

Increasing the efficiency of post-selection in direct measurement of the quantum wave function

Yong-Li Wen(温永立), Shanchao Zhang(张善超), Hui Yan(颜辉), and Shi-Liang Zhu(朱诗亮)
Citation: Chin. Phys. B, 2022, 31 (**3**): 034206. DOI: 10.1088/1674-1056/ac1f04
Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Topological phases and type-II edge state in two-leg-coupled Su-Schrieffer-Heeger chains

Tianqi Luo(罗天琦), Xin Guan(关欣), Jingtao Fan(樊景涛), Gang Chen(陈刚), and Suo-Tang Jia(贾锁堂)
Chin. Phys. B, 2022, 31 (**1**): 014208. DOI: 10.1088/1674-1056/ac3650

Parameter accuracy analysis of weak-value amplification process in the presence of noise

Jiangdong Qiu(邱疆冬), Zhaoxue Li(李兆雪), Linguo Xie(谢林果), Lan Luo(罗兰), Yu He(何宇), Changliang Ren(任昌亮), Zhiyou Zhang(张志友), and Jinglei Du(杜惊雷)
Chin. Phys. B, 2021, 30 (**6**): 064216. DOI: 10.1088/1674-1056/abcfa0

Reversion of weak-measured quantum entanglement state

Shao-Jiang Du(杜少将), Yonggang Peng(彭勇刚), Hai-Ran Feng(冯海冉), Feng Han(韩峰), Lian-Wu Yang(杨连武), Yu-Jun Zheng(郑雨军)
Chin. Phys. B, 2020, 29 (**7**): 074202. DOI: 10.1088/1674-1056/ab8c3e

Extended validity of weak measurement

Jiangdong Qiu(邱疆冬), Changliang Ren(任昌亮), Zhaoxue Li(李兆雪), Linguo Xie(谢林果), Yu He(何宇), Zhiyou Zhang(张志友), Jinglei Du(杜惊雷)
Chin. Phys. B, 2020, 29 (**6**): 064214. DOI: 10.1088/1674-1056/ab8622

Optimizing quantum correlation dynamics by weak measurement in dissipative environment

Du Shao-Jiang, Xia Yun-Jie, Duan De-Yang, Zhang Lu, Gao Qiang
Chin. Phys. B, 2015, 24 (**4**): 044205. DOI: 10.1088/1674-1056/24/4/044205

Increasing the efficiency of post-selection in direct measurement of the quantum wave function

Yong-Li Wen(温永立)[1], Shanchao Zhang(张善超)[2,3], Hui Yan(颜辉)[2,3], and Shi-Liang Zhu(朱诗亮)[2,3,†]

[1] *National Laboratory of Solid State Microstructures and School of Physics, Nanjing University, Nanjing* 210093, *China*
[2] *Guangdong Provincial Key Laboratory of Quantum Engineering and Quantum Materials, School of Physics and Telecommunication Engineering, South China Normal University, Guangzhou* 510006, *China*
[3] *Guangdong–Hong Kong Joint Laboratory of Quantum Matter, Frontier Research Institute for Physics, South China Normal University, Guangzhou* 510006, *China*

(Received 10 June 2021; revised manuscript received 11 July 2021; accepted manuscript online 19 August 2021)

Direct weak or strong measurement of quantum wave function based on post-selections has been widely explored; however, the efficiency of the measurement is heavily limited by the success probability of post-selection. Here we propose a modified scheme to directly measure photon's wave function by simply inserting a liquid crystal plate before the post-selection stage. Numerical simulations demonstrate that our modified method can significantly increase the efficiency of post selection. Our proposal would speed up the quantum wave function measurement with high resolution and high fidelity.

Keywords: weak measurement, quantum state

PACS: 42.50.Dv **DOI:** 10.1088/1674-1056/ac1f04

1. Introduction

A wave function is a fundamental concept in quantum mechanics,[1,2] and its determination is therefore of crucial importance in many applications. However, with the traditional quantum measurement methods developed by Von Neumann,[1] we can only obtain the modulus square of the wave function, which determines the probabilities of all kinds of measurement outcomes. For a long time, quantum-state tomography has been considered as an operational measurement method to reconstruct the wave function.[3–7] Quantum-state tomography is a conventional way to construct a quantum wave function, but it usually requires plenty of projection measurements on several bases sets, and thus it is also considered as an indirect measurement method with a limitation in efficiency and concision.

Recently, an alternative method of measuring the wave function was proposed in Ref. [8] based on the concepts of weak measurement[9–21] and post-selection. This method was named "direct measurement" in Refs. [8,22]. The word "direct" here expresses that the real and imaginary components of the wave-function are proportional to the signal of the experimental apparatus and no complicated computation is required.

The method has been widely applied for measurement of quantum states.[23–33] The method has also been generalized from the weak measurement to the strong measurement.[34–38] On the other hand, an alternative direct measurement strategy based on a δ-quench probe was developed: quenching its complex probability amplitude one by one (δ quench) in a given basis, one can directly obtain the quantum wave function of a pure ensemble by projecting the quenched state onto a post-selection state.[39,40] However, the efficiency of these direct measurement methods is limited by the success probability of post-selection,[41] which is usually pretty low in a practical experiment, and thus increasing the efficiency of post-selection is highly needed.

Analyzing carefully the measurement efficiency of the experiment in which the transverse wave function of a photon was measured with the direct measurement method,[8] we find that the efficiency is mainly dependent on the success probability of post-selection. In this article, we propose a modified method that can significantly increase the efficiency of directly measuring photon's wavefunction by inserting a polarization dependent Fourier transformation (Pol-FT) lens before post-selection stage. In the previous direct measurement scheme,[8,34] the photons would be post-selected to a zero transverse momentum state, but the distribution in zero momentum is pretty low, especially under the conditions of high resolution and high measurement fidelity. With a liquid crystal plate as the Pol-FT lens, almost all the input photons can contribute to the zero momentum components in post-selection stage. Therefore, the post-selection would not decrease the measurement efficiency. Numerical simulations demonstrate that our proposal can have greater efficiency in higher resolution and higher fidelity. Our modified method can thus greatly increase the measurement efficiency of the direct measurement of the quantum wave function.

This paper is organized as follows. In Section 2, we review a general method of direct measurement of the wave function, including the previous weak and strong measurements as examples. The efficiency of the method is discussed.

[†]Corresponding author. E-mail: slzhu@nju.edu.cn

© 2022 Chinese Physical Society and IOP Publishing Ltd

In Section 3, we propose a modified method to improve the efficiency of post-selection. Section 4 investigates the performances of our proposed method. Finally, short discussion and conclusion are given in Section 5.

2. Direct measurement of the wave function

In this section, we present a general direct measurement method which includes the case under a weak coupling proposed in Ref. [8] and strong coupling proposed in Ref. [34], and then discuss the efficiency of this method.

We consider to measure a quantum wave function $|\psi\rangle = \sum_x \psi(x)|x\rangle$, where the basis $\{|x\rangle\}$ are position eigenstates of a discretized segment. A measurement can be considered as the coupling between an apparatus and a physical system that results in the translation of a pointer. Here we study the case of a photon spatial wave function, and the pointer can be represented by the polarization of the photons. The coupling between an observable and the pointer can be written as a Hamiltonian $\hat{H} = g\hat{\pi}_x\hat{\sigma}_y$, where $\hat{\pi}_x = |x\rangle\langle x|$, g is the coupling strength, and $\hat{\sigma}_{x,y,z}$ in this paper represent the Pauli matrices. The evolution operator with arbitrary coupling strength can be written as

$$\begin{aligned}\hat{U}(\theta) &= \exp[-\mathrm{i}g\hat{\pi}_x \otimes \hat{\sigma}_y t/\hbar] \\ &= \hat{I}_\pi \otimes \hat{I}_\sigma - 2\sin^2\frac{\theta}{2}\hat{\pi}_x \otimes \hat{I}_\sigma - \sin\theta\,(\mathrm{i}\hat{\pi}_x \otimes \hat{\sigma}_y), \quad (1)\end{aligned}$$

where $\theta = gt/\hbar$, $\hat{I}_\pi$ is the identity operator of position, and $\hat{I}_\sigma$ is the identity operator of the pointer. Suppose the initial state of the pointer is $|0\rangle$, and the whole system after the coupling can then be written as

$$\begin{aligned}\hat{U}(\theta)|\psi\rangle \otimes |0\rangle &= [|\psi\rangle - 2\sin^2\frac{\theta}{2}\psi(x)|x\rangle] \otimes |0\rangle \\ &\quad - \sin\theta\,\psi(x)|x\rangle \otimes |1\rangle. \quad (2)\end{aligned}$$

If the system is projected to a momentum state $|p\rangle$, we have

$$\begin{aligned}|f\rangle &= \langle p|\hat{U}(\theta)|\psi\rangle \otimes |0\rangle \\ &= \left[\varphi_p - 2\beta\sin^2\frac{\theta}{2}\psi(x)\right] \otimes |0\rangle - \beta\sin\theta\,\psi(x) \otimes |1\rangle, \quad (3)\end{aligned}$$

where $\varphi_p = \langle p|\psi\rangle$ and $\beta = \langle p|x\rangle$. The post-selection can be considered as a projective measurement. Therefore, the quantum state expressed by Eq. (3) is not normalized. The success probability of projective measurement also infers the efficiency of transforming the measurement resource to final readout. After the post-selection, the wave function can be obtained as[22]

$$\begin{aligned}&\frac{1}{2}\left[\langle f|\hat{\sigma}_x|f\rangle - \mathrm{i}\left\langle f\middle|\hat{\sigma}_y\middle|f\right\rangle\right] - \frac{2\sin^2\frac{\theta}{2}}{\sin\theta}\langle f|\hat{P}_1|f\rangle \\ &= \frac{1}{2}[(P_+ - P_-) - \mathrm{i}(P_L - P_R)] - \frac{2\sin^2\frac{\theta}{2}}{\sin\theta}P_1 \\ &= -\beta\sin\theta\,\varphi_p^*\psi(x), \quad (4)\end{aligned}$$

where $P_+ = \langle f|+\rangle\langle +|f\rangle$, $P_- = \langle f|-\rangle\langle -|f\rangle$, $P_L = \langle f|L\rangle\langle L|f\rangle$, $P_R = \langle f|R\rangle\langle R|f\rangle$, and $P_1 = \langle f|1\rangle\langle 1|f\rangle$ are the probabilities of $|f\rangle$ projecting on the following pointer states:

$$\begin{aligned}|+\rangle &= \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle), \\ |L\rangle &= \frac{1}{\sqrt{2}}(|1\rangle + \mathrm{i}|0\rangle), \quad |R\rangle = \frac{1}{\sqrt{2}}(|1\rangle - \mathrm{i}|0\rangle). \quad (5)\end{aligned}$$

If the coupling strength is sufficiently weak to neglect the second order of θ, Eq. (4) will lead to the result of weak measurement studied in Ref. [8],

$$\frac{1}{2}[(P_+ - P_-) - \mathrm{i}(P_L - P_R)] = -\beta\sin\theta\,\varphi_p^*\psi(x). \quad (6)$$

However, if θ is too small, the intensity of the final readout of the measurement may be very low. When $\theta = \pi/2$, it becomes the direct strong measurement case investigated in Ref. [34], and the result can be obtained as

$$\tfrac{1}{2}[(P_+ - P_-) - \mathrm{i}(P_L - P_R)] - P_1 = -\beta\varphi_p^*\psi(x). \quad (7)$$

Benefit from the higher coupling strength, this direct strong measurement method has better precision in wave function measurement. We will focus on the strong coupling case in this paper.

The prefactors $\beta\varphi_p^*$ of the wave function $\psi(x)$ in Eq. (7) determine the efficiency of the direct measurement, which averagely describes the magnitude of input photon energy that can contribute to the final detectable signal. In the experiment reported in Ref. [8], φ_p^* is the probability amplitude at zero momentum of the initial state $|\psi\rangle$. Thus $|\varphi_p^*|^2$ is the success probability of projecting $|\psi\rangle$ to the post-selection state $|p_0\rangle$. β is the inner product of a momentum eigenstate and a position eigenstate, and its square modulus $|\beta|^2 = |\langle p_0|x\rangle|^2$ is the probability of $|x\rangle$ to be in $|p_0\rangle$. Strictly speaking, the momentum uncertainty of a position eigenstate is infinite, so $|\beta|^2$ approaches to zero. However, in a real experiment, both $|p_0\rangle$ and $|x\rangle$ have widths which are relative to their measuring steps. Therefore, $|\beta|^2$ is practically a small value instead of zero. The greater the prefactor $|\beta\varphi_p^*|$, the higher signal intensity in the measurement, which leads to a higher signal-to-noise ratio. Because $|\varphi_p^*|$ is determined by the unknown state $|\psi\rangle$, it is unable to increase without the knowledge of this initial state. But β is independent from $|\psi\rangle$, so it is able to be modified. Therefore, optimizing the value of $|\beta|$ plays a key role in improving the efficiency of a direct measurement experiment.

3. Proposal to improve the probability of post-selection

As a concrete example, we address our method to optimize the efficiency of post-selection based on a previous optical experiment,[8] which is schematically shown in Fig. 1. In the experiment, the researchers considered a photon travelling along the z direction, and the aim is to directly measure the x wave function of the photons.

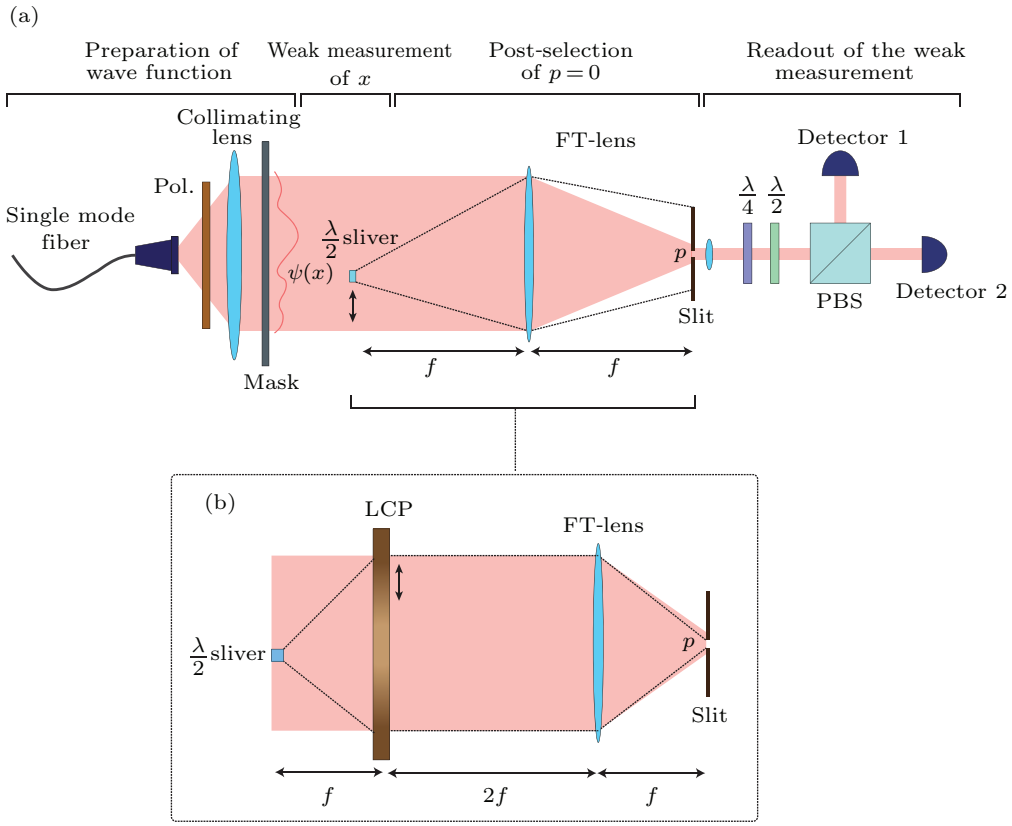


Fig. 1. (a) Schematic setup of the experiment proposed in Ref. [8] to directly measure the quantum wave function. The experiment has four sequential steps: state preparation, interaction of the probe system and the pointer, post-selection of the system, and readout the measurement results. (b) Our modified method to increase the efficiency of the post-seletion.

An ensemble of single photons are emitted from a single mode fiber and collimated. The polarization of the photons is chosen as the pointer in the measurement. The polarizer (Pol.) is used to initialize the polarization state of the photons to be $|0\rangle$. The photons pass through a collimating lens to make the transverse mode of the photons a nearly parallel propagating Gaussian beam. A phase mask placing after the collimating lens generates the initial quantum wave function $\psi(x)$. A half-wave plate ($\lambda/2$) sliver placing at the transverse position x performs the coupling between the pointer and the wave function $\psi(x)$. We can consider that the photons have been divided into two parts: the unperturbed part in the pointer state $|0\rangle$, and the signal part in the pointer state $|1\rangle$. As Eq. (3) shows, the signal part is carrying the information of the wave function $\psi(x)$. In the previous direct measurement scheme,[8,34] the photons would be post-selected to the transverse momentum state $|p=0\rangle$ after the coupling. The dashed line in Fig. 1(a) shows the propagation of the signal part. The photons modulated by the $\lambda/2$ sliver evolve approximately like a point-source light. Then the Fourier transformation lens (FT-lens) transforms the light to a momentum spectrum at the focal plane. As the measuring region δx is small, the uncertainty of the transverse momentum of the signal part is very large compared with the post-selected momentum $p \approx 0$. It leads to a consequence that only a small portion of the signal part can be successfully selected by the slit. Thus it largely limits the efficiency of the measurement.

After the slit, a lens is used to collimate the outcoming light. Then the quarter-wave plate (QWP), half-wave plate (HWP) and the PBS are used to project the final pointer state to four diagonal bases respectively. This step is to read out the expectation values $P_{\pm}$, $P_{L,R}$, and P_1 in Eq. (7).

We here propose a modified method to increase the efficiency of post-selection. Our modification of the experimental proposal is shown in Fig. 1(b): a liquid crystal plate (LCP) is inserted between the $\lambda/2$ sliver and the FT-lens. In our modified scheme, the photons first transmit the LCP after the coupling of the half wave plate sliver. The LCP is filled with oriented liquid crystal polymers. These polymers perform a phase retardation at the light whose polarization is parallel to the polymer orientation, while the LCP does not affect the light if its polarization is perpendicular to the polymer orientation. Thus the LCP can be considered as a Fourier transformation lens that can perform a Fourier transformation only on the polarization eigen state $|1\rangle$ (see the details in Appendix B). The longitudinal distance between the $\lambda/2$ sliver and the LCP is f, which is the focal length of this LCP. In this process, the photon evolves like a non-relativistic particle with the equivalent mass of $m = \frac{2\pi\hbar}{\lambda c}$, where λ is the wavelength of the photon and c is the velocity of light (see Appendix A). The transverse center of LCP is aligned with the $\lambda/2$ sliver. We denote the transverse position at the LCP as x'. Thus we can use a unitary

operator to describe the evolution of the photon,

$$\hat{T} = \mathrm{e}^{-\mathrm{i}\frac{\hat{p}^2}{2m}\frac{t}{\hbar}} \otimes \hat{I}_\sigma. \tag{8}$$

Here t is proportional to the propagation distance which can be written as $t = f/c$ in this case. Then, the LCP performs spatial phase modulation on the wavefront of the photons. The transverse center of LCP is aligned with the $\lambda/2$ sliver. We can also use a unitary operator to describe this modulation

$$\hat{L} = \hat{I}_\pi \otimes \hat{P}_0 + \sum_{x'} \mathrm{e}^{-\mathrm{i}\frac{m\hbar(x'-x)^2}{2t}} \hat{\pi}_{x'} \otimes \hat{P}_1, \tag{9}$$

where $\hat{\pi}_{x'} = |x'\rangle\langle x'|$ is the projective operator at position x'. Since the phase modulation of the LCP only acts on the polarization state $|1\rangle$, $\hat{L}$ includes a projective operator $\hat{P}_1 = |1\rangle\langle 1|$. Now we perform these two operators, $\hat{T}$ and $\hat{L}$, to the state describe by Eq. (2) with $\theta = \pi/2$, we can get the state at the LCP

$$\begin{aligned} |c\rangle &= \hat{L}\hat{T}\hat{U}(\pi/2)|\psi\rangle \otimes |0\rangle \\ &= [\mathrm{e}^{-\mathrm{i}\frac{\hat{p}^2}{2m}\frac{t}{\hbar}}|\psi\rangle - \psi(x)\mathrm{e}^{-\mathrm{i}\frac{\hat{p}^2}{2m}\frac{t}{\hbar}}|x\rangle] \otimes |0\rangle \\ &\quad - \psi(x)\sum_{x'} \mathcal{A}|x'\rangle \otimes |1\rangle, \end{aligned} \tag{10}$$

where $\mathcal{A}$ is a x' independent normalization factor. Actually, the spatial state $\sum_{x'} \mathcal{A}|x'\rangle$ is a plane wave which is the eigen state of zero momentum. Therefore we can rewrite Eq. (10) as

$$\begin{aligned} |c\rangle &= [\mathrm{e}^{-\mathrm{i}\frac{\hat{p}^2}{2m}\frac{t}{\hbar}}|\psi\rangle - \psi(x)\mathrm{e}^{-\mathrm{i}\frac{\hat{p}^2}{2m}\frac{t}{\hbar}}|x\rangle] \\ &\quad \otimes |0\rangle - \psi(x)|p_0\rangle \otimes |1\rangle. \end{aligned} \tag{11}$$

Remarkably, the momentum of the component in polarization state $|1\rangle$ becomes $p_0 = 0$, so the $|p_0\rangle$ state appears in Eq. (11), in contrast to any moment p in Eq. (3). Then the FT-lens and the slit perform a post-selection of the zero momentum state $|p_0\rangle$, and the final state of the pointer can be expressed as

$$\begin{aligned} |f'\rangle &= [\langle p_0|\mathrm{e}^{-\mathrm{i}\frac{\hat{p}^2}{2m}\frac{t}{\hbar}}|\psi\rangle - \psi(x)\langle p_0|\mathrm{e}^{-\mathrm{i}\frac{\hat{p}^2}{2m}\frac{t}{\hbar}}|x\rangle]|0\rangle \\ &\quad - \psi(x)\langle p_0|p_0\rangle|1\rangle \\ &= \left[\int \mathrm{d}p\langle p_0|\mathrm{e}^{-\mathrm{i}\frac{\hat{p}^2}{2m}\frac{t}{\hbar}}|p\rangle\langle p|\psi\rangle \right. \\ &\quad \left. - \int \mathrm{d}p\langle p_0|\mathrm{e}^{-\mathrm{i}\frac{\hat{p}^2}{2m}\frac{t}{\hbar}}|p\rangle\langle p|x\rangle\psi(x)\right]|0\rangle - \psi(x)|1\rangle \\ &= [\delta(p-p_0)\langle p|\psi\rangle - \psi(x)\delta(p-p_0)\langle p|x\rangle]|0\rangle - \psi(x)|1\rangle \\ &= [\varphi_p - \beta\psi(x)]|0\rangle - \psi(x)|1\rangle. \end{aligned} \tag{12}$$

Similar to Eq. (4), we have

$$\begin{aligned} &\frac{1}{2}\left[\langle f'|\hat{\sigma}_x|f'\rangle - \mathrm{i}\langle f'|\hat{\sigma}_y|f'\rangle\right] - |\beta|\langle f'|\hat{P}_1|f'\rangle \\ &= -\varphi^*(0)\psi(x). \end{aligned} \tag{13}$$

In this case, the probe wave function can be obtained as

$$\frac{1}{2}\left[(P'_+ - P'_-) - \mathrm{i}(P'_L - P'_R)\right] - |\beta|P'_1 = -\varphi^*(0)\psi(x). \tag{14}$$

Similarly, in the weak measurement case where θ is small, we have

$$\frac{1}{2}[(P_+ - P_-) - \mathrm{i}(P_L - P_R)] = -\sin\theta\varphi_p^*\psi(x). \tag{15}$$

Comparing Eqs. (7) and (14) [or Eqs. (6) and (15)], we notice that the coefficient β in the right hand side of Eq. (14) [and (13)] is eliminated. In other words, our scheme will have a magnification

$$\mathcal{M} = 1/|\beta| = 1/|\langle p_0|x\rangle| \tag{16}$$

in the measurement, see the details in Appendix C.

4. Performances of the improved method

We now turn to address the key magnification factor $\mathcal{M} = 1/|\beta|$ with $\beta = \langle p_0|x\rangle$. For convenience, we usually choose a post-selected state to make β a constant. one of the states meets this condition is mutually unbiased bases. In a discrete system, two bases $\{b\}$ and $\{a\}$ are mutually unbiased[42] if the square of the magnitude of the inner product between any basis states equals the inverse of the dimension d, i.e., $|\langle b|a\rangle|^2 = 1/d$. For a typical mutually unbiased bases in an infinite dimensional Hilbert space, the inner product of the position state $|x\rangle$ and momentum state $|p\rangle$ is in a plane wave formulation, $\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}\mathrm{e}^{\mathrm{i}px/\hbar}$. In previous works of direct measuring the position wave function, $\{|x\rangle\}$ are recast to d-dimension position eigenstates. Then the position quantum state is written in a discrete formulation, $|\psi(x)\rangle = \sum_{x=1}^{d}\psi_x|x\rangle$. After this discretization, the post-selected state of zero momentum can be expressed as a mutually unbiased bases, $|p_0\rangle = (1/\sqrt{d})\sum_{x=1}^{d}|x\rangle$, so we can obtain $\beta = 1/\sqrt{d}$. Therefore, the magnification of our modified proposal is $\mathcal{M} = \sqrt{d}$. Since d is the number of the measuring points, the magnification $\mathcal{M}$ is higher in high resolution of measurement.

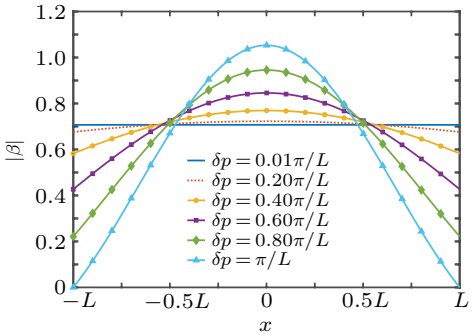


Fig. 2. (a). The factor $|\beta|$ as a function of x. δp can be chosen in $[0, \pi/L]$, where $[-L, L]$ is the measuring range of $\psi(x)$ in x. The function $|\beta|$ of each curves has been normalized in the measuring range.

In a practical experiment, $|p_0\rangle$ is not an exact zero momentum state, it should be an integral of momentum eigen states in the range $[-\delta p, \delta p]$. δp is related to the width of

the slit which can be expressed as $\delta p = \pi l_{\rm slit}/f\lambda$. Under this condition, β can be recast as

$$\beta = \int_{-\delta p}^{\delta p} \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} dp. \tag{17}$$

In Fig. 2, we show β as a function of x for different values of δp. It is shown that $|\beta|$ has a good flatness in the range $x \in [-L, L]$ when δp is much smaller than π/L. So to make β a nearly x independent value, δp has to be sufficiently small. Simultaneously, the success probability of post-selection would decrease as δp decreases. Therefore, δp affects the precision of measuring the wave function.

There is a trade off between high efficiency of post-selection and the precision in direct measurement method. We thus further evaluate how does the value δp affect the accuracy of the measurement. We denote the ideal wave function in the preparation as $\psi_{\rm G}(x)$, which is a Gaussian wave packet $\psi_{\rm G}(x) = [\frac{a}{a+it/m}]^{\frac{3}{2}} \exp[-\frac{x^2}{2(a+it/m)}]$. Here a is the waist of the wave packet, t is the propagation time from the preparation to the moment of the coupling on $\lambda/2$ sliver, $m = \frac{2\pi\hbar}{\lambda c}$ with c being the velocity of light and λ being the wave length of the light is the effective mass of the light. If we only consider the errors caused by the width of the post-selection δp, the measured wave function can then be written as $\psi_{\rm m}(x) = \beta(x)\psi_{\rm G}(x)$. The fidelity of these two wave functions can be used to describe the accuracy of the measurement, i.e.,

$$F = |\langle \psi_{\rm m} | \psi_{\rm G} \rangle|^2. \tag{18}$$

We calculate the fidelity F as a function of δp for various waists a, and the results are shown in Fig. 3. The range of δp is $[0, \pi/L]$, and the measuring position $x \in [-L, L]$. These two ranges satisfy the condition of $2\pi/(2L) = \pi/L$, which is required for the discrete Fourier transformation. It is clearly that the fidelity decreases with increasing δp.

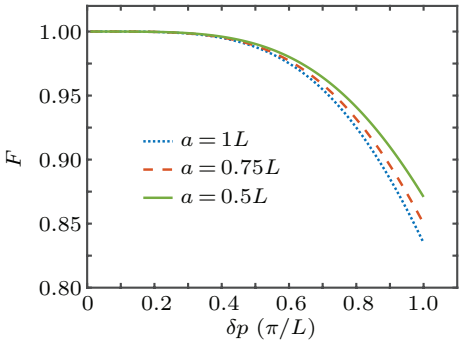


Fig. 3. The fidelity F of the measured wave function as a function of δp. The waists of the Gaussian wave packets $a = 1.00L, 0.75L, 0.50L$.

On the other hand, the width of the measuring step δx also affects β. δx can be considered as the uncertainty of transverse position. According to Heisenberg's uncertainty principle, the uncertainty of transverse momentum becomes larger when δx decreases. For a given δp, higher resolution of the measurement will lead to a lower efficiency of post-selection.

For a fixed δx and δp, β can be written as a double integral formulation

$$\beta = \int_{-\delta x}^{\delta x} dx \int_{-\delta p}^{\delta p} \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} dp. \tag{19}$$

Figure 4 shows the magnification factor $\mathcal{M} = 1/|\beta|$ as a function of number N which is the measuring points with a given measuring range $x \in [-L, L]$. The magnification factor is higher when the number N increases. Furthermore, our proposed method has a higher efficiency for higher resolution and fidelity.

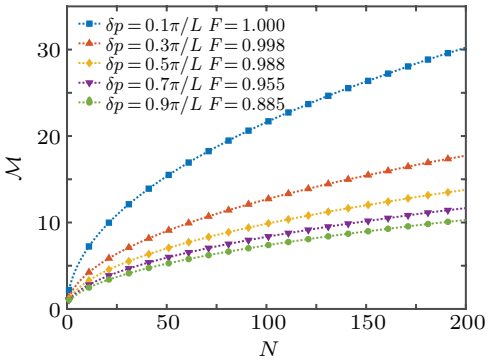


Fig. 4. (a). The magnification factor $\mathcal{M}$ as a function of the measuring point N. The waist of the Gaussian wave packet $a = L$.

5. Discussion and conclusion

We now discuss the improvement of the efficiency under the conditions in Ref. [8], where the parameters $L = 30$ mm, $\delta x = 1$ mm, $f = 1000$ mm, $l_{\rm slit} = 15$ μm, and $\lambda = 800$ nm. We can estimate the width of measuring step $\delta x \approx 0.033L$ and the post-selection $\delta p \approx 0.563(\pi/L)$ by using the formula $\delta p = \pi l_{\rm slit}/f\lambda$. With these two parameters (δx and δp), we can obtain $|\beta| \approx 0.138$ by numerical calculation. Therefore, if we modify the experimental setup in Ref. [8] with our proposal, it would have a magnification of $\mathcal{M} = 1/|\beta| \approx 7.24$ in efficiency increasing.

In summary, we have proposed a modified scheme to directly measure the quantum wave function by simply adding a LCP after the coupling between the pointer and the wave function. Our modified method can significantly increase the efficiency of post selection. Numerical simulations have shown that our proposal can have a greater efficiency in higher resolution and fidelity.

Acknowledgments

This work was supported by the Key-Area Research and Development Program of Guangdong Province, China (Grant No. 2019B030330001), the National Natural Science Foundation of China (Grant Nos. 12074180, 11804105,

U20A2074, and U1801661), the Key Project of Science and Technology of Guangzhou (Grant No. 2019050001), the National Key Research and Development Program of China (Grant No. 2020YFA0309500), and the Natural Science Foundation of Guangdong Province, China (Grant No. 2018A0303130066).

Appendix A: The equivalent mass of the photon

Since the photon is a relativistic particle, the Hamiltonian of the photon can be written as

$$H = c|\boldsymbol{p}| = c\sqrt{p_x^2 + p_y^2 + p_z^2}. \tag{A1}$$

Here, x and y are two transverse coordinate, and z is the propagation direction. The momentum p_z is related to the wave length of the light, $p_z = \hbar k_z = 2\pi\hbar/\lambda$. In our experiment, the initial wave function is a Gaussian wave packet in x–y plane, $\psi(x,y) = \mathcal{N}\exp(-x^2/a_x^2)\exp(-y^2/a_y^2)$. The uncertainty of p_x and p_y is $\hbar/a_x$ and $\hbar/a_y$. In our case, $p_z \gg p_x$, p_y, so the Hamiltonian can be approximately rewritten as

$$H \approx \frac{cp_x^2}{2p_z} + \frac{cp_y^2}{2p_z} + cp_z. \tag{A2}$$

Here we only focus on the x and z direction. Besides, p_z is nearly a constant and the z is proportional to the propagation time $t = z/c$. Then, the evolution of the spatial wave function $\psi(x,t)$ can be written as a Schrödinger like equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(x,t) = \frac{cp_x^2}{2p_z}\psi(x,t), \tag{A3}$$

As shown from this equation, $m = \frac{p_z}{c} = \frac{2\pi\hbar}{\lambda c}$ can be considered as an equivalent mass and the photon acts as a non-relativistic particle in transverse dimension when the propagation momentum is much greater than the uncertainty of the transverse momentum.

Appendix B: Phase modulation of the LCP

In the theory of Fourier optics, the Fourier transformation lens (FT-lens) performs a phase retardation on a wave front. In dimension x, the phase transformation function is given by

$$U_{ft} = \mathrm{e}^{-\mathrm{i}\frac{\pi x^2}{\lambda f}}, \tag{B1}$$

where λ is the wave length of the light and f is the focal length of the FT-lens. The LCP can be considered as a FT-lens only acting on the polarization state $|1\rangle$ (a spatial light modulator can also realize this Fourier transformation). We now address our experimental proposal. The signal part in Eq. (3) with $\theta = \pi/2$ is $|\psi_s\rangle = \psi(x)|x\rangle$. Besides, the free evolution Hamiltonian of the photons is $\hat{H} = \frac{\hat{p}^2}{2m}$. Then the signal part evolves for a distance of $f = tc$, and the wave function after this evolution is given by

$$\begin{aligned}\psi'(x') &= \langle x'|\,\mathrm{e}^{-\mathrm{i}\frac{p^2}{2m}\frac{t}{\hbar}}\psi(x)|x\rangle \\ &= \mathcal{A}\psi(x)\,\mathrm{e}^{\mathrm{i}\frac{mc(x'-x)^2}{2\hbar f}},\end{aligned} \tag{B2}$$

where $\mathcal{A}$ is the normalization factor and x' is the transverse position at f away from the $\lambda/2$ sliver. At this moment, if the center of the LCP is at the position of x, it will perform a phase retardation of $\exp[-\mathrm{i}\frac{\pi(x'-x)^2}{\lambda f}]$ at x'. The wave function after the modulation of LCP is given by

$$\begin{aligned}\psi_L(x') &= \mathrm{e}^{-\mathrm{i}\frac{\pi(x'-x)^2}{\lambda f}}\psi'(x') \\ &= \mathcal{A}\psi(x)\,\mathrm{e}^{-\mathrm{i}\frac{\pi(x'-x)^2}{\lambda f}}\,\mathrm{e}^{\mathrm{i}\frac{mc(x'-x)^2}{2\hbar f}} \\ &= \mathcal{A}\psi(x).\end{aligned} \tag{B3}$$

We notice that $\psi_L(x')$ is a constant in respect of x'. At this moment, the state in x' can be written as $|\psi_L\rangle = \psi(x)\sum_{x'}\mathcal{A}|x'\rangle$. If we recast Eq. (B3) into the momentum representation, it becomes the eigenstate with zero momentum $|\psi_L\rangle = \psi(x)|p = 0\rangle$. With the modulation of the LCP, the eigenstate of transverse position x is transformed to a zero transverse momentum state, which is also the post-selection state in the measurement. Notably, state $|\psi_L\rangle = \psi(x)|p = 0\rangle$ still carries the wave function $\psi(x)$ as a coefficient.

Appendix C: The magnification factor

In strong measurement case represented in Eq. (4), we need to measure the pointer in three bases, namely $\{|L\rangle, |R\rangle\}$, $\{|+\rangle, |-\rangle\}$ and $\{|1\rangle\}$. We can obtained the real and imaginary parts of $\psi(x)$ from the final detected signals of these bases,

$$\begin{aligned}&(P_+ - P_-) = 2|\beta\varphi_p^*|\,\mathrm{Re}[\psi(x)] - 2|\beta|^2|\psi(x)|^2, \\ &(P_L - P_R) = 2|\beta\varphi_p^*|\,\mathrm{Im}[\psi(x)], \\ &P_1 = |\beta|^2|\psi(x)|^2.\end{aligned} \tag{C1}$$

For simplicity, here we use the module $|\beta\varphi_p^*|$ instead of $\beta\varphi_p^*$, because we can normalize the wave function $\psi(x)$ by an arbitrary global phase. In the modified case after using LCP, we have

$$\begin{aligned}&(P'_+ - P'_-) = 2|\varphi_p^*|\,\mathrm{Re}[\psi(x)] - 2|\beta||\psi(x)|^2, \\ &(P'_L - P'_R) = 2|\varphi_p^*|\,\mathrm{Im}[\psi(x)], \\ &P'_1 = |\psi(x)|^2.\end{aligned} \tag{C2}$$

Here $(P_+ - P_-)$ and $(P_L - P_R)$ represent the imbalance in signal between two detectors, and P_1 represents the signal only on P_1. For a given value of $\psi(x)$ we can calculate the magnifications of the signal after the modification of the LCP

$$\frac{(P'_+ - P'_-)}{(P_+ - P_-)} = \frac{1}{|\beta|}, \quad \frac{(P'_L - P'_R)}{(P_L - P_R)} = \frac{1}{|\beta|}, \quad \frac{P'_1}{P_1} = \frac{1}{|\beta|^2}. \tag{C3}$$

From these results we notice that, the signals after the modification, $(P'_+ - P'_-)$ and $(P'_L - P'_R)$, have been amplified with a magnification factor of $\mathcal{M} = 1/|\beta|$. When measuring P'_1, the magnification factor is even greater. Therefore the magnification factor of our modification scheme is at least $\mathcal{M} = 1/|\beta|$.

References

[1] von Neumann J, Beyer R and Wheeler N 2018 *Mathematical Foundations of Quantum Mechanics: New Edition* (Princeton University Press)

[2] Struppa D C and Tollaksen J M (eds.) 2014 *Quantum theory: a two-time success story: Yakir Aharonov festschrift* (Springer)

[3] Cramer M, Plenio M B, Flammia S T, Somma R, Gross D, Bartlett S D, Landon-Cardinal O, Poulin D and Liu Y K 2010 *Nat. Commun.* **1** 149

[4] Resch K J, Walther P and Zeilinger A 2005 *Phys. Rev. Lett.* **94** 070402

[5] Smithey D T, Beck M, Raymer M G and Faridani A 1993 *Phys. Rev. Lett.* **70** 1244

[6] Hradil Z 1997 *Phys. Rev. A* **55** R1561

[7] Torlai G, Mazzola G, Carrasquilla J, Troyer M, Melko R and Carleo G 2018 *Nat. Phys.* **14** 447

[8] Lundeen J S, Sutherland B, Patel A, Stewart C and Bamber C 2011 *Nature* **474** 188

[9] Aharonov Y, Albert D Z and Vaidman L 1988 *Phys. Rev. Lett.* **60** 1351

[10] Dressel J, Malik M, Miatto F M, Jordan A N and Boyd R W 2014 *Rev. Mod. Phys.* **86** 307

[11] Ritchie N W M, Story J G and Hulet R G 1991 *Phys. Rev. Lett.* **66** 1107

[12] Dixon P B, Starling D J, Jordan A N and Howell J C 2009 *Phys. Rev. Lett.* **102** 173601

[13] Starling D J, Dixon P B, Jordan A N and Howell J C 2010 *Phys. Rev. A* **82** 063822

[14] Hosten O and Kwiat P 2008 *Science* **319** 787

[15] Kedem Y and Vaidman L 2010 *Phys. Rev. Lett.* **105** 230401

[16] Zilberberg O, Romito A and Gefen Y 2011 *Phys. Rev. Lett.* **106** 080405

[17] Huang J 2017 *Acta Phys. Sin.* **66** 010301 (in Chinese)

[18] Wang M J and Xia Y J 2015 *Acta Phys. Sin.* **64** 240303 (in Chinese)

[19] Wu C W, Zhang J, Xie Y, Ou B Q, Chen T, Wu W and Chen P X 2019 *Phys. Rev. A* **100** 062111

[20] Pan Y, Zhang J, Cohen E, Wu C W, Chen P X and Davidson N 2020 *Nat. Phys.* **16** 1206

[21] Qiu J D, Ren C L and Li Z X 2020 *Chin. Phys. B* **29** 064214

[22] Lundeen J and Resch K 2005 *Phys. Lett. A* **334** 337

[23] Salvail J Z, Agnew M, Johnson A S, Bolduc E, Leach J and Boyd R W 2013 *Nat. Photon.* **7** 316

[24] Malik M, Mirhosseini M, Lavery M P J, Leach J, Padgett M J and Boyd R W 2014 *Nat. Commun.* **5** 3115

[25] Kocsis S, Braverman B, Ravets S, Stevens M J, Mirin R P, Shalm L K and Steinberg A M 2011 *Science* **332** 1170

[26] Thekkadath G S, Giner L, Chalich Y, Horton M J, Banker J and Lundeen J S 2016 *Phys. Rev. Lett.* **117** 120401

[27] Lundeen J S and Bamber C 2012 *Phys. Rev. Lett.* **108** 070402

[28] Bolduc E, Gariepy G and Leach J 2016 *Nat. Commun.* **7** 10439

[29] Qin L, Xu L, Feng W and Li X Q 2017 *New J. Phys.* **19** 033036

[30] Bamber C and Lundeen J S 2014 *Phys. Rev. Lett.* **112** 070405

[31] Shojaee E, Jackson C S, Riofrío C A, Kalev A and Deutsch I H 2018 *Phys. Rev. Lett.* **121** 130404

[32] Fischbach J and Freyberger M 2012 *Phys. Rev. A* **86** 052110

[33] Mirhosseini M, Maga na Loaiza O S, Hashemi Rafsanjani S M and Boyd R W 2014 *Phys. Rev. Lett.* **113** 090402

[34] Vallone G and Dequal D 2016 *Phys. Rev. Lett.* **116** 040502

[35] Denkmayr T, Geppert H, Lemmel H, Waegell M, Dressel J, Hasegawa Y and Sponar S 2017 *Phys. Rev. Lett.* **118** 010402

[36] Calderaro L, Foletto G, Dequal D, Villoresi P and Vallone G 2018 *Phys. Rev. Lett.* **121** 230501

[37] Pan W W, Xu X Y, Kedem Y, Wang Q Q, Chen Z, Jan M, Sun K, Xu J S, Han Y J, Li C F and Guo G C 2019 *Phys. Rev. Lett.* **123** 150402

[38] Zhang C R, Hu M J, Hou Z B, Tang J F, Zhu J, Xiang G Y, Li C F, Guo G C and Zhang Y S 2020 *Phys. Rev. A* **101** 012119

[39] Zhang S, Zhou Y, Mei Y, Liao K, Wen Y L, Li J, Zhang X D, Du S, Yan H and Zhu S L 2019 *Phys. Rev. Lett.* **123** 190402

[40] Wen Y L, Zhang S, Yan H and Zhu S L 2021 *Acta Phys. Sin.* **70** 110301 (in Chinese)

[41] Maccone L and Rusconi C C 2014 *Phys. Rev. A* **89** 022122

[42] Durt T, Englert B G, Bengtsson I and Życzkowski K 2010 *Inter. J. Quantum Inf.* **8** 535