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# Residual symmetries, consistent-Riccati-expansion integrability, and interaction solutions of a new (3+1)-dimensional generalized Kadomtsev–Petviashvili equation

Jian-Wen Wu(吴剑文), Yue-Jin Cai(蔡跃进), and Ji Lin(林机)[†]

*Department of Physics, Zhejiang Normal University, Jinhua* 321004, *China*

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With the aid of the Painlevé analysis, we obtain residual symmetries for a new (3+1)-dimensional generalized Kadomtsev–Petviashvili (gKP) equation. The residual symmetry is localized and the finite transformation is proposed by introducing suitable auxiliary variables. In addition, the interaction solutions of the (3+1)-dimensional gKP equation are constructed via the consistent Riccati expansion method. Particularly, some analytical soliton-cnoidal interaction solutions are discussed in graphical way.

**Keywords:** residual symmetry, interaction solutions, (3+1)-dimensional generalized Kadomtsev–Petviashvili equation

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## 1. Introduction

As is known, nonlinear integrable evolution equations have been used in many nonlinear fields, such as fluid mechanics,[1–3] nonlinear optics,[4–6] plasma physics,[7,8] ferrimagnetics[9] and Bose–Einstein condensation.[10] A class of physically important integrable equations, such as the KP equation and the cubic gKP equation, has been applied to describe long wave propagation of small amplitude propagating in plasma physics[11,12] and the collapse of ultrashort spatiotemporal pulses in nonlinear optics.[13,14] In 1970, Kadomtsev and Petviashvili proposed the standard KP equation

$$(u_t+6uu_x+u_{xxx})_x-u_{yy}=0, \tag{1}$$

which exhibits integrability, owing to the existence of infinitely many Lax pairs,[15] infinitely many conservation laws[16] and multiple soliton solutions.[17] Recently, the (3+1)-dimensional KP equation has attracted a great deal of attention from many mathematical physicists.[18–20] In Refs. [21,22], researchers presented a new (3+1)-dimensional gKP equation

$$\begin{aligned}&(u_t+6uu_x+u_{xxx})_x-u_{yy}\\&-\frac{\alpha^2}{4}u_{tt}+\alpha u_{yt}+\beta u_{xt}+\gamma u_{xz}=0,\end{aligned} \tag{2}$$

where $u=u(x,y,z,t)$ is a function, $\alpha$, $\beta$ and $\gamma$ are nonzero constants. The nonlinear equation, which is added these dispersion items $\frac{\alpha^2}{4}u_{tt}$, $\alpha u_{yt}$, $\beta u_{xt}$ and $\gamma u_{xz}$ to the standard KP equation, can describe the surface waves and internal waves with more dispersion effect. Without loss of generality, we choose $\alpha=1$ for Eq. (2), which is rewritten as

$$\begin{aligned}&(u_t+6uu_x+u_{xxx})_x-u_{yy}\\&-\frac{1}{4}u_{tt}+u_{yt}+\beta u_{xt}+\gamma u_{xz}=0.\end{aligned} \tag{3}$$

For the new model, the Painlevé integrability, bilinear form, multiple-soliton and lump solutions are investigated, while the interaction wave solutions are unsolved. It is well known that several systematic methods to construct exact and analytical solutions have been proposed, including inverse scattering method,[23,24] Hirota bilinear method,[25–30] and Lie group analysis.[31] The purpose of this paper stems from the observation of interaction waves in laboratory experiments, such as Fermionic quantum plasma[32] and elastic woodpile periodic structures.[33] In recent years, a number of papers have been devoted to research of interaction waves between solitons and other nonlinear waves.[34–36] Researchers have provided the nonlocal symmetry method,[37–42] the consistent tanh expansion method[43–45] and the consistent Riccati expansion (CRE) method,[46–50] which can be used to construct abundant interaction waves such as soliton-Painlevé Ⅱ waves and soliton-cnoidal waves. Based on the nonlocal symmetry method which is related to the infinitesimal forms of Darboux transformation and Bäcklund transformation (BT), many new interaction waves have been discussed for various integrable models, such as Korteweg–de Vries (KdV) equation,[51] sine-Gordon (sG) equation[52] and modified KdV-sG equation.[53] In addition, for the Painlevé integrable systems, the interaction solution and $n$-th Bäcklund transformation has been obtained in Refs. [54–56]. Moreover, for some nonintegrable systems such as the Kraenkel–Manna–Merle system, the interaction waves can be constructed.[57] In this work, we employ the residual symmetries and the auto Bäcklund transformation of the gKP equation (3) to obtain analytic soliton-cnoidal wave solutions, which can describe a soliton moving on a cnoidal

[†]Corresponding author. E-mail: linji@zjnu.edu.cn

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wave background.

The structure of the paper is as follows. In Section 2, we demonstrate the existence of the residual symmetry of the gKP equation (3) with the Painlevé analysis. Then, we take advantage of introducing suitable auxiliary variables to construct the localization of the residual symmetry. Based on the results, we construct the finite symmetry transformation via solving the initial value problem. In Section 3, with the help of the CRE method we derive the CRE solvability for the gKP equation (3). In Section 4, some analytical soliton-cnoidal interaction solutions are studied in graphical way. Section 5 contains a conclusion and discussion.

## 2. Nonlocal symmetry of (3+1)-dimensional gKP equation

Previously, Bluman found that the Lie point symmetries of the potential system, namely nonlocal symmetries, were derived by conservation laws.[58] In addition, researchers constructed nonlocal symmetries by the corresponding Bäcklund transformation and Lax pair.[37,59,60] Thus, along the way, how to construct a new nonlocal symmetry of the gKP equation (3) is interesting. In this paper, we start with the truncating Painlevé expansion of the gKP equation (3)

$$u=\frac{u_0}{f^2}+\frac{u_1}{f}+u_2 \tag{4}$$

with the singular manifold $f$ and the analytic functions $u_0, u_1$ and $u_2$ related to the variables $x$, $y$, $z$ and $t$. By substituting Eq. (4) into the gKP equation (3) and balancing the coefficients of $f^{-6}$ and $f^{-5}$, we obtain

$$u_0=-2f_x^2, \quad u_1=2f_{xx}. \tag{5}$$

Collecting the coefficients of $f^{-4}$, the explicit value of $u_2$ is constructed as follows:

$$u_2=\frac{f_{xx}^2}{2f_x^2}-\frac{2f_{xxx}}{3f_x}-\frac{(\beta+1)f_t+\gamma f_z}{6f_x}+\frac{(f_y-\frac{1}{2}f_t)^2}{6f_x^2}. \tag{6}$$

Further, for the coefficients of $f^{-3}$, $f^{-2}$, $f^{-1}$ and $f^0$, we obtain the Schwarzian form of the gKP equation (3) as follows:

$$\begin{aligned}&C_t-4(C_y+S_x-K_y)-(4\beta-C+4)C_x\\&+4(K-C)K_x-4\gamma L_x=0,\end{aligned} \tag{7}$$

where the Schwarzian variables are $S=\frac{f_{xxx}}{f_x}-\frac{3}{2}\frac{f_{xx}^2}{f_x^2}$, $C=\frac{f_t}{f_x}$, $K=\frac{f_y}{f_x}$ and $L=\frac{f_z}{f_x}$. The Schwarzian variables are all invariant under the Möbius transformation

$$f\to\frac{a+bf}{c+df}, \quad (ad\neq bc). \tag{8}$$

It is very interesting that $\sigma_u=u_1$ in Eq. (4) is a symmetry of the gKP equation (3), which can be checked by substituting it into the linearized form of the gKP equation (3)

$$\begin{aligned}&(\sigma_{u,t}+6u\sigma_{u,x}+6\sigma_u u_x-\sigma_{u,xxx})_x-\sigma_{u,yy}\\&-\frac{1}{4}\sigma_{u,tt}+\sigma_{u,yt}+\beta\sigma_{u,xt}+\gamma\sigma_{u,xz}=0,\end{aligned} \tag{9}$$

with Eq. (7). It is obvious that the symmetry $u_1$ is a residue in the expansion (4). Based on the above characteristics, the symmetry $\sigma_u=u_1$ is also called the residual symmetry of the gKP equation (3). In addition, the auto-BT (4) is obtained with Eqs. (5) and (6). On the basis of the standard Lie algebra theory, the finite transformation can be constructed by solving the corresponding initial value problem

$$\frac{\mathrm{d}\bar{u}(\varepsilon)}{\mathrm{d}\varepsilon}=2f_{xx}, \quad \bar{u}(0)=u. \tag{10}$$

Due to the nonlocal property of the residual symmetry, the initial value problem is very difficult. In order to overcome this puzzle, we introduce the auxiliary functions $g(x,y,z,t)$ and $h(x,y,z,t)$, and the original system is prolonged to a suitable system

$$\begin{aligned}&(u_t+6uu_x+u_{xxx})_x-u_{yy}-\frac{1}{4}u_{tt}+u_{yt}+\beta u_{xt}+\gamma u_{xz}=0,\\&6uf_x^2-3f_{xx}^2+4f_xf_{xxx}+[(\beta+1)f_t+\gamma f_z]f_x-(f_y-\frac{1}{2}f_t)^2=0,\\&g=f_x, \quad h=g_x.\end{aligned} \tag{11}$$

Under the infinitesimal transformation $u\to u+\varepsilon\sigma_u$, $f\to f+\varepsilon\sigma_f$, $g\to g+\varepsilon\sigma_g$ and $h\to h+\varepsilon\sigma_h$, the linearized equations of the prolonged system (11) are obtained as follows:

$$\begin{aligned}&(\sigma_{u,t}+6u\sigma_{u,x}+6\sigma_u u_x-\sigma_{u,xxx})_x-\sigma_{u,yy}\\&-\frac{1}{4}\sigma_{u,tt}+\sigma_{u,yt}+\beta\sigma_{u,xt}+\gamma\sigma_{u,xz}=0,\\&\sigma_{f,x}[2(\beta+1)f_xf_t+2\gamma f_xf_z+8f_xf_{xxx}+4f_yf_t\\&-12f_{xx}^2-f_t^2-4f_y^2]-2\sigma_{f,y}f_x(f_t-2f_y)\\&+\sigma_{f,t}f_x[2f_xf_tf_y-2(\beta+1)f_x+f_t]\\&+12\sigma_{f,xx}f_xf_{xx}-8\sigma_{f,xxx}f_x^2-2\gamma\sigma_{f,z}f_x^2-12\sigma_uf_x^3=0,\\&\sigma_g=\sigma_{f,x}, \quad \sigma_h=\sigma_{g,x}.\end{aligned} \tag{12}$$

Fortunately, the above linearized system has a solution

$$\sigma_u=2h, \quad \sigma_f=-f^2, \quad \sigma_g=-2fg, \quad \sigma_h=-2(g^2+fh). \tag{13}$$

Consequently, the corresponding vector field of the prolonged system (11) is

$$V=2h\frac{\partial}{\partial u}-f^2\frac{\partial}{\partial f}-2fg\frac{\partial}{\partial g}-2(g^2+fh)\frac{\partial}{\partial h}. \tag{14}$$

Thanks to Lie's first fundamental theorem,[61] using the following initial value condition:

$$\begin{aligned}&\frac{\mathrm{d}\bar{u}(\varepsilon)}{\mathrm{d}\varepsilon}=2\bar{h}(\varepsilon), \quad \bar{u}(\varepsilon)|_{\varepsilon=0}=u,\\&\frac{\mathrm{d}\bar{f}(\varepsilon)}{\mathrm{d}\varepsilon}=-\bar{f}(\varepsilon)^2, \quad \bar{f}(\varepsilon)|_{\varepsilon=0}=f,\\&\frac{\mathrm{d}\bar{g}(\varepsilon)}{\mathrm{d}\varepsilon}=-2\bar{f}(\varepsilon)\bar{g}(\varepsilon), \quad \bar{\psi}(\varepsilon)|_{\varepsilon=0}=\psi,\end{aligned}$$

$$\frac{\mathrm{d}\bar{h}(\varepsilon)}{\mathrm{d}\varepsilon} = -2[\bar{g}(\varepsilon)^2 + \bar{f}(\varepsilon)\bar{h}(\varepsilon)], \quad \bar{h}(\varepsilon)|_{\varepsilon=0} = h, \quad (15)$$

we can obtain the corresponding finite symmetry transformation

$$\begin{aligned} \bar{u} &= u - \frac{2g^2\varepsilon^2}{(f\varepsilon+1)^2} + \frac{2h\varepsilon}{f\varepsilon+1}, \quad \bar{f} = \frac{f}{f\varepsilon+1}, \\ \bar{g} &= \frac{g}{(f\varepsilon+1)^2}, \quad \bar{h} = \frac{h}{(f\varepsilon+1)^2} - \frac{2g^2\varepsilon}{(f\varepsilon+1)^3}. \end{aligned} \quad (16)$$

Here, $\{u, f, g, h\}$ is a solution of the prolonged system (11), and another solution $\{\bar{u}, \bar{f}, \bar{g}, \bar{h}\}$ can be obtained by the finite symmetry transformation (16).

## 3. CRE solvability of the (3+1)-dimensional gKP equation

In this section, based on the CRE method, we prove the CRE solvability of the gKP equation (3). Using the leading order analysis, we can realize that the truncated solution of the gKP equation (3) reads

$$u = u_0 + u_1 R(\phi) + u_2 R(\phi)^2, \quad (17)$$

with the analytic functions $u_0$, $u_1$, $u_2$ and $\phi$ related to the variables $x$, $y$, $z$ and $t$, and $R$ is the solution of the Riccati equation

$$R_\phi = a_0 + a_1 R + a_2 R^2, \quad (18)$$

where $a_0$, $a_1$ and $a_2$ are arbitrary constants. Substituting Eq. (17) into the gKP equation (3) and vanishing the coefficients of $R^6$, $R^5$ and $R^4$, we obtain

$$\begin{aligned} u_2 &= -2a_2^2\phi_x^2, \quad u_1 = -2a_2\phi_{xx} - 2a_1a_2\phi_x^2, \quad \delta = a_1^2 - 4a_0a_2, \\ u_0 &= \frac{\phi_t^2 - 4\phi_t\phi_y + 12\phi_{xx}^2 + 4\phi_y^2}{24\phi_x^2} - \frac{(\beta+1)\phi_t + \gamma\phi_z + 4\phi_{xxx}}{6\phi_x} \\ &\quad - \frac{1}{2}a_1^2\phi_x^2 - a_1\phi_{xx} + \frac{1}{3}\delta\phi_x^2. \end{aligned} \quad (19)$$

Then from the coefficient of $R^3$, the consistent equation of the gKP equation (3) is constructed,

$$\begin{aligned} &K_y - S_x - C_y - \gamma L_x - (\beta+1)C_x + (K-C)K_x \\ &+ \frac{1}{4}(C_t + CC_x) + \delta\phi_x\phi_{xx} = 0. \end{aligned} \quad (20)$$

Finally, it can be checked that the coefficients of $R^2$, $R$ and $R^0$ are identically zero with Eq. (20), which means the CRE solvability of the gKP equation (3). The solution of the Riccati equation (18) is obtained as follows:

$$R = \tanh(\phi), \quad \delta = 4. \quad (21)$$

If the solution of Eq. (20) is given, the explicit expression for $u$ will be deduced by

$$\begin{aligned} u &= -2\phi_x^2\tanh(\phi)^2 + 2\phi_{xx}\tanh(\phi) \\ &\quad + \frac{\phi_t^2 - 4\phi_t\phi_y + 12\phi_{xx}^2 + 4\phi_y^2}{24\phi_x^2} \\ &\quad + \frac{(\beta+1)\phi_t + \gamma\phi_z + 4\phi_{xxx}}{6\phi_x} + \frac{4}{3}\phi_x^2. \end{aligned} \quad (22)$$

## 4. Soliton-cnoidal wave solutions

In this section, the solutions of the (3+1)-dimensional gKP equation will be obtained, including trivial straight-line solutions and soliton-cnoidal wave solutions. As we all know, a trivial straight-line solution is formed as

$$\phi = k_1x + l_1y + m_1z + \omega_1t, \quad (23)$$

where $k_1$, $l_1$, $m_1$ and $\omega_1$ are arbitrary constants. Substituting Eq. (23) into Eq. (22) the one soliton solution reads

$$\begin{aligned} u &= -2k_1^2\tanh(k_1x + l_1y + m_1z + \omega_1t) \\ &\quad + \frac{4}{3}k_1^2 - \frac{\beta\omega_1 + \gamma m_1 + \omega_1}{6k_1} + \frac{1}{24k_1^2}(2l_1 - \omega_1)^2. \end{aligned} \quad (24)$$

It is very interesting that the interaction solutions between soliton and cnoidal wave are displayed in the Fermionic quantum plasma.[32] Consequently, to seek out the soliton-cnoidal wave solutions, one can assume $\phi$ in the following form:

$$\begin{aligned} \phi &= k_1x + l_1y + m_1z + \omega_1t + W(\xi), \\ \xi &= k_2x + l_2y + m_2z + \omega_2t. \end{aligned} \quad (25)$$

By substituting Eq. (25) into Eq. (20), the result reads

$$\begin{aligned} W_{1,\xi}^2 &= C_0 + C_1W_1 + C_2W_1^2 + C_3W_1^3 + C_4W_1^4, \\ W_1 &= W_1(\xi) \equiv W_\xi, \end{aligned} \quad (26)$$

with

$$\begin{aligned} C_0 &= \frac{5k_1^2}{12k_2^6}(2l_2 - \omega_2)^2 - \frac{k_1}{3k_2^5}(3\beta k_1\omega_2 \\ &\quad + 3\gamma k_1m_2 + 3k_1\omega_2 + 4l_1l_2 - 2l_1\omega_2 - 2l_2\omega_1 + \omega_1\omega_2) \\ &\quad + \frac{1}{k_2^4}(12k_1^4 + \beta k_1\omega_1 + \gamma k_1m_1 + k_1\omega_1 - \frac{1}{3}l_1^2 \\ &\quad + \frac{1}{3}l_1\omega_1 - \frac{1}{12}\omega_1^2) - \frac{2C_3k_1^3}{k_2^3} + \frac{C_2k_1^2}{k_2^2}, \\ C_1 &= \frac{k_1^2}{2k_2^5}(2l_2 - \omega_2)^2 - \frac{1}{k_2^4}\left(\beta k_1\omega_2 + \gamma k_1m_2 + k_1\omega_2 \right. \\ &\quad \left. + 2l_1l_2 - l_1\omega_2 - l_2\omega_1 + \frac{1}{2}\omega_1\omega_2\right) \\ &\quad + \frac{1}{k_2^3}(16k_1^3 + \beta\omega_1 + \gamma m_1 + \omega_1) - \frac{3C_3k_1^2}{k_2^2} + \frac{2C_2k_1}{k_2}, \\ C_4 &= 4. \end{aligned} \quad (27)$$

Then the explicit solution of the gKP equation (3) is

$$\begin{aligned} u &= -2(k_1 + k_2W_1)^2\tanh\left(k_1x + l_1y + m_1z + \omega_1t + \int W_1\mathrm{d}\xi\right)^2 \\ &\quad + 2k_2^2W_{1,\xi}\tanh(k_1x + l_1y + m_1z + \omega_1t \\ &\quad + \int W_1\mathrm{d}\xi) - \frac{1}{24(k_1 + k_2W_1)^2} \end{aligned}$$

$$\times\{16k_2^3(k_1+k_2W_1)W_{1,\xi\xi}-12k_2^4W_{1,\xi}^2-32k_2^4W_1^4$$
$$-128k_1k_2^3W_1^3+[4k_2(\beta\omega_2+\gamma m_2+\omega_2)$$
$$-(2l_2-\omega_2)^2-192k_1^2k_2^2]W_1^2+[4k_1(\beta\omega_2+\gamma m_2+\omega_2)$$
$$+4k_2(\beta\omega_1+\gamma m_1+\omega_1)$$
$$-2(2l_2-\omega_2)(2l_1-\omega_1)-128k_1^3k_2]W_1$$
$$+4k_1(\beta\omega_2+\gamma m_2+\omega_2)-(2l_1-\omega_1)^2-32k_1^4\}. \quad (28)$$

It is well known that the general solution of Eq. (26) can be written in terms of Jacobi elliptic functions. To show more in detail for this kind of solution, we list three special types in the following.

**Type 1** The solution of Eq. (26) is given as

$$W_1=b_0+b_1\text{JacobiSN}(K\xi,m), \quad (29)$$

where $b_0$, $b_1$, $m$ and $K$ are constants, and JacobiSN$(K\xi,m)$ is the usual Jacobi elliptic sine function. Substituting Eq. (29) into Eq. (26) and vanishing the coefficients of JacobiDN$(K\xi,m)$, JacobiSN$(K\xi,m)$, the solution of the gKP equation (3) is constructed,

$$u=-2(b_0k_2+k_1)^2(m\text{sn}+1)^2\tanh(\Theta)^2$$
$$+4m(b_0k_2+k_1)^2\text{cn}\,\text{dn}\tanh(\Theta)-2m^2(b_0k_2+k_1)^2\text{sn}^2$$
$$+4m(b_0k_2+k_1)^2\text{sn}+\frac{2}{3}(b_0k_2+k_1)^2(m^2+4)$$
$$-\frac{1}{6k_2}(\beta\omega_2+\gamma m_2+\omega_2)+\frac{1}{24k_2^2}(2l_2-\omega_2)^2,$$
$$\Theta=\frac{1}{2}\ln(\text{dn}-m\text{cn})+(b_0k_2+k_1)x$$
$$+\left(b_0l_2+\frac{1}{2}\omega_1+\frac{k_1l_2}{k_2}-\frac{k_1\omega_2}{2k_2}\right)y+(b_0\omega_2+\omega_1)t$$
$$+\left[16k_2(m^2-1)(b_0k_2+k_1)^3+2\gamma m_2(b_0k_2+k_1)\right.$$
$$\left.+2\beta(k_1\omega_2-k_2\omega_1)+2k_1\omega_2-\frac{\omega_1}{\gamma}\right]z \quad (30)$$

with the functions sn = JacobiSN$(K\xi,m)$, cn = JacobiCN$(K\xi,m)$, dn = JacobiDN$(K\xi,m)$ and the constant $K=-\frac{2}{k_2}(b_0k_2+k_1)$. Figures 1(a) and 1(b) show the interaction solution of a soliton with cnoidal wave (30) by fixing parameters

$$m=\frac{1}{20},\ b_0=1,\ \beta=\frac{1}{20},\ \gamma=4,\ k_1=\frac{9}{10},$$
$$k_2=-\frac{12}{25},\ l_2=\frac{1}{5},\ m_2=\frac{1}{10},\ \omega_1=\frac{1}{3},\ \omega_2=-\frac{1}{4}, \quad (31)$$

in the plane $(x,z)$ and the plane $(y,z)$. Figures 1(c) and 1(d) clearly display the wave travel mode along the $y$ and $z$ axes, respectively. The type of interaction solution can describe the soliton scattering from cnoidal wave train in a fiber.[62]

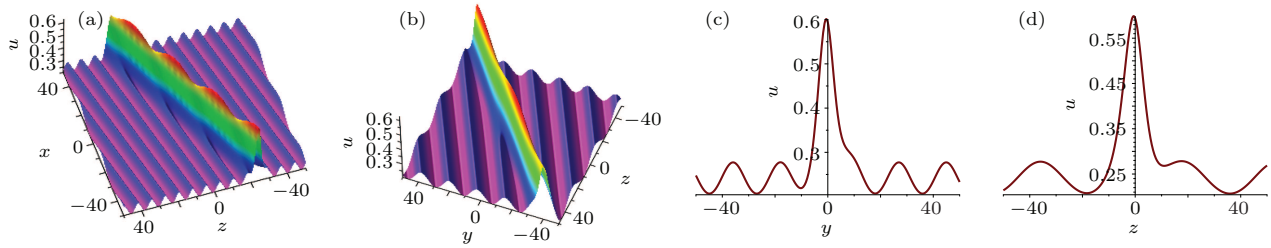



**Fig. 1.** The type of a bright soliton on a cnoidal wave background expressed by Eq. (30) with Eq. (31): (a) the corresponding three-dimensional plot of the soliton–cnoidal wave in the plane $(x,z)$ at $t=0$, (b) the three-dimensional plot of the soliton–cnoidal wave in the plane $(y,z)$ at $t=0$, (c) the wave travel mode along the $y$ axis, (d) the wave travel mode along the $z$ axis.

**Type 2** We take the function $W(\xi)$ of Eq. (25) as

$$W=-\frac{1}{2}\text{arctanh}[\text{JacobiSN}(k_2x+l_2y+m_2z+\omega_2t,m)], \quad (32)$$

and substituting Eq. (32) with Eq. (23) and conditions $\text{cn}^2=1-\text{sn}^2$ and $\text{dn}^2=1-m^2\text{sn}^2$ into Eq. (20) and vanishing all the coefficients of powers of sn, we obtain

$$m=\sqrt{\frac{[48k_1^4+4(l_1-\frac{1}{2}\omega_1)^2]k_2^2-12k_1^2k_2^4-8(l_1-\frac{1}{2}\omega_1)(l_2-\frac{1}{2}\omega_2)k_1k_2+4k_1^2(l_2-\frac{1}{2}\omega_2)^2}{12k_1^2k_2^4-3k_2^6}},$$
$$m_1=\frac{1}{24\gamma k_1^2k_2^2-6\gamma k_2^4}[-12k_1k_2^6+(96k_1^3+6\beta\omega_1+6\omega_1)k_2^4-(6\beta k_1\omega_2+6\gamma k_1m_2+6k_1\omega_2$$
$$+12l_1l_2-6l_1\omega_2-6l_2\omega_1+3\omega_1\omega_2)k_2^3-k_1(192k_1^4+24\beta k_1\omega_1+24k_1\omega_1-16l_1^2+16l_1\omega_1$$
$$-12l_2^2+12l_2\omega_2-4\omega_1^2-3\omega_2^2)k_2^2+4k_1^2(6\beta k_1\omega_2+6\gamma k_1m_2+6k_1\omega_2+4l_1l_2-2l_1\omega_2$$
$$-2l_2\omega_1+\omega_1\omega_2)k_2-8k_1^3(2l_2-\omega_2)^2]. \quad (33)$$

Therefore, we obtain the following soliton-cnoidal wave interaction solution of the gKP equation (3):

$$u=-\frac{\Sigma}{2\Lambda^2}\tanh(\Theta)^2-\frac{k_2^2}{\Lambda^2}\text{sn}\,(m^2-1)(m^2k_2^2\text{sn}^2-4k_1\text{sn}^2+4k_1\Lambda+4k_1^2-k_2^2)\tanh(\Theta)-\frac{\Delta}{24k_2\Lambda^2}, \quad (34)$$

where

$$\begin{aligned}
\Sigma &= (k_2^2m^2+12k_1^2)(k_2^2m^2-4k_1^2)\mathrm{sn}^4+(8\Lambda k_1k_2^2m^2-8k_1^2k_2^2m^2-2k_2^4m^2+32\Lambda k_1^3+96k_1^4\\
&\quad -8k_1^2k_2^2)\mathrm{sn}^2-32\Lambda k_1^3-8\Lambda k_1k_2^2-48k_1^4+8k_1^2k_2^2+k_2^4,\\
\Theta &= k_1x+l_1y+m_1z+\omega_1t-\frac{1}{2}\mathrm{arctanh}(\mathrm{sn}),\quad \Lambda=k_2\mathrm{cn}\,\mathrm{dn}+2k_1\mathrm{sn}^2-2k_1,\\
\Delta &= [8k_2^5m^2(m^2-2)-64k_1^2k_2^3(2m^2-1)+4m^2k_2^2(\beta\omega_2+\gamma m_2+\omega_2)+(384k_1^4-4l_2^2m^2+4l_2m^2\omega_2\\
&\quad -m^2\omega_2^2-16l_1^2+16l_1\omega_1-4\omega_1^2)k_2-8k_1(2\beta k_1\omega_2+2\gamma k_1m_2+2k_1\omega_2-4l_1l_2+2l_1\omega_2+2l_2\omega_1-\omega_1\omega_2)]\mathrm{sn}^4\\
&\quad +[4k_2^5(m^2+1)^2-32k_1k_2^3(m^2+1)(\Lambda-2k_1)-4(m^2+1)(\beta\omega_2+\gamma m_2+\omega_2)k_2^2\\
&\quad +(4l_2^2m^2-256\Lambda k_1^3-768k_1^4-4l_2m^2\omega_2+m^2\omega_2^2+8\Lambda\beta\omega_1+8\Lambda\gamma m_1+8\Lambda\omega_1+32l_1^2-32l_1\omega_1\\
&\quad +4l_2^2-4l_2\omega_2+8\omega_1^2+\omega_2^2)k_2+4(2\beta k_1\omega_2+2\gamma k_1m_2+2k_1\omega_2-4l_1l_2+2l_1\omega_2+2l_2\omega_1-\omega_1\omega_2)(\Lambda+4k_1)]\mathrm{sn}^2\\
&\quad +(8-16m^2)k_2^5+32k_1k_2^3(\Lambda m^2+2k_1m^2+\Lambda-4k_1)+(4\beta\omega_2+4\gamma m_2+4\omega_2)k_2^2\\
&\quad +(256\Lambda k_1^3+384k_1^4-8\Lambda\beta\omega_1-8\Lambda\gamma m_1-8\Lambda\omega_1-16l_1^2+16l_1\omega_1-4l_2^2+4l_2\omega_2-4\omega_1^2-\omega_2^2)k_2\\
&\quad -4(2\beta k_1\omega_2+2\gamma k_1\omega_2-4l_1l_2+2l_1\omega_2+2l_2\omega_1-\omega_1\omega_2)(\Lambda+2k_1),
\end{aligned}\tag{35}$$

with the arbitrary constants $k_1$, $k_2$, $l_1$, $l_2$, $\omega_1$, $\omega_2$, $m_2$, $\beta$ and $\gamma$. Meanwhile, the constants $m_1$ and $m$ are given by Eq. (33), and $\mathrm{sn}=\mathrm{JacobiSN}(k_2x+l_2y+m_2z+\omega_2t,m)$, $\mathrm{cn}=\mathrm{JacobiCN}(k_2x+l_2y+m_2z+\omega_2t,m)$ and $\mathrm{dn}=\mathrm{JacobiDN}(k_2x+l_2y+m_2z+\omega_2t,m)$. From Fig. (2), we find that one soliton resides on the cnoidal wave instead of plane wave background by choosing suitable coefficients

$$\begin{aligned}
&m_2=\frac{1}{2},\ \beta=1,\ \gamma=2,\ k_1=-\frac{2}{3},\ k_2=-\frac{6}{5}\\
&l_1=\frac{1}{20},\ l_2=-\frac{1}{2},\ \omega_1=2,\ \omega_2=2.
\end{aligned}\tag{36}$$

**Type 3** Another special solution of Eq. (26) reads

$$W=\frac{1}{K}E_\pi[\mathrm{sn}\,(K\xi,m),\nu,m],\tag{37}$$

where $E_\pi$ is the third type of incomplete elliptic integral. Thus, the interaction solution of the gKP equation (3) is obtained,

$$\begin{aligned}
u &= -\frac{2k_2^2}{(\nu\mathrm{sn}^2-1)^2}\tanh(\Theta)^2+\frac{4Kk_2^2\nu\,\mathrm{cn}\,\mathrm{dn}\,\mathrm{sn}}{(\nu\mathrm{sn}^2-1)^2}\tanh(\Theta)\\
&\quad -\frac{\Sigma}{k_2^2(\nu-1)(\nu\mathrm{sn}^2-1)^2},
\end{aligned}\tag{38}$$

where

$$\begin{aligned}
\Theta &= \frac{1}{K}E_\pi[\mathrm{sn}\,\nu,m]-\frac{\omega_1}{2}(2t+y)+\frac{\omega_1z}{\gamma}(\beta+1)\\
&\quad +\frac{8k_2^2K^2z}{\gamma}-\frac{8k_2^2z}{\gamma(\nu-1)},\\
m &= \sqrt{\nu-\frac{\nu}{K^2(\nu-1)}},\\
\Sigma &= [16k_2^4(3K^2\nu-K^2\nu^2-2K^2+\nu-3)\\
&\quad +4(\nu-1)(\beta\omega_2+\gamma m_2+\omega_2)k_2\\
&\quad -(2l_2-\omega_2)^2(\nu-1)]\nu^2\mathrm{sn}^4\\
&\quad -2[8k_2^4(K^2\nu^2-K^2-4\nu+3)\\
&\quad +4k_2(\nu-1)(\beta\omega_2+\omega_2+\gamma m_2)\\
&\quad -4\left(l_2-\frac{1}{2}\omega_2\right)^2(\nu-1)]\nu\mathrm{sn}^2\\
&\quad +16k_2^4(2K^2\nu^2-3K^2\nu+K^2-2\nu+3)\\
&\quad +4k_2(\nu-1)(\beta\omega_2+\omega_2+\gamma m_2)\\
&\quad -4(l_2-\frac{1}{2}\omega_2)^2(\nu-1),
\end{aligned}\tag{39}$$

with the functions $\mathrm{sn}=\mathrm{JacobiSN}(K\xi,m)$, $\mathrm{cn}=\mathrm{JacobiCN}(K\xi,m)$, $\mathrm{dn}=\mathrm{JacobiDN}(K\xi,m)$. This kind of interaction solutions is more useful in plasmas.[32]

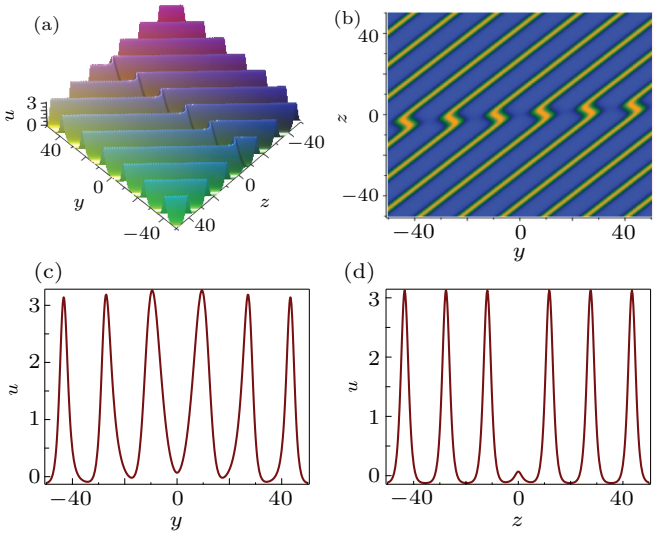



**Fig. 2.** The type of a soliton resides on a cnoidal wave background expressed by Eq. (34) with Eqs. (33) and (36): (a) the corresponding three-dimensional plots for components $u$ in the plane $(y,z)$ at $t=0$, (b) the density plot of the soliton–cnoidal wave in the plane $(y,z)$ at $t=0$, (c) the wave travel mode along the $y$ axis, (d) the wave travel mode along the $z$ axis.

## 5. Conclusion and discussion

In conclusion, the interaction waves of the 3+1-dimensional gKP equation (3) are studied via its residual symmetries related to its Painlevé integrability. Firstly, a special

residual symmetry is constructed for the gKP equation (3) by requiring the prolong system (11). Then, due to Lie's first fundamental theorem, the initial value problem has been solved, which leads to a finite symmetry transformation, and with the help of the finite symmetry transformation, one can find some new solutions of the gKP equation (3) from a given one. Furthermore, according to the CRE method, the consistent equation (20) of the gKP equation (3) is obtained. Finally, the interaction solutions for soliton-cnoidal wave are unearthed, which can be explicitly constructed by the Jacobi elliptic functions.

It would be very interesting to search for nonlocal symmetries in any other integrable systems. There are other more methods to obtain the nonlocal symmetries, such as the Darboux transformation[59] and the Bäcklund transformation.[37] Many questions remain open in the domain of interaction solutions among different waves, which need be investigated in the future.

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