gamma^2, \quad a_{1616} = \delta^2. \end{aligned} \quad (6)$$

4) When all observers are accelerated, the nonzero matrix elements are

$$\begin{aligned} &a_{11} = \beta^2\gamma^8, \quad a_{116} = a_{161} = \alpha\beta\gamma^4, \\ &a_{22} = a_{33} = a_{55} = a_{99} = \beta^2\gamma^6\delta^2, \\ &a_{44} = a_{66} = a_{77} = a_{1010} = a_{1111} = a_{1313} = \beta^2\gamma^4\delta^4, \\ &a_{88} = a_{1212} = a_{1414} = a_{1515} = \beta^2\gamma^2\delta^6, \\ &a_{1616} = \alpha^2 + \beta^2\delta^8. \end{aligned} \quad (7)$$

In what follows, we are going to make use of these results to calculate the negativity and von Neumann entropy.

3. Negativity and whole entanglement measures

The negativity as one of the entanglement measures for describing the entanglement property of the entangled systems is defined by $N_{\alpha(\beta\gamma\delta)} = ||\rho^{{\rm T}_\alpha}_{\alpha(\beta\gamma\delta)}|| - 1$ and $N_{\alpha\beta} = ||\rho^{{\rm T}_\alpha}_{\alpha\beta}|| - 1$, which represent the 1-3 tangle and 1-1 tangle entanglements, respectively. Here we use symbols $||\rho^{{\rm T}_\alpha}_{\alpha(\beta\gamma\delta)}||$ and $||\rho^{{\rm T}_\alpha}_{\alpha\beta}||$ to denote the trace-norm of each partial transpose matrix. For a Hermitian operator O, the quantity $||O|| = {\rm tr}\sqrt{O^\dagger O}$,[53] and thus one has $||M|| - 1 = 2\sum_{i=1}^{N}|\lambda_M^{(-)}|^i$, where $\lambda_M^{(-)}$ are the negative eigenvalues of the corresponding matrix M. Now, our main task is about to find the negative eigenvalues of each density matrix, which are used to study the negativity including the 1-1, 1-3 and 2-2 tangles.

After tedious calculation, the explicit expressions of 1-3 tangles are given by

$$
\begin{aligned}
N_{\mathrm{A(BCDI)}} &= \sin(2\theta)\cos(r),\\
N_{\mathrm{DI(ABC)}} &= \frac{1}{4}\Big\{\Big[\cos^2(\theta)[\cos(2\theta)(-20\cos(2r)+\cos(4r)-13)+2(\cos(2r)+3)^2]\Big]^{1/2}-4\cos^2(\theta)\sin^2(r)\Big\}, \qquad (8)\\
N_{\mathrm{A(BCIDI)}} &= \sin(2\theta)\cos^2(r),\\
N_{\mathrm{DI(ABCI)}} &= \frac{1}{8}\Big\{\cos^2(\theta)(\cos(4r)-1)+2\Big[\cos^2(\theta)\cos^4(r)[-8\cos^2(\theta)\cos(2r)\\
&\quad +\cos(2\theta)(\cos(4r)-29)+\cos(4r)+35]\Big]^{1/2}\Big\}, \qquad (9)\\
N_{A(\mathrm{BICIDI})} &= \sqrt{\sin^2(2\theta)\cos^6(r)+\cos^4(\theta)\sin^{12}(r)}-\cos^2(\theta)\sin^6(r),\\
N_{\mathrm{BI(ACIDI)}} &= \frac{1}{8}\Big\{\sqrt{2}\Big[\cos^2(\theta)\left(-\cos^6(r)\right)[63\cos(2\theta)+\cos^2(\theta)(\cos(2r)+2\cos(4r)-\cos(6r))-65]\Big]^{1/2}\\
&\quad -8\cos^2(\theta)\sin^2(r)\cos^4(r)\Big\}, \qquad (10)\\
N_{\mathrm{AI(BICIDI)}} &= \frac{1}{64}\Big\{\cos^2(\theta)(4\cos(4r)+\cos(8r)-5)+8\{\cos^2(\theta)\cos^4(r)[2\cos(2\theta)(2\cos(2r)+\cos(4r)+5)(-6\cos(2r)\\
&\quad +\cos(4r)-3)+52\cos(2r)+24\cos(4r)-4\cos(6r)+\cos(8r)+55]\}^{1/2}\Big\}, \qquad (11)
\end{aligned}
$$

where

$$
\begin{aligned}
&N_{\mathrm{A(BCDI)}}=N_{\mathrm{B(ACDI)}}=N_{\mathrm{C(ABDI)}},\\
&N_{\mathrm{A(BCIDI)}}=N_{\mathrm{B(ACIDI)}},\\
&N_{\mathrm{DI(ABCI)}}=N_{\mathrm{CI(ABDI)}},\\
&N_{\mathrm{BI(ACIDI)}}=N_{\mathrm{CI(ABIDI)}}=N_{\mathrm{DI(ABICI)}},\\
&N_{\mathrm{AI(BICIDI)}}=N_{\mathrm{BI(AICIDI)}}=N_{\mathrm{CI(AIBIDI)}}=N_{\mathrm{DI(AIBICI)}}.
\end{aligned}
$$

Even though 2-2 tangle defined by $N_{\alpha\beta(\gamma\delta)}=||\rho^{\mathrm{T}_{\alpha\beta}}_{\alpha\beta(\gamma\delta)}||-1$ has no any direct relation to the whole entanglement measures, as before we present them for completeness, i.e.,

$$
\begin{aligned}
N_{\mathrm{AB(CDI)}} &= \sin(2\theta)\cos(r),\\
N_{\mathrm{AB(CIDI)}} &= \sqrt{\cos^4(\theta)\sin^8(r)+\sin^2(2\theta)\cos^4(r)}\\
&\quad -\cos^2(\theta)\sin^4(r),\\
N_{\mathrm{ADI(BCI)}} &= \sin(2\theta)\cos^2(r),\\
N_{\mathrm{ABI(CIDI)}} &= \frac{1}{16}\Big\{\Big[\cos^2(\theta)\cos^4(r)[-221\cos(2\theta)\\
&\quad +2\cos^2(\theta)(28\cos(4r)-8\cos(6r)+\cos(8r))\\
&\quad +8(25-39\cos(2\theta))\cos(2r)+291]\Big]^{1/2}\\
&\quad -16\cos^2(\theta)\sin^4(r)\cos^2(r)\Big\},\\
N_{\mathrm{AIBI(CIDI)}} &= \cos^4(r)\left[\sin(2\theta)-2\cos^2(\theta)\sin^4(r)\right]. \qquad (12)
\end{aligned}
$$

Also, we observe some useful relations

$$
\begin{aligned}
N_{\mathrm{AB(CDI)}} &= N_{\mathrm{AC(BDI)}}=N_{\mathrm{BC(ADI)}}=N_{\mathrm{ADI(BC)}}\\
&= N_{\mathrm{BDI(AC)}}=N_{\mathrm{CDI(AB)}},\\
N_{\mathrm{ACI(BDI)}} &= N_{\mathrm{ADI(BCI)}}=N_{\mathrm{BCI(ADI)}}=N_{\mathrm{BDI(ACI)}},\\
N_{\mathrm{AB(CIDI)}} &= N_{\mathrm{CIDI(AB)}},\\
N_{\mathrm{ABI(CIDI)}} &= N_{\mathrm{ACI(BIDI)}}=N_{\mathrm{BICI(ADI)}}=N_{\mathrm{ADI(BICI)}}\\
&= N_{\mathrm{BIDI(ACI)}}=N_{\mathrm{CIDI(ABI)}},\\
N_{\mathrm{AIBI(CIDI)}} &= N_{\mathrm{AICI(BIDI)}}=N_{\mathrm{BICI(AIDI)}}=N_{\mathrm{AIDI(BICI)}}\\
&= N_{\mathrm{BIDI(AICI)}}=N_{\mathrm{CIDI(AIBI)}}.
\end{aligned}
$$

Except for these, we observe that all 1-1 tangles, which include the $N_{\kappa\zeta}$, $N_{\kappa_I\zeta}$ and $N_{\kappa_I\zeta_I}$ [$\kappa,\zeta\in(\mathrm{A,B,C,D})$], are equal to zero. This implies that there is no any entanglement between any two qubits. This result is different from other cases such as the initial entangled W-class state,[38,39,46] in which the entanglement 1-1 tangles are never equal to zero.

As shown in Fig. 1 about the 1-3 tangles, we find that they first increase with θ and then decrease, but they always decrease with the increasing acceleration parameter r. How to explain this phenomenon? As we know, the degree of entanglement for the whole entangled system becomes weaker as the acceleration parameter r increases so that the entangled tetrapartite system becomes more disordered and less stable. We also notice that the entanglement of all them has never been disappeared even in the particular case $r\to\pi/4$. On the other hand, we observe that the $N_{\mathrm{A(BCDI)}}$ is symmetric with respect to $\theta=\pi/4$, but there is no such a symmetry for other negativities $N_{\mathrm{B(ACIDI)}}$, $N_{\mathrm{CI(ABDI)}}$, $N_{\mathrm{BI(ACIDI)}}$ and $N_{\mathrm{AI(BICIDI)}}$.

Based on the obtained negativity, let us study the whole entanglement measures. One of them is called the algebraic average π_4. Among the four components, the respective definition has the following form:[54]

$$
\pi_\kappa = N^2_{\kappa(\zeta\o\xi)}-N^2_{\kappa\zeta}-N^2_{\kappa\o}-N^2_{\kappa\xi}=N^2_{\kappa(\zeta\o\xi)}, \qquad (13)
$$

$$
\pi_\zeta = N^2_{\zeta(\kappa\o\xi)}-N^2_{\zeta\kappa}-N^2_{\zeta\o}-N^2_{\zeta\xi}=N^2_{\zeta(\kappa\o\xi)}, \qquad (14)
$$

$$\pi_{\emptyset} = N^2_{\emptyset(\kappa\zeta\xi)} - N^2_{\emptyset\kappa} - N^2_{\emptyset\zeta} - N^2_{\emptyset\xi} = N^2_{\emptyset(\kappa\zeta\xi)}, \quad (15)$$

$$\pi_{\xi} = N^2_{\xi(\kappa\zeta\emptyset)} - N^2_{\xi\kappa} - N^2_{\xi\zeta} - N^2_{\xi\emptyset} = N^2_{\xi(\kappa\zeta\emptyset)}, \quad (16)$$

from which we are able to calculate the algebraic average π_4 as $\pi_4 = \left(\pi_\kappa + \pi_\zeta + \pi_\emptyset + \pi_\zeta\right)/4$. We plot in Fig. 2 the algebraic averages $\pi_{\mathrm{A(BCDI)}}$, $\pi_{\mathrm{DI(ABC)}}$, $\pi_{\mathrm{A(BCIDI))}}$, $\pi_{\mathrm{CI(ABDI)}}$, $\pi_{\mathrm{A(BICIDI)}}$, $\pi_{\mathrm{BI(ACIDI)}}$ and $\pi_{\mathrm{AI(BICIDI)}}$ as the functions of both angle θ and acceleration parameter r. The π_{A}, π_{B}, π_{C}, and π_{D}, which include different number of the accelerated observers are displayed in Fig. 3. Beside this, another whole entanglement measurement is named as the geometric average[55] $\Pi_4 = \left(\pi_\kappa \pi_\zeta \pi_\emptyset \pi_\zeta\right)^{1/4}$.

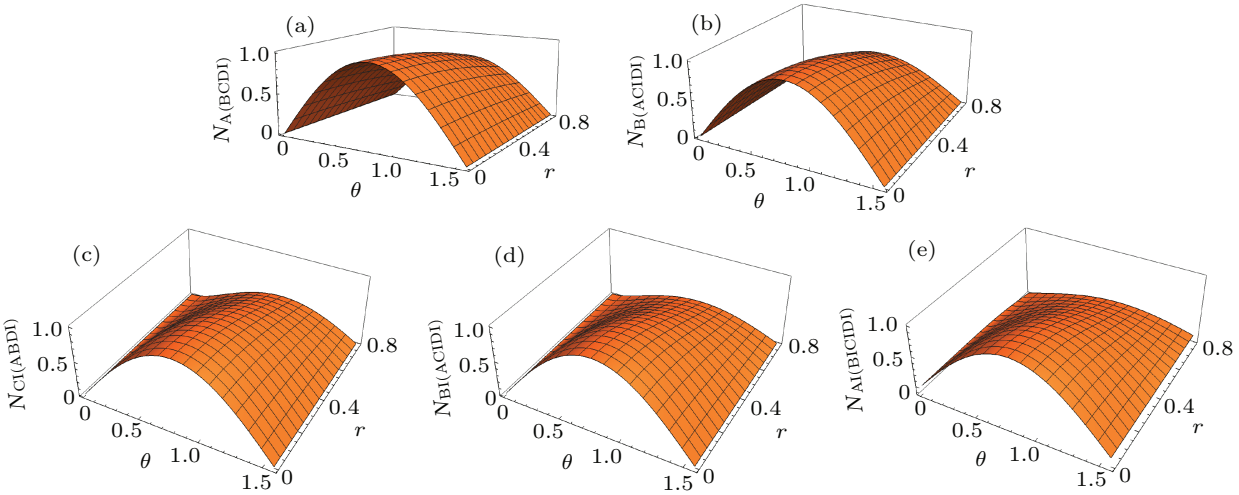


Fig. 1. The 1-3 tangles $N_{\mathrm{A(BCDI)}}$, $N_{\mathrm{B(ACIDI)}}$, $N_{\mathrm{CI(ABDI)}}$, $N_{\mathrm{BI(ACIDI)}}$ and $N_{\mathrm{AI(BICIDI)}}$ shown in (a)–(e), respectively.

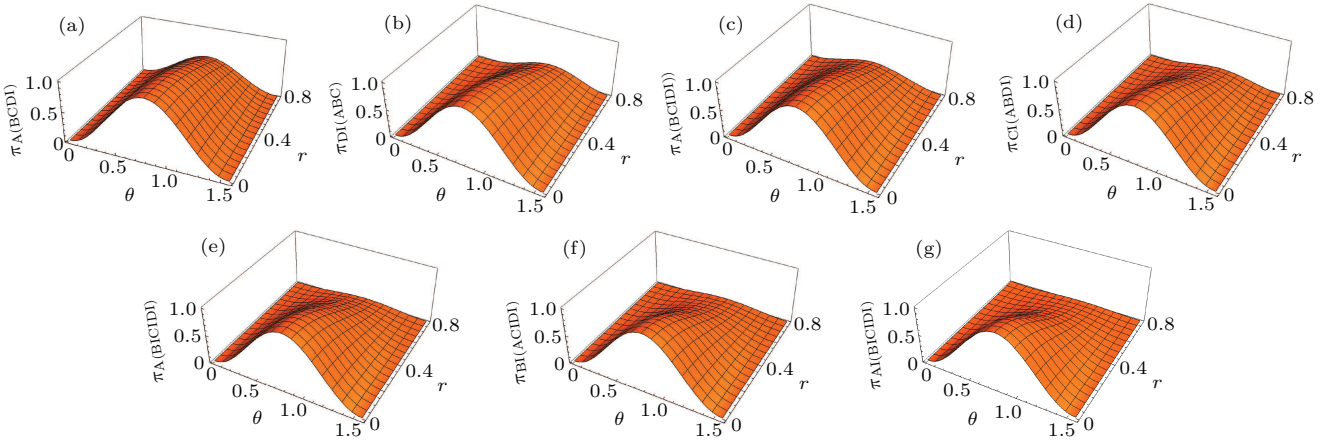


Fig. 2. Algebraic average $\pi_{\mathrm{A(BCDI)}}$, $\pi_{\mathrm{DI(ABC)}}$, $\pi_{\mathrm{A(BCIDI))}}$, $\pi_{\mathrm{CI(ABDI)}}$, $\pi_{\mathrm{A(BICIDI)}}$, $\pi_{\mathrm{BI(ACIDI)}}$ and $\pi_{\mathrm{AI(BICIDI)}}$.

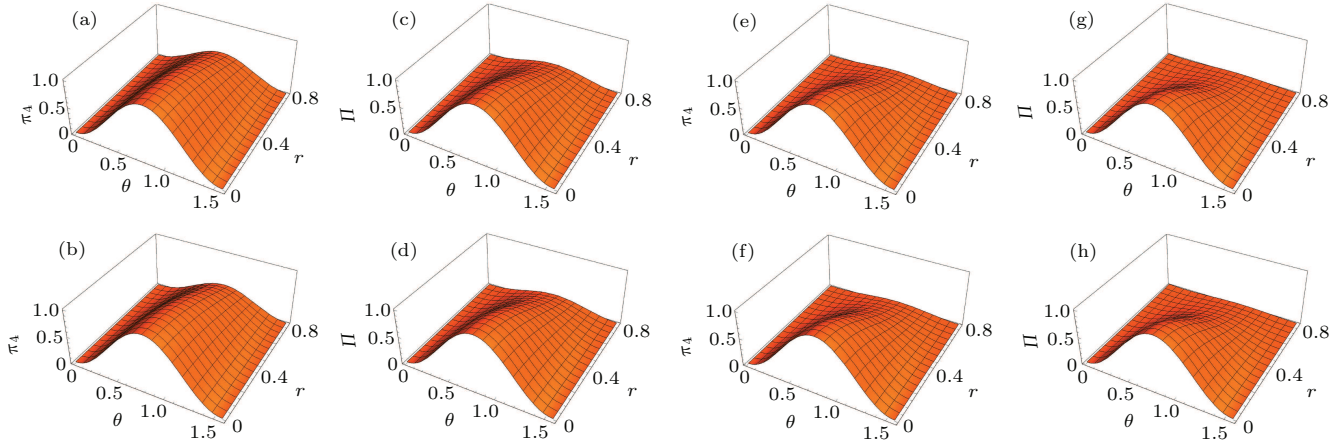


Fig. 3. Algebraic average measure π_4 and geometric average measure Π as the functions of both angle θ and acceleration parameter r: (a) and (b) for only one observer accelerated; (c) and (d) for two observers accelerated; (e) and (f) for three observers accelerated; (g) and (h) for all observers are accelerated.

We show the algebraic and geometric averages π_4 and Π_4 in Fig. 3 and notice that the difference between them is very small. This means that either π_4 or Π_4 can be used to describe the entanglement property of this system. How to explain this phenomenon? It is known from Eqs. (13)–(16) that two (two accelerated qubits) and three (one or three accelerated qubits) of algebraic averages $\pi_\kappa, \pi_\zeta, \pi_\emptyset$ and π_ξ are equal to each other if considering the symmetric properties of the negativity. This results in a small difference between the algebraic average and the geometric average. In particular, they are totally equal for the particular case, i.e., the four qubits are accelerated simultaneously.

4. The von Neumann entropy

To know the information measure for an entangled quantum system, let us finally study the von Neumann entropy[56]

$$S = -\mathrm{Tr}(\rho \log_2 \rho) = -\sum_{i=1}^{n} \lambda^{(i)} \log_2 \lambda^{(i)}, \tag{17}$$

where $\lambda^{(i)}$ denotes the i-th nonzero eigenvalue of the density matrix ρ.

In what follows, we are going to present eigenvalues of all bipartite, tripartite subsystems and tetrapartite systems, i.e.,

$$\begin{aligned}
&\lambda^{(1)}_{\mathrm{AB}} = \lambda^{(1)}_{\mathrm{ABC}} = \cos^2(\theta),\\
&\lambda^{(2)}_{\mathrm{AB}} = \lambda^{(2)}_{\mathrm{ABC}} = \sin^2(\theta),\\
&\lambda^{(1)}_{\mathrm{ADI}} = \lambda^{(1)}_{\mathrm{ABDI}} = \cos^2(\theta)\cos^2(r),\\
&\lambda^{(2)}_{\mathrm{ADI}} = \lambda^{(2)}_{\mathrm{ABDI}} = \cos^2(\theta)\sin^2(r),\\
&\lambda^{(3)}_{\mathrm{ADI}} = \lambda^{(3)}_{\mathrm{ABDI}} = \sin^2(\theta),
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\lambda^{(1)}_{\mathrm{CIDI}} = \cos^2(\theta)\cos^4(r),\\
&\lambda^{(2,3)}_{\mathrm{CIDI}} = \cos^2(\theta)\sin^2(r)\cos^2(r),\\
&\lambda^{(4)}_{\mathrm{CIDI}} = \sin^2(\theta) + \cos^2(\theta)\sin^4(r),
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\lambda^{(1)}_{\mathrm{ACIDI}} = \cos^2(\theta)\cos^4(r),\\
&\lambda^{(2,3)}_{\mathrm{ACIDI}} = \cos^2(\theta)\sin^2(r)\cos^2(r),\\
&\lambda^{(4)}_{\mathrm{ACIDI}} = \cos^2(\theta)\sin^4(r),\\
&\lambda^{(5)}_{\mathrm{ACIDI}} = \sin^2(\theta),
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\lambda^{(1)}_{\mathrm{BICIDI}} = \cos^2(\theta)\cos^6(r),\\
&\lambda^{(2,3,4)}_{\mathrm{BICIDI}} = \cos^2(\theta)\sin^2(r)\cos^4(r),\\
&\lambda^{(5,6,7)}_{\mathrm{BICIDI}} = \cos^2(\theta)\sin^4(r)\cos^2(r),\\
&\lambda^{(8)}_{\mathrm{BICIDI}} = \sin^2(\theta) + \cos^2(\theta)\sin^6(r),
\end{aligned} \tag{21}$$

$$\begin{aligned}
\lambda^{(1,2)}_{\mathrm{AIBICIDI}} &= \frac{1}{256}\Big[-29\cos(2\theta) + 2\cos^2(\theta)(28\cos(4r)+\cos(8r)) \mp 4\Big\{16\cos(4\theta)\sin^4(2r)(\cos(4r)+7)\\
&\quad -16\cos(2\theta)\sin^2(2r)(16\cos(4r)+\cos(8r)+47) + 301\cos(4r) + 46\cos(8r) + 3\cos(12r) + 674\Big\}^{1/2} + 99\Big],\\
\lambda^{(3,4,5,6)}_{\mathrm{AIBICIDI}} &= \cos^2(\theta)\sin^2(r)\cos^6(r),\quad \lambda^{(7,8,9,10,11,12)}_{\mathrm{AIBICIDI}} = \cos^2(\theta)\sin^4(r)\cos^4(r),\quad \lambda^{(13,14,15,16)}_{\mathrm{AIBICIDI}} = \cos^2(\theta)\sin^6(r)\cos^2(r),
\end{aligned} \tag{22}$$

where the symbols $\mp$ appearing in $\lambda^{(1,2)}_{\mathrm{AIBICIDI}}$ correspond to $\lambda^{(1)}_{\mathrm{AIBICI}}$ and $\lambda^{(2)}_{\mathrm{AIBICI}}$, respectively. The eigenvalues of the bipartite subsystems S_{AIBI} (S_{AICI}) are as same as those given by ρ_{BICI}. The nonzero eigenvalues of $\rho_{\mathrm{BC_I}}$ are as same as those of $\rho_{\mathrm{AC_I}}$, and so are other cases.

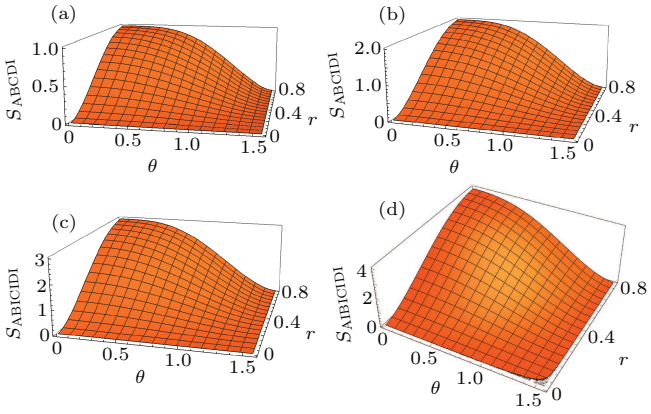


Fig. 4. The von Neumann entropies S_{ABCDI}, S_{ABCIDI}, S_{ABICIDI} and S_{AIBICIDI} plotted as the functions of both parameters.

As displayed in Fig. 4, we see that the von Neumann entropy becomes larger with the increasing number of accelerated observers. This is because the entangled system becomes more disordered and less stable as the accelerated observers increase and the quantum correlation among them becomes stronger. In general, the entropy increases with the increasing acceleration r but does not always increase with it. For example, as illustrated in Fig. 5, we notice that the entropies $S_{3-0\ \mathrm{non}}$ and $S_{2-0\ \mathrm{non}}$ are totally independent of the acceleration parameter r. Here we denote the number of the accelerated observers by the subscripts $0,1,2,3$-non. Let us explain this. As we know, the entropies $S_{3-0\ \mathrm{non}}$ and $S_{2-0\ \mathrm{non}}$ correspond to a particular case, i.e., without acceleration, and thus it is not strange that they are independent of the acceleration r. On the other hand, we observe for $\theta > \pi/8$ that the $S_{3-3\ \mathrm{non}}$ and $S_{2-2\ \mathrm{non}}$ first increase with the acceleration parameter r and then decrease with it, but $S_{3-2\ \mathrm{non}}$, $S_{3-1\ \mathrm{non}}$, and $S_{2-1\ \mathrm{non}}$ always increase with the acceleration parameter r. How to explain this kind of phenomenon? When the tetrapartite system is reduced to a tripartite system with all three accelerated observers, these entropies first reach a maximum value and then

decrease when the acceleration r increases. This implies that the stability of this entangled system is closely related with the controllable angel. This system will pass through a most disordered state and then gradually decreases with the increasing r. However, this did not occur only if one of three qubits, say $S_{3-2\ \mathrm{non}}$, $S_{3-1\ \mathrm{non}}$, and $S_{2-1\ \mathrm{non}}$, was not accelerated. In addition, we find that the von Neumann entropy S_{ABCDI}, S_{ABCIDI}, S_{ABICIDI} and S_{AIBICIDI} always decrease with the controllable angle θ, while the entropies $S_{3-3\ \mathrm{non}}$, $S_{3-2\ \mathrm{non}}$, $S_{3-1\ \mathrm{non}}$ and $S_{3-0\ \mathrm{non}}$ first increase with the angle θ and then decrease with it, but $S_{3-0\ \mathrm{non}}$ and $S_{2-0\ \mathrm{non}}$ are symmetric to the variable θ. Let us explain these interesting phenomena. For a fixed acceleration $r \neq 0$, as shown in Fig. 4, the von Neumann entropy of the whole entangled system changes gradually from the most disordered state (the controllable angle $\theta \simeq 0$) to a stable state ($\theta \simeq \pi/2$). On the contrary, that of the tripartite system in which one of four qubits disappears is different from that of the whole tetrapartite case (see Fig. 5). These tripartite entropies will arrive to the most disordered state around $\theta \approx \pi/4$ (exactly $\pi/4$ for the particular cases $S_{3-0\ \mathrm{non}}$ and $S_{2-0\ \mathrm{non}}$) when the controllable angel θ increases from the initial value of zero and then becomes more stable when it reaches to $\theta \approx \pi/2$. This can also be explained by the fact that the initial GHZ state becomes maximally entangled when $\theta = \pi/4$ as mentioned in Section 2. This means that the degree of entanglement in tripartite case is totally different from the tetrapartite case when one of the qubits disappears.

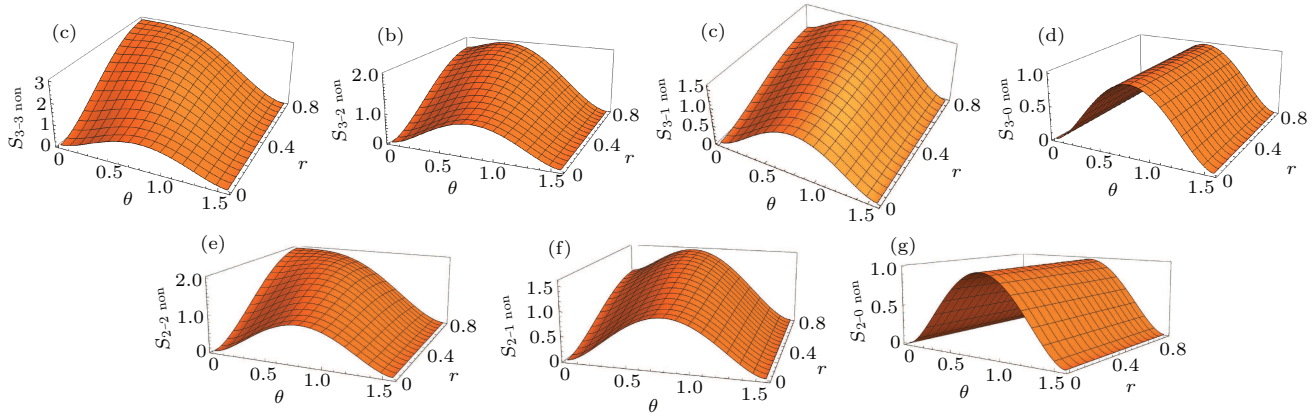


Fig. 5. The von Neumann entropies $S_{3-3\ \mathrm{non}}$, $S_{3-2\ \mathrm{non}}$, $S_{3-1\ \mathrm{non}}$, $S_{3-0\ \mathrm{non}}$, $S_{2-2\ \mathrm{non}}$, $S_{2-1\ \mathrm{non}}$ and $S_{2-0\ \mathrm{non}}$.

5. Concluding remarks

In this work, we have calculated the negativity (say the 1-1, 2-2 and 1-3 tangles) of the generalized tetrapartite GHZ state. Based on these results, the algebraic average and geometric average π_4 and Π_4 are computed. It is seen that all 1-1 tangles are equal to zero. This is different from the entangled W-class case,[38,39] in which there exists an entanglement sudden death. When compared between the arithmetic average π_4 and the geometric average Π_4, it is found that their difference is very small. This implies that one of them can be used to describe this entangled system.

As far as the von Neumann entropy, we have noticed that it increases with the increasing number of accelerated qubits and first increases and then decreases with the angle θ. Generally speaking, the von Neumann entropy for the tetrapartite system increases with the acceleration parameter r but that of the tripartite, say $S_{3-3\ \mathrm{non}}$ and $S_{2-2\ \mathrm{non}}$, does not always increase with it. The $S_{3-0\ \mathrm{non}}$ and $S_{2-0\ \mathrm{non}}$ are independent of the acceleration parameter r and symmetric to variable θ.

Before ending this work, we are going to give a few useful remarks. Firstly, compared the present results with those of the tripartite case,[44] it is found that the results obtained here are more complicated than those in the tripartite case. For example, the expressions of the respective density matrix in tetrapartite entangled system become more complicated because of the increasing number of qubits. This is the reason why we do not write them out explicitly in this work unlike the tripartite case.[44] On the other hand, the 1-3 and 2-2 tangles do not exist at all for the tripartite case. Undoubtedly, the results obtained in this work will reduce to those of tripartite case when one of entangled qubits, say David, disappears. Secondly, the von Neumann entropy in tetrapartite system becomes larger than that in tripartite case because of the increasing of the number of qubits. Thirdly, with respect to the recent study in Refs. [51,52], it is possible to consider in the future how to carry out the entanglement properties of this system when the general non-single-mode approximation is used.

Acknowledgement

This work was partially supported by the 20210414-SIP-IPN, Mexico.

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