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Citation: Chin. Phys. B, 2022, 31 (**2**): 020303. DOI: 10.1088/1674-1056/ac1e1c
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Optical wavelet-fractional squeezing combinatorial transform

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(Received 9 June 2021; revised manuscript received 1 August 2021; accepted manuscript online 17 August 2021)

By virtue of the method of integration within ordered product (IWOP) of operators we find the normally ordered form of the optical wavelet-fractional squeezing combinatorial transform (WFrST) operator. The way we successfully combine them to realize the integration transform kernel of WFrST is making full use of the completeness relation of Dirac's ket–bra representation. The WFrST can play role in analyzing and recognizing quantum states, for instance, we apply this new transform to identify the vacuum state, the single-particle state, and their superposition state.

Keywords: wavelet transform, fractional squeezing transform, combinatorial transform, IWOP technique

PACS: 03.65.–w, 03.65.Db, 02.30.Uu **DOI:** 10.1088/1674-1056/ac1e1c

1. Introduction

In Fourier optics and information optics, the fractional Fourier transform (FrFT) is a very useful tool.[1] The concept of the FrFT was originally introduced for signal processing in 1980 by Namias[2,3] as a Fourier transform of fractional order. However, the FrFT did not have a significant impact on optics until it was defined physically based on propagation in quadratic graded-index media (GRIN media)[4,5] by Mendlovic, Ozaktas, and Lohmann. The one-dimensional FrFT of the α-th order (for a real α angle) is defined as

$$F_\alpha[f](p)=\int_{-\infty}^{\infty}K_\alpha(p,x)f(x)\,\mathrm{d}x, \tag{1}$$

where

$$K_\alpha(p,x)\equiv\frac{\mathrm{e}^{\mathrm{i}(\alpha/2-\pi/4)}}{\sqrt{2\pi\sin\alpha}}\exp\left[\frac{\mathrm{i}\left(x^2+p^2\right)}{2\tan\alpha}-\frac{\mathrm{i}xp}{\sin\alpha}\right] \tag{2}$$

is the transform integration kernel. It can be described in the text of quantum mechanics,

$$K_\alpha(p,x)=\langle p|\exp\left[\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)a^\dagger a\right]|x\rangle, \tag{3}$$

where ket $|x\rangle$ and bra $\langle p|$ are Dirac's coordinate and momentum eigenstates, respectively, they are mutually conjugate, when $\alpha=\pi/2$, equation (3) $=(1/\sqrt{2\pi})\langle p|x\rangle=(1/\sqrt{2\pi})\mathrm{e}^{-\mathrm{i}xp}$ reduces to the usual Fourier transform, from coordinate representation to momentum representation. Due to this, FrFT is characteristic of additive property, *i.e.*, $F_\beta F_\alpha[f(x)]=F_{\beta+\alpha}[f(x)]$. Nowadays, the FrFT has been widely employed in optical communication, image manipulation, and signal analysis.[6,7]

By converting the triangular functions in the integration kernel $K_\alpha(p,x)$ of the FrFT to the hyperbolic functions, equation (1) becomes

$$\begin{aligned}\mathfrak{F}_\alpha[g(x)]&=\frac{1}{\sqrt{2\pi\mathrm{i}\sinh\alpha}}\\&\quad\times\int_{-\infty}^{\infty}\exp\left[\frac{\mathrm{i}(x^2+p^2)}{2\tanh\alpha}-\frac{\mathrm{i}xp}{\sinh\alpha}\right]g(x)\,\mathrm{d}x\\&=\int_{-\infty}^{\infty}\mathfrak{K}_\alpha(p,x)g(x)\,\mathrm{d}x\\&=G(p).\end{aligned} \tag{4}$$

By using the properties of the hyperbolic function, one can prove that this kind of integration transform is also additive,[8] that is, $\mathfrak{F}_\beta\mathfrak{F}_\alpha[g(x)]=\mathfrak{F}_{\beta+\alpha}[g(x)]$. As will be shown shortly later that the quantum-mechanical unitary operator for generating transform (4) involves a squeezing operator $\exp\left[-(\mathrm{i}\alpha/2)\left(a^2+a^{\dagger2}\right)\right]$ and α embodies fractionality. The squeezing transform has been a major topic in quantum optics because it produces squeezed state which exhibits less quantum fluctuation than a coherent state in one quadrature at the expense of more fluctuation in another quadrature.[9–11] Thus we name $\mathfrak{F}_\alpha[g(x)]$ the fractional squeezing transform (FrST).

On the other hand, the wavelet transform (WT) is powerful for signal analysis and Fourier optics.[12,13] It is a local wave about time and frequency. Since the WT can overcome some shortcomings of classical Fourier analysis, it has been widely used for many signal processing applications, such as wavelet-based denoising and image compression.[14–16] The continuous WT of a wave function $f(x)\in L^2(R)$ by a mother wavelet is defined by the integral

$$W_\psi f(\mu,s)=\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}f(x)\psi^*\left(\frac{x-s}{\mu}\right)\mathrm{d}x, \tag{5}$$

where the mother wavelet ψ generates other wavelets of the family (μ is a scaling parameter, $\mu>0$, and s is a real translation parameter for generating wavelet family) by change of scale and the translation in x,

$$\frac{1}{\sqrt{\mu}}\psi^*\left(\frac{x-s}{\mu}\right)=\psi^*_{(\mu,s)}(x). \tag{6}$$

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An interesting question thus naturally arises: can we make up a joint wavelet-fractional squeezing transform (WFrST)? If yes, how do we build this theory? In the following we shall tackle this problem quantum-mechanically, *i.e.*, the way we successfully combine them to realize the integration transform kernel of WFrST is making full use of the completeness relation of Dirac's ket-bra representation.[17] We start with recasting the WT and the FrST into the context of quantum-mechanics in Sections 2 and 3. In Section 4, by combining the WT and the FrST quantum mechanically, we propose the optical wavelet-fractional squeezing combinatorial transform and derive its corresponding operator by using the method of integration within ordered product (IWOP) of operators.[18,19] The WFrST can play role in analyzing and recognizing quantum states, so in Section 5, we apply this new transform to identify the vacuum state, the single-particle state and their superposition state. Our conclusions and discussions are involved in Section 6.

2. The quantum optical version of FrST

For combining the wavelet transform and the fractional squeezing transform as a whole, we appeal to the completeness relation of Dirac's ket-bra representation. Firstly, we simply introduce how to obtain the unitary operator responsible for the FrST. Multiplying $\mathfrak{K}_\alpha(p,x)$ in Eq. (4) by $|x'|_{x'=p}\rangle$ from the left and $\langle x|$ from the right, we consider the following integration[12]

$$\begin{aligned}
S_\alpha &\equiv \iint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}p\, |x'|_{x'=p}\rangle\, \mathfrak{K}_\alpha(p,x)\, \langle x| \\
&= \frac{1}{\sqrt{2\pi \mathrm{i}\sinh\alpha}} \iint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}p\, |x'|_{x'=p}\rangle \exp\left[\frac{\mathrm{i}(x^2+p^2)}{2\tanh\alpha} - \frac{\mathrm{i}xp}{\sinh\alpha}\right] \langle x| \\
&= \frac{1}{\sqrt{2\pi \mathrm{i}\sinh\alpha}} \iint_{-\infty}^{\infty} \frac{\mathrm{d}x\mathrm{d}p}{\pi^{1/2}} \exp\left[\frac{\mathrm{i}(x^2+p^2)}{2\tanh\alpha} - \frac{\mathrm{i}xp}{\sinh\alpha}\right] : \exp\left(-\frac{p^2}{2} + \sqrt{2}pa^\dagger - \frac{a^{\dagger 2}}{2} - \frac{x^2}{2} + \sqrt{2}xa - \frac{a^2}{2} - a^\dagger a\right) : \\
&= \sqrt{\mathrm{sech}\alpha} : \exp\left[\mathrm{i}\frac{a^{\dagger 2}}{2}\tanh\alpha + a^\dagger a(\mathrm{sech}\alpha - 1) + \mathrm{i}\frac{a^2}{2}\tanh\alpha\right] : \\
&= \exp\left[\mathrm{i}\frac{a^{\dagger 2}}{2}\tanh\alpha\right] \exp\left[\left(a^\dagger a + \frac{1}{2}\right)\ln\mathrm{sech}\alpha\right] \exp\left[\mathrm{i}\frac{a^2}{2}\tanh\alpha\right] \\
&= \exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2 + a^{\dagger 2}\right)\right].
\end{aligned} \tag{7}$$

In the above calculation, we performed the integration within : : (note $a^\dagger$ and a are commutable within : : and can be taken as c-number parameters) by virtue of the IWOP technique.[12] In additon, we also used the normally ordered form of the vacuum projector $|0\rangle\langle 0| = :\mathrm{e}^{-a^\dagger a}:$ as well as the following formula $:\mathrm{e}^{a^\dagger a(\mathrm{sech}\alpha - 1)}: = \mathrm{e}^{a^\dagger a \ln\mathrm{sech}\alpha}$. S_α is Abelian with respect to the parameter α and is really a squeezing operator. Further, due to $|x'|_{x'=p}\rangle = \mathrm{e}^{-\mathrm{i}\pi a^\dagger a/2}|p\rangle$ and $\langle p|\,\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2} = \langle x'|_{x'=p}|$, we can rewrite Eq. (7) as

$$S_\alpha = \frac{1}{\sqrt{2\pi \mathrm{i}\sinh\alpha}} \mathrm{e}^{-\mathrm{i}\pi a^\dagger a/2} \iint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}p\, |p\rangle \exp\left[\frac{\mathrm{i}(x^2+p^2)}{2\tanh\alpha} - \frac{\mathrm{i}xp}{\sinh\alpha}\right] \langle x|\,. \tag{8}$$

Thus

$$\frac{1}{\sqrt{2\pi \mathrm{i}\sinh\alpha}} \iint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}p\, |p\rangle \exp\left[\frac{\mathrm{i}(x^2+p^2)}{2\tanh\alpha} - \frac{\mathrm{i}xp}{\sinh\alpha}\right] \langle x| = \mathrm{e}^{\mathrm{i}\pi a^\dagger a/2} S_\alpha = \mathrm{e}^{\mathrm{i}\pi a^\dagger a/2} \exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2 + a^{\dagger 2}\right)\right]. \tag{9}$$

Due to $\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2} a \mathrm{e}^{-\mathrm{i}\pi a^\dagger a/2} = a\mathrm{e}^{-\mathrm{i}\pi/2} = -\mathrm{i}a$ and $\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2} a^\dagger \mathrm{e}^{-\mathrm{i}\pi a^\dagger a/2} = a^\dagger \mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{i}a^\dagger$, we see

$$\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2} \exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2 + a^{\dagger 2}\right)\right] = \exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2 + a^{\dagger 2}\right)\right] \mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}. \tag{10}$$

Comparing Eq. (9) with Eq. (10), we can obtain the matrix element

$$\langle p| \exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2 + a^{\dagger 2}\right)\right] \mathrm{e}^{\mathrm{i}\pi a^\dagger a/2} |x\rangle = \frac{1}{\sqrt{2\pi \mathrm{i}\sinh\alpha}} \exp\left[\frac{\mathrm{i}(x^2+p^2)}{2\tanh\alpha} - \frac{\mathrm{i}xp}{\sinh\alpha}\right] = \mathfrak{K}_\alpha(p,x), \tag{11}$$

which is just the transform integration kernel of FrST. We name $\exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2 + a^{\dagger 2}\right)\right] \mathrm{e}^{\mathrm{i}\pi a^\dagger a/2} \equiv \mathfrak{K}_\alpha$ the fractional squeezing operator which is a composite operator and $\mathfrak{K}_\alpha$ is the quantum-mechanical unitary operator for generating the fractional squeezing

transformation. As a result, expressing $g(x)=\langle x|\,g\rangle$ and $\mathfrak{K}_\alpha(p,x)$ by $\langle p|\,\mathfrak{K}_\alpha\,|x\rangle$, we reexpress the FrST in the form of quantum-mechanics

$$\mathfrak{F}_\alpha[g(x)]=\int_{-\infty}^{\infty}\mathrm{d}x\,\langle p|\,\mathfrak{K}_\alpha\,|x\rangle\,\langle x|\,g\rangle=\langle p|\exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}\,|g\rangle=G(p). \tag{12}$$

Employing Eq. (12), we can prove the additive property of the FrST as

$$\begin{aligned}
&\mathfrak{F}_\beta\mathfrak{F}_\alpha[g(x)]\\
&=\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}x'\,\langle p|\,\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}\exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]|x'\rangle_{p'=x'}\,\langle p'|\,\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}\exp\left[\frac{\mathrm{i}\beta}{2}\left(a^2+a^{\dagger 2}\right)\right]|x\rangle\,\langle x|\,g\rangle\\
&=\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}x'\,\langle p|\,\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}\exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]|x'\rangle\,\langle x'|\exp\left[\frac{\mathrm{i}\beta}{2}\left(a^2+a^{\dagger 2}\right)\right]|x\rangle\,\langle x|\,g\rangle\\
&=\langle p|\,\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}\exp\left[\frac{\mathrm{i}(\beta+\alpha)}{2}\left(a^2+a^{\dagger 2}\right)\right]|g\rangle=\mathfrak{F}_{\beta+\alpha}[g(x)].
\end{aligned} \tag{13}$$

3. The quantum optical version of WT

Now we convert the WT into quantum mechanical transform matrix element engendered by some squeezing-displacing operator. In the context of quantum mechanics, function f turns to quantum state $|f\rangle$, the value $f(x)$ of f at given point x turns to the matrix element $\langle x|\,f\rangle$, $|f\rangle$ is the state vector of the signal function. Using Dirac's symbolic method, we obtain

$$\psi^*\left(\frac{x-s}{\mu}\right)=\langle\psi|\,\frac{x-s}{\mu}\Big\rangle, \tag{14}$$

$\langle\psi|$ is the state vector corresponding to the given mother wavelet. Then equation (5) becomes

$$\begin{aligned}
&\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}f(x)\psi^*\left(\frac{x-s}{\mu}\right)\mathrm{d}x\\
&=\langle\psi|\,\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}\mathrm{d}x\left|\frac{x-s}{\mu}\right\rangle\langle x|\,f\rangle,
\end{aligned} \tag{15}$$

where

$$U(\mu,s)=\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}\left|\frac{x-s}{\mu}\right\rangle\langle x|\,\mathrm{d}x \tag{16}$$

is the squeezing-displacing operator. Thus equation (5) can be expressed as

$$W_\psi f(\mu,s)=\langle\psi|U(\mu,s)\,|f\rangle, \tag{17}$$

which states that the classical wavelet transform of a signal function $f(x)$ can be recast to a matrix element of the squeezing-displacing operator $U(\mu,s)$ between the mother wavelet vector $\langle\psi|$ and the quantum state vector $|f\rangle$ to be transformed.

We can further convert the above theory into the momentum representation. Firstly, we transform $U(\mu,s)$ into

$$U(\mu,s)=\sqrt{\mu}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}sp}\,|up\rangle\,\langle p|\,\mathrm{d}p, \tag{18}$$

thus equation (5) can be expressed as follows:

$$W_\psi f(\mu,s)=\sqrt{\mu}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}sp}\psi^*(up)\,f(p)\,\mathrm{d}p, \tag{19}$$

where

$$\psi^*(up)=\langle\psi|\,up\rangle,\quad f(p)=\langle p|\,f\rangle. \tag{20}$$

We can further express Eq. (18) as

$$\begin{aligned}
U(\mu,s)&=\sqrt{\mu}\int_{-\infty}^{\infty}|up\rangle\,\langle p|\,\mathrm{d}p\,\mathrm{e}^{\mathrm{i}sP}\\
&=\exp\left[\frac{\lambda}{2}(a^2-a^{\dagger 2})\right]\mathrm{e}^{\mathrm{i}sP},
\end{aligned} \tag{21}$$

where $\exp[\frac{\lambda}{2}(a^2-a^{\dagger 2})]$ is the squeezing operator, $\mathrm{e}^{\mathrm{i}sP}$ is the displacing operator.

4. The optical wavelet-fractional squeezing combinatorial transform

Now we propose the optical wavelet-fractional squeezing combinatorial transform by combining Eq. (12) with Eqs. (18) and (19), in so doing we have

$$\begin{aligned}
W_\psi[G(u,s)]&=\sqrt{\mu}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}sp}\psi^*(up)\,G(p)\,\mathrm{d}p\\
&=\sqrt{\mu}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}sp}\psi^*(up)\int_{-\infty}^{\infty}\mathfrak{K}_\alpha(p,x)\,g(x)\,\mathrm{d}x\mathrm{d}p\\
&=\langle\psi|\,U(\mu,s)\exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]\\
&\quad\times\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}\,|g\rangle,
\end{aligned} \tag{22}$$

which means that the optical wavelet-fractional squeezing combinatorial transform operator is

$$U(\mu,s)\exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}$$

$$
\begin{aligned}
&= \sqrt{\mu}\int_{-\infty}^{\infty} \mathrm{d}p \mathrm{e}^{\mathrm{i}sp} |\mu p\rangle\langle p| \frac{1}{\sqrt{2\pi \mathrm{i}\sinh\alpha}} \iint_{-\infty}^{\infty} \mathrm{d}x''\mathrm{d}p'' |p''\rangle \exp\left[\frac{\mathrm{i}(x''^2+p''^2)}{2\tanh\alpha} - \frac{\mathrm{i}x''p''}{\sinh\alpha}\right]\langle x''| \\
&= \sqrt{\frac{\mu}{2\pi \mathrm{i}\sinh\alpha}} \iiint_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sp} |\mu p\rangle \delta(p-p'') \exp\left[\frac{\mathrm{i}(x''^2+p''^2)}{2\tanh\alpha} - \frac{\mathrm{i}x''p''}{\sinh\alpha}\right]\langle x''| \mathrm{d}p\mathrm{d}x''\mathrm{d}p'' \\
&= \sqrt{\frac{\mu}{2\pi \mathrm{i}\sinh\alpha}} \iint_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sp''} |\mu p''\rangle \exp\left[\frac{\mathrm{i}(x''^2+p''^2)}{2\tanh\alpha} - \frac{\mathrm{i}x''p''}{\sinh\alpha}\right]\langle x''| \mathrm{d}p''\mathrm{d}x'' \\
&= \sqrt{\frac{\mu}{2\pi \mathrm{i}\sinh\alpha}} \iint_{-\infty}^{\infty} \frac{\mathrm{d}p''\mathrm{d}x''}{\sqrt{\pi}} :\ \exp\left[\mathrm{i}sp'' - \frac{1}{2}\mu^2 p''^2 + \mathrm{i}\sqrt{2}\mu p'' a^\dagger + \frac{1}{2}a^{\dagger 2} + \frac{\mathrm{i}x''^2}{2\tanh\alpha}\right. \\
&\quad \left. + \frac{\mathrm{i}p''^2}{2\tanh\alpha} - \frac{\mathrm{i}x''p''}{\sinh\alpha} - \frac{1}{2}x''^2 + \sqrt{2}x''a - \frac{1}{2}a^2 - a^\dagger a\right] : \\
&= \sqrt{\frac{\mu}{2\pi \mathrm{i}\sinh\alpha}} \iint_{-\infty}^{\infty} \frac{\mathrm{d}p''\mathrm{d}x''}{\sqrt{\pi}} :\ \exp\left[-\frac{1}{2}p''^2\left(\mu^2 - \frac{\mathrm{i}}{\tanh\alpha}\right) + \mathrm{i}p''\left(s + \sqrt{2}\mu a^\dagger - \frac{x''}{\sinh\alpha}\right)\right] \\
&\quad \times \exp\left[-\frac{1}{2}x''^2 + \frac{\mathrm{i}x''^2}{2\tanh\alpha} + \sqrt{2}x''a + \frac{1}{2}a^{\dagger 2} - \frac{1}{2}a^2 - a^\dagger a\right] : .
\end{aligned} \tag{23}
$$

Using the IWOP technique, we can derive

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\mathrm{d}p''}{\sqrt{\pi}} \exp\left[-\frac{1}{2}p''^2\left(\mu^2 - \frac{\mathrm{i}}{\tanh\alpha}\right) + \mathrm{i}p''\left(s + \sqrt{2}\mu a^\dagger - \frac{x''}{\sinh\alpha}\right)\right] \\
&= \sqrt{\frac{2\tanh\alpha}{\mu^2\tanh\alpha - \mathrm{i}}} \exp\left[-\frac{\tanh\alpha}{2\mu^2\tanh\alpha - 2\mathrm{i}}\left(s + \sqrt{2}\mu a^\dagger - \frac{x''}{\sinh\alpha}\right)^2\right],
\end{aligned} \tag{24}
$$

thus we can further derive

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}x'' :\ \exp\left(-\frac{1}{2}x''^2 + \frac{\mathrm{i}x''^2}{2\tanh\alpha} + \sqrt{2}x''a + \frac{1}{2}a^{\dagger 2} - \frac{1}{2}a^2 - a^\dagger a\right) \\
&\times \exp\left[-\frac{\tanh\alpha}{2\mu^2\tanh\alpha - 2i}\left(s + \sqrt{2}\mu a^\dagger - \frac{x''}{\sinh\alpha}\right)^2\right] : \\
&= \sqrt{\frac{2\pi(\mu^2\tanh\alpha - \mathrm{i})\sinh^2\alpha\tanh\alpha}{\sinh^2\alpha(\mu^2\tanh\alpha - \mathrm{i})(\tanh\alpha - \mathrm{i}) + \tanh^2\alpha}} \\
&\times \exp\left\{\frac{\{\tanh^2\alpha - [\sinh^2\alpha(\mu^2\tanh\alpha - \mathrm{i})(\tanh\alpha - \mathrm{i}) + \tanh^2\alpha]\}\tanh\alpha}{2[\sinh^2\alpha(\mu^2\tanh\alpha - \mathrm{i})(\tanh\alpha - \mathrm{i}) + \tanh^2\alpha](\mu^2\tanh\alpha - \mathrm{i})} s^2\right\} \\
&\times \exp\left\{\frac{2\mu^2\tanh\alpha\{\tanh^2\alpha - [\sinh^2\alpha(\mu^2\tanh\alpha - \mathrm{i})(\tanh\alpha - \mathrm{i}) + \tanh^2\alpha]\}}{2[\sinh^2\alpha(\mu^2\tanh\alpha - \mathrm{i})(\tanh\alpha - \mathrm{i}) + \tanh^2\alpha](\mu^2\tanh\alpha - \mathrm{i})} a^{\dagger 2}\right. \\
&\quad \left. + \frac{1}{2}a^{\dagger 2} + \frac{\sqrt{2}s\mu\{\tanh^2\alpha - [\sinh^2\alpha(\mu^2\tanh\alpha - \mathrm{i})(\tanh\alpha - \mathrm{i}) + \tanh^2\alpha]\}\tanh\alpha}{[\sinh^2\alpha(\mu^2\tanh\alpha - \mathrm{i})(\tanh\alpha - \mathrm{i}) + \tanh^2\alpha](\mu^2\tanh\alpha - \mathrm{i})} a^\dagger\right\} \\
&\times \exp\left[\ln\left(\frac{2\mu\sinh\alpha\tanh^2\alpha}{\sinh^2\alpha(\mu^2\tanh\alpha - \mathrm{i})(\tanh\alpha - \mathrm{i}) + \tanh^2\alpha}\right)a^\dagger a\right] \\
&\times \exp\left\{\left[\frac{2(\mu^2\tanh\alpha - \mathrm{i})\sinh^2\alpha\tanh\alpha}{2[\sinh^2\alpha(\mu^2\tanh\alpha - \mathrm{i})(\tanh\alpha - \mathrm{i}) + \tanh^2\alpha]}\right]a^2 - \frac{1}{2}a^2\right. \\
&\quad \left. + \frac{\sqrt{2}s\sinh\alpha\tanh^2\alpha}{[\sinh^2\alpha(\mu^2\tanh\alpha - \mathrm{i})(\tanh\alpha - \mathrm{i}) + \tanh^2\alpha]}a\right\}.
\end{aligned} \tag{25}
$$

Let

$$
A = \sinh^2\alpha(\mu^2\tanh\alpha - \mathrm{i})(\tanh\alpha - \mathrm{i}) + \tanh^2\alpha, \tag{26}
$$

$$
B = \mu^2\tanh\alpha - \mathrm{i},\ \ C = (\tanh^2\alpha - A)\tanh\alpha, \tag{27}
$$

$$
D = \sinh\alpha\tanh^2\alpha,\ \ E = \sinh^2\alpha\tanh\alpha, \tag{28}
$$

and using the relation

$$:\exp\left\{\frac{2\mu\sinh\alpha\tanh^2\alpha-[\sinh^2\alpha(\mu^2\tanh\alpha-\mathrm{i})(\tanh\alpha-\mathrm{i})+\tanh^2\alpha]}{[\sinh^2\alpha(\mu^2\tanh\alpha-\mathrm{i})(\tanh\alpha-\mathrm{i})+\tanh^2\alpha]}a^\dagger a\right\}:=\exp\left[\ln\frac{2\mu D}{A}a^\dagger a\right], \tag{29}$$

we can get the optical wavelet-fractional squeezing combinatorial transform operator

$$\begin{aligned}&U(\mu,s)\exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger2}\right)\right]\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}\\&=\sqrt{\frac{2\mu D}{iA}}\exp\left[\frac{C}{2AB}s^2\right]\exp\left(\frac{2\mu^2C+AB}{2AB}a^{\dagger2}+\frac{\sqrt{2}s\mu C}{AB}a^\dagger\right)\exp\left[\ln\frac{2\mu D}{A}a^\dagger a\right]\exp\left(\frac{2BE-A}{2A}a^2+\frac{\sqrt{2}sD}{A}a\right).\end{aligned} \tag{30}$$

5. Some applications

In this section, we further consider some applications of the optical wavelet-fractional squeezing combinatorial transform. For example, when the mother wavelet is the Mexican hat wavelet

$$\psi(x)=\mathrm{e}^{-x^2/2}(1-x^2), \tag{31}$$

its corresponding state vector is

$$\begin{aligned}|\psi\rangle&=\int_{-\infty}^{\infty}\mathrm{d}x\,|x\rangle\langle x|\,\psi\rangle\\&=\pi^{-1/4}\int_{-\infty}^{\infty}\mathrm{d}x(1-x^2)\\&\quad\times\exp\left(-x^2+\sqrt{2}xa^\dagger-\frac{1}{2}a^{\dagger2}\right)|0\rangle\\&=\frac{\pi^{1/4}}{2}(1-a^{\dagger2})\,|0\rangle,\end{aligned} \tag{32}$$

we can derive the joint wavelet-fractional squeezing transform for the vacuum state $|0\rangle$,

$$\begin{aligned}&\langle\psi|U(\mu,s)\exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger2}\right)\right]\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}|0\rangle\\&=\pi^{1/4}\sqrt{\frac{\mu D}{2\mathrm{i}A}}\exp\left[\frac{C}{2AB}s^2\right]\langle0|\,(1-a^2)\\&\quad\times\exp\left(\frac{2\mu^2C+AB}{2AB}a^{\dagger2}+\frac{\sqrt{2}s\mu C}{AB}a^\dagger\right)|0\rangle\\&=\pi^{1/4}\sqrt{\frac{\mu D}{2\mathrm{i}A}}\exp\left[\frac{C}{2AB}s^2\right]\langle0|\\&\quad\times\left[1-\left(a+\frac{2\mu^2C+AB}{2AB}a^\dagger+\frac{\sqrt{2}s\mu C}{AB}\right)^2\right]|0\rangle\\&=\pi^{1/4}\sqrt{\frac{\mu D}{2\mathrm{i}A}}\exp\left[\frac{C}{2AB}s^2\right]\\&\quad\times\left[\frac{1}{2}-\frac{\mu^2C(AB+2s^2C)}{A^2B^2}\right],\end{aligned} \tag{33}$$

where we have used the operator formula

$$a^2\mathrm{e}^{-ra^{\dagger2}-ta^\dagger}=\mathrm{e}^{-ra^{\dagger2}-ta^\dagger}(a-2ra^\dagger-t)^2. \tag{34}$$

We further examine the wavelet-fractional squeezing combinatorial transform for the state $|1\rangle$,

$$\begin{aligned}&\langle\psi|U(\mu,s)\exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger2}\right)\right]\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}|1\rangle\\&=\pi^{1/4}\sqrt{\frac{\mu D}{2iA}}\exp\left[\frac{C}{2AB}s^2\right]\langle0|\,(1-a^2)\\&\quad\times\exp\left(\frac{2\mu^2C+AB}{2AB}a^{\dagger2}+\frac{\sqrt{2}s\mu C}{AB}a^\dagger\right)\exp\left[\ln\frac{2\mu D}{A}a^\dagger a\right]\\&\quad\times\exp\left(\frac{2BE-A}{2A}a^2+\frac{\sqrt{2}sD}{A}a\right)|1\rangle,\end{aligned} \tag{35}$$

where

$$\begin{aligned}&\langle0|\,(1-a^2)\exp\left(\frac{2\mu^2C+AB}{2AB}a^{\dagger2}+\frac{\sqrt{2}s\mu C}{AB}a^\dagger\right)\\&=\left[\frac{1}{2}-\frac{\mu^2C(AB+2s^2C)}{A^2B^2}\right]\langle0|\\&\quad-\frac{2\sqrt{2}s\mu C}{AB}\langle1|-\sqrt{2}\langle2|,\end{aligned} \tag{36}$$

$$\begin{aligned}&\exp\left[\ln\frac{2\mu D}{A}a^\dagger a\right]\exp\left(\frac{2BE-A}{2A}a^2+\frac{\sqrt{2}sD}{A}a\right)|1\rangle\\&=\exp\left[\ln\frac{2\mu D}{A}\right]|1\rangle+\frac{\sqrt{2}sD}{A}|0\rangle.\end{aligned} \tag{37}$$

Combining Eqs. (36) and (37), we can get the wavelet-fractional squeezing combinatorial transform for the state $|1\rangle$

$$\begin{aligned}&\langle\psi|U(\mu,s)\exp\left[-\frac{i\alpha}{2}\left(a^2+a^{\dagger2}\right)\right]\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}|1\rangle\\&=\pi^{1/4}\sqrt{\frac{\mu D}{2iA}}\exp\left[\frac{C}{2AB}s^2\right]\\&\quad\times\left\{\frac{\sqrt{2}sD}{A}\left[\frac{1}{2}-\frac{\mu^2C(AB+2s^2C)}{A^2B^2}\right]-\frac{4\sqrt{2}s\mu^2CD}{A^2B}\right\}.\end{aligned} \tag{38}$$

To visualize the wavelet-fractional squeezing combinatorial transform spectrum of the number state, the resulting values

$$\langle\psi|U(\mu,s)\exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger2}\right)\right]\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}|0\rangle,$$

$$\langle\psi|U(\mu,s)\exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]\mathrm{e}^{\mathrm{i}\pi a^{\dagger}a/2}|1\rangle$$

are plotted in Figs. 1(a)–1(f), with the two horizontal axes denoting scaling parameter and translation parameter, respectively. These figures exhibit the common behavior that all peaks are narrow and local, which are characteristics of the wavelet-fractional squeezing combinatorial transform. Since the wavelet transform is a local wave with respect to time and frequency, the resulting graphs are both narrow and local. We combine the wavelet transform and the fractional squeezing transform to obtain a wavelet-fractional squeezing combinatorial transform, which is equivalent to compress the wavelet transform. So the picture we get is even narrower. We can also find that all the peaks are symmetrical to the $s=0$ plane. In the figures of the state $|0\rangle$ and $|1\rangle$, the location of the peaks change with different α, the peak is largest when $\alpha=\pi/2$, and the peak becomes smaller when α becomes lager. For the same α, the peaks near the $s=0$ plane of state $|1\rangle$ become smaller compared with state $|0\rangle$, while the peaks far away the $s=0$ plane become larger. At the same time, comparing with state $|0\rangle$, the peaks of state $|1\rangle$ becomes scattered and easily identifiable. These different peaks and shapes of the pictures from different quantum states imply that the wavelet-fractional squeezing combinatorial transform spectrum may play a important role in identifying different quantum optical states, and there may be some potential quantum applications.

For the superposition state of the vacuum state and the single-particle state, we calculated the wavelet-fractional squeezing combinatorial transform results which is plotted in Figs. 2(a)–2(e). In the figures, we see that the characteristic of the state $|0\rangle$ become more obvious with the increase of the coefficient $x_1=1$, meanwhile, the characteristic of the state $|1\rangle$ become more significant with the increase of x_2. These results has an auxiliary function in analyzing and recognizing quantum states.

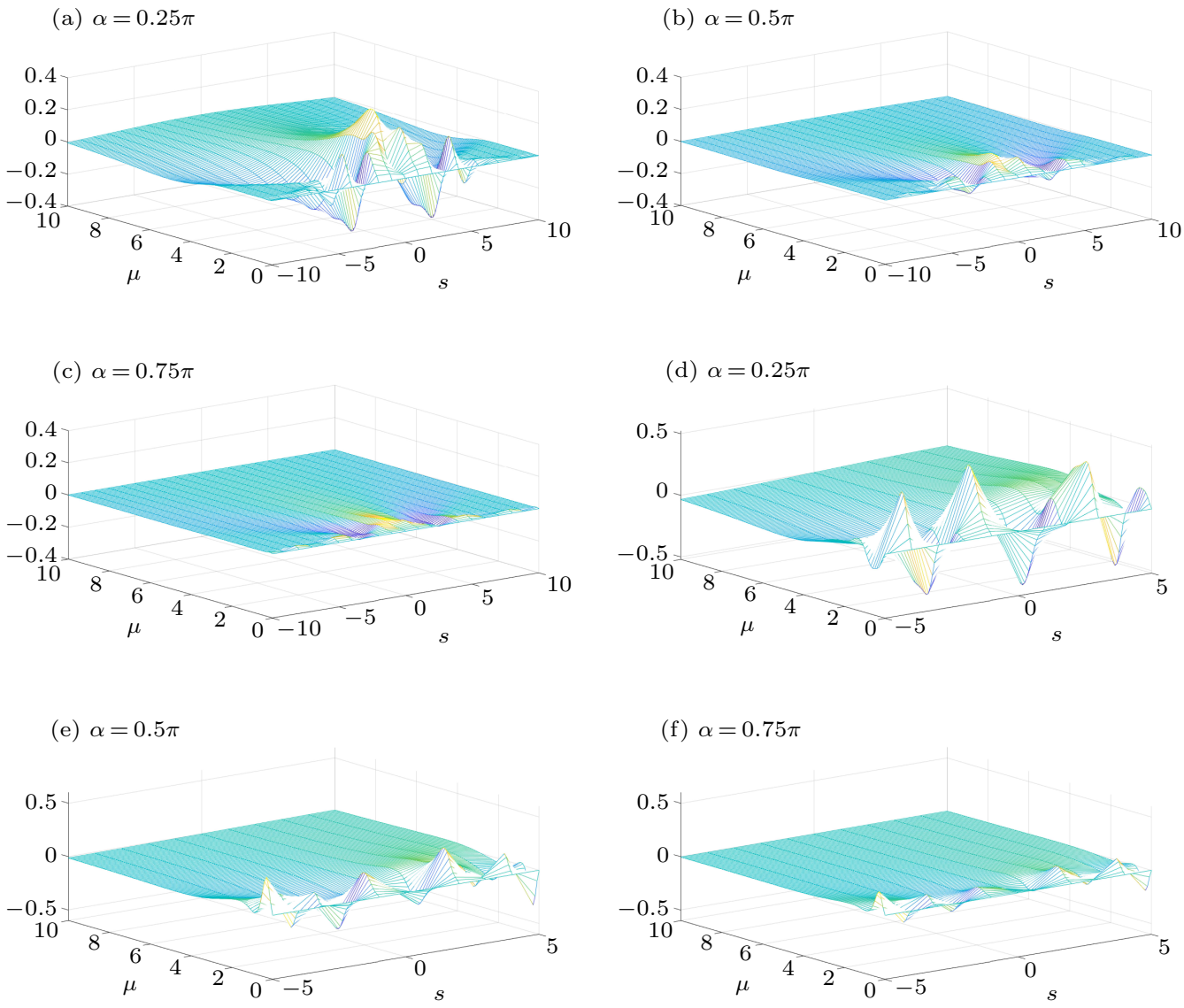


Fig. 1. The optical wavelet-fractional squeezing combinatorial transform results of state $|0\rangle$ and state $|1\rangle$: (a) state $|0\rangle$, $\alpha=\pi/4$; (b) state $|0\rangle$, $\alpha=\pi/2$; (c) state $|0\rangle$, $\alpha=3\pi/4$; (d) state $|1\rangle$, $\alpha=\pi/4$; (e) state $|1\rangle$, $\alpha=\pi/2$; (f) state $|1\rangle$, $\alpha=3\pi/4$.

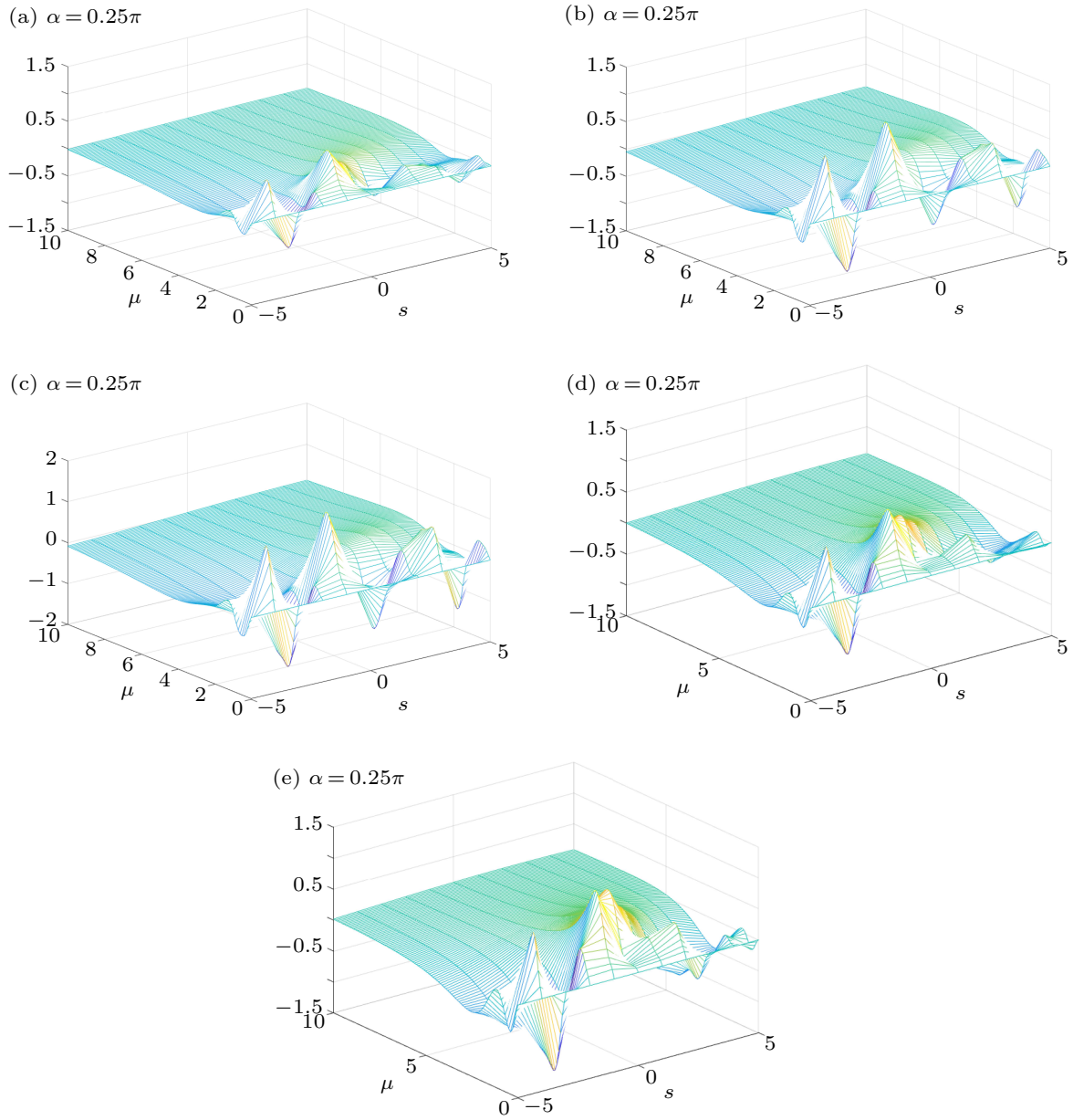


Fig. 2. The optical wavelet-fractional squeezing combinatorial transform results of the superposition state: (a) $x_1 = 1$, $x_2 = 1$; (b) $x_1 = 1$, $x_2 = 2$; (c) $x_1 = 1$, $x_2 = 3$; (d) $x_1 = 2$, $x_2 = 1$; (e) $x_1 = 3$, $x_2 = 1$.

6. Conclusion

Based on Dirac's quantum-mechanics, by converting the WT and the FrST into the framework of quantum mechanical operator transform theory, we have successfully combined the wavelet transform and fractional squeezing transform to realize a new quantum transform: the optical wavelet-fractional squeezing combinatorial transform. The unitary operator for this new transform is found and its normally ordered form is deduced. In addition, we have calculated the wavelet-fractional squeezing combinatorial transform for the vacuum state, the single-particle state and their superposition state. This kind of transform may be used to analyze and identify quantum states. Moreover, by analyzing the wavelet-fractional squeezing combinatorial transform spectrum, one may obtain other information which may represent new quantum optical states beneficial to quantum computing engineering. We also expect that the wavelet-fractional squeezing combinatorial transform can be implemented by experimentalists.[20,21] Wavelet transform has a pivotal position in many fields, and its application is also very extensive. Wavelet transform can compress the image. The new transform given in this paper is based on wavelet transform. Combined with the characteristics of squeezing transform, can we believe that this new transform may have an advantage in the application of wavelet transform? We are also looking forward to having experimenters to confirm this.

Acknowledgments

Project supported by the National Natural Science Foundation of China (Grant No. 11304126) and the

College Students' Innovation Training Program (Grant No. 202110299696X).

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