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Wenjie Yang(杨文杰)

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Bifurcation and dynamics in double-delayed Chua circuits with periodic perturbation

Wenjie Yang(杨文杰)†

School of Science, Xuchang University, Xuchang 461000, *China*

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Rank-1 attractors play a vital role in biological systems and the circuit systems. In this paper, we consider a periodically kicked Chua model with two delays in a circuit system. We first analyze the local stability of the equilibria of the Chua system and obtain the existence conditions of supercritical Hopf bifurcations. Then, we derive some explicit formulas about Hopf bifurcation, which could help us find the form of Hopf bifurcation and the stability of bifurcating period solutions through the Hassards method. Also, we show that rank-1 chaos occurs when the Chua model with two delays undergoes a supercritical Hopf bifurcation and encounters a periodic kick, which shows the effect of two delays on the circuit system. Finally, we illustrate the theoretical analysis by simulations and try to explain the mechanism of delay in our system.

Keywords: rank one chaos, periodically kicked Chua model, time delay

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1. Introduction

Research of chaos is a challenge in theory of nonlinear dynamical systems, especially, analyses of the global bifurcations and chaotic dynamics[1] and multi-pulse chaotic dynamics.[2] Moreover, more complex chaos attractors, such as hidden attractors, rank-1 attractors have been studied recently. For example, a rapidly growing number of studies have been published, in which hidden chaotic attractors are shown in some systems, including a single equilibrium,[3,4] stable equilibria,[5,6] as well as other systems.[7,8] Wang and Young[9] illustrated the existence of rank-1 attractors when only one direction of instability and certain external force exist, and they obtained the dual property of occurring naturally of a class of strange attractors.[10] Then Chen and Han found that a periodically kicked planar system with heteroclinic cycle having rank-1 strange attractors.[11] Also, Yang *et al.* proposed the rank-1 chaotic theory to solve the chaotic behavior of delayed systems, and they obtained some conditions for rank-1 chaos in a periodically kicked system.[12]

The Chua model is a well-known model to investigate the bifurcation and chaos, which has attracted more attention in recent years.[13] Tsafack and Kengne[15] pointed out that the Chua model is more flexible to show the rich dynamics behaviors. Meanwhile, Ozkaynak[16] developed a fractional-order chaotic Chua model and proved that it is crucial for the practical application of fractional-order chaotic systems. Yu *et al.*[17] considered some phenomena observed in a Chua system with multiple delays.

Mathematical modeling is a vital tool to study different systems and is often used to explain some novel phenomena.[18–21] Meanwhile, the concept of rank-1 attractors is used to illustrate some mechanisms and plays a vital role in biological systems and circuit systems. However, the Chua model with two delays has been seldom investigated in theory. To show the effect of two delays and a periodic kick on the Chua model, we consider a periodically kicked Chua model with two delays in a circuit system. We first analyze the local stability of the equilibria of the Chua system and obtain the existence conditions of supercritical Hopf bifurcations. Then, we derive some explicit formulas about Hopf bifurcation, which could help us find the form of Hopf bifurcation and the stability of bifurcating period solutions through the Hassards method. Also, we show that rank-1 chaos occurs when the Chua model with two delays undergoes a supercritical Hopf bifurcation and encounters a periodic kick, which shows the effect of two delays on the circuit system. Finally, we illustrate the theoretical analysis by simulations and try to explain the mechanism of delay in our system.

2. Analysis of Hopf bifurcation and rank-1 chaos in Chua model with two delays

In Ref. [20], Wang considered rank-1 chaos of the Chua model as follows:

$$\begin{aligned} \dot{x} &= a[y-(cx+dx^3)]+\varepsilon P_{\mathrm{T}} x, \\ \dot{y} &= x-y+z, \\ \dot{z} &= -by. \end{aligned} \tag{1}$$

To investigate the rank-1 attractor in the Chua system with both delays and periodically kick, we add time delays τ_1, τ_2 to Eq. (1), and obtain the following form:

$$\dot{x}(t) = a[y(t-\tau_1)-(cx(t)+dx^3(t))]+\varepsilon P_{\mathrm{T}} x(t),$$

†Corresponding author. E-mail: ywjwm123@foxmail.com

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$$\dot{y}(t) = x(t) - y(t-\tau_1) + z(t-\tau_2),$$
$$\dot{z}(t) = -by(t-\tau_1), \tag{2}$$

where a, b, c, d, τ_1, $\tau_2 \in R$, and τ_1, $\tau_2 \geq 0$ are time delays for y and z, respectively. The undisturbed system is

$$\dot{x}(t) = a[y(t-\tau_1) - (cx(t) + dx^3(t))],$$
$$\dot{y}(t) = x(t) - y(t-\tau_1) + z(t-\tau_2),$$
$$\dot{z}(t) = -by(t-\tau_1). \tag{3}$$

In this section, we consider system (3). It always has equilibrium $E_i^*(x_i^*, y_i^*, z_i^*)$, $i = 1,2,3$, where

$$x_1^* = 0,\ y_1^* = 0,\ z_1^* = 0,$$
$$x_2^* = \sqrt{\frac{-c}{d}},\ y_2^* = 0,\ z_2^* = -x_2^*,$$
$$x_3^* = -\sqrt{\frac{-c}{d}},\ y_3^* = 0,\ z_3^* = -x_3^*.$$

Let $E^* = (x^*, y^*, z^*)$ denote the arbitrary equilibrium, $\bar{x}(t) = x(t) - x^*$, $\bar{y}(t) = y(t) - y^*$, $\bar{z}(t) = z(t) - z^*$, still written $\bar{x}(t)$, $\bar{y}(t)$, $\bar{z}(t)$ as $x(t)$, $y(t)$, $z(t)$, respectively, then the linearized system at E^* is

$$\dot{u} = Au(t) + B_1 u(t-\tau_1) + B_2 u(t-\tau_2), \tag{4}$$

where $u(t) = [x(t), y(t), z(t)]^{\mathrm{T}}$, and

$$A = \begin{bmatrix} -ac - 3adx^{*2} & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0 & a & 0 \\ 0 & -1 & 0 \\ 0 & -b & 0 \end{bmatrix},\quad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we think about the following three kinds of circumstances.

I. $E^* = E_1^*$.

The characteristic equation of (4) is

$$\lambda^3 + (\alpha_1 - 1)\lambda^2 + (\lambda^2 + (\alpha_2 - b)\lambda)\,\mathrm{e}^{-\lambda\tau_1}$$
$$+ (b\lambda + \alpha_3)\,\mathrm{e}^{-\lambda(\tau_1+\tau_2)} = 0, \tag{5}$$

where $\alpha_1 = ac + 1$, $\alpha_2 = ac - a + b$, $\alpha_3 = abc$. In order to study the distribution of roots of the transcendental Eq. (5), we need to use the Corollary 2.4 of Ruan and Wei.[24] As Eq. (4) has two time delays τ_1 and τ_2, we consider the following six cases.

Case 1: $\tau_1 = \tau_2 = 0$, Eq. (5) becomes

$$\Gamma(\lambda) = \lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3 = 0. \tag{6}$$

The Routh–Hurwitz criterion implies if

(H_1) $\alpha_1 > 0$, $\alpha_3 > 0$, and $\alpha_1(\alpha_2 - b) + b > 0$.

Then all roots of Eq. (5) have negative real parts. Namely, the equilibrium point E_1^* is locally asymptotically stable when the condition (H_1) is satisfied.

Case 2: $\tau_1 > 0, \tau_2 = 0$, Eq. (5) becomes

$$\lambda^3 + (\alpha_1 - 1)\lambda^2 + (\lambda^2 + \alpha_2\lambda + \alpha_3)\,\mathrm{e}^{-\lambda\tau_1} = 0. \tag{7}$$

Suppose that $\mathrm{i}\omega_1 (\omega_1 > 0)$ is a root of Eq. (7) in the imaginary axis. Substituting it to Eq. (7) and separating the real and imaginary parts, we have

$$\alpha_2\omega_1\cos(\omega_1\tau_1) - (\alpha_3 - \omega_1^2)\sin(\omega_1\tau_1) = \omega_1^3,$$
$$\alpha_2\omega_1\sin(\omega_1\tau_1) + (\alpha_3 - \omega_1^2)\cos(\omega_1\tau_1) = (\alpha_1 - 1)\omega_1^2, \tag{8}$$

which is equivalent to

$$\omega_1^6 + e_{11}\omega_1^4 + e_{12}\omega_1^2 + e_{13} = 0. \tag{9}$$

Let $z_1 = \omega_1^2$, then Eq. (9) becomes

$$G(z_1) \triangleq z_1^3 + e_{11}z_1^2 + e_{12}z_1 + e_{13} = 0, \tag{10}$$

where $e_{11} = \alpha_1^2 - 2\alpha_2, e_{12} = 2\alpha_3 - \alpha_2^2, e_{13} = -\alpha_3^2$.

Since $G(0) = -\alpha_3^2 < 0$, $G(+\infty) = +\infty$, we can obtain that Eq. (10) has at least one positive root since $e_{13} < 0$.

We assume that Eq. (10) has three positive roots, z_{11}, z_{12}, z_{13}, and $\omega_{11} = \sqrt{z_{11}}$, $\omega_{12} = \sqrt{z_{12}}$, $\omega_{13} = \sqrt{z_{13}}$.

According to Eq. (8), we have

$$\tau_{1k}^{(j)} = \frac{1}{\omega_{1k}}\left\{\arcsin\frac{\omega_{1k}^5 + (\alpha_1\alpha_2 - \alpha_2 - \alpha_3)\omega_{1k}^3}{\omega_{1k}^4 + (\alpha_2^2 - 2\alpha_3)\omega_{1k}^2 + \alpha_3^2} + 2j\pi\right\},$$

where $k = 1,2,3$ and $j = 0,1,\ldots$, then $\pm\mathrm{i}\omega_{1k}$ is a pair of imaginary roots of Eq. (7) with $\tau_{1k}^{(j)}$. Define

$$\tau_{10} = \tau_{1k_0}^{(0)} = \min_{k=1,2,3}\{\tau_{1k}^{(0)}\},\quad \omega_{10} = \omega_{1k_0}.$$

According to the lemma in Ref. [23], we can investigate the distribution of the roots of Eq. (7). Since $e_{13} < 0$, for the polynomial Eq. (7), we can obtain that all roots with positive real parts of Eq. (7) have the same sum to those of the polynomial Eq. (6) for $\tau_1 \in [0, \tau_{10})$.

Let $\lambda(\tau_1) = \alpha(\tau_1) \pm \mathrm{i}\omega_1(\tau_1)$ be the root of Eq. (7) near $\tau_1 = \tau_{1j}^{(k)}$ $(k = 1,2,3)$ satisfying $\alpha(\tau_{1j}^{(k)}) = 0, \omega_1(\tau_{1j}^{(k)}) = \omega_{1j}$ $(k = 1,2,3;\ j = 0,1,2,\ldots)$. For the transversality condition, we have the following lemma.

Lemma 1 Suppose that $z_{1k} = \omega_{1k}^2$ and $h'(z_{1k}) \neq 0$, then $\frac{\mathrm{Re}\lambda(\tau_{1k}^{(j)})}{d\tau_1}$ has the same sign with $h'(z_{1k})$.

Proof Substituting $\lambda(\tau_1)$ into Eq. (7) and taking the derivative with respect to τ_1, we obtain

$$\left\{3\lambda^2 + 2(\alpha_1 - 1)\lambda + \mathrm{e}^{-\lambda\tau_1}(2\lambda + \alpha_2)\right.$$
$$\left. - \tau_1\,\mathrm{e}^{-\lambda\tau_1}(\lambda^2 + \alpha_2\lambda + \alpha_3)\right\}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1}$$
$$= \lambda\,\mathrm{e}^{-\lambda\tau_1}(\lambda^2 + \alpha_2\lambda + \alpha_3).$$

Therefore,

$$\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1}\right]^{-1} = \frac{(3\lambda^2 + 2(\alpha_1 - 1)\lambda)\,\mathrm{e}^{\lambda\tau_1}}{\lambda(\lambda^2 + \alpha_2\lambda + \alpha_3)}$$

$$+\frac{2\lambda+\alpha_2}{\lambda(\lambda^2+\alpha_2\lambda+\alpha_3)}-\frac{\tau_1}{\lambda}, \tag{11}$$

which leads to

$$\begin{aligned}
&\left[(3\lambda^2+2\lambda(\alpha_1-1))\,\mathrm{e}^{\lambda\tau_1}\right]_{\tau_1=\tau_{1k}^{(j)}}\\
&=-[2\omega_{1k}(\alpha_1-1)\sin(\omega_{1k}\tau_{1k}^{(j)})+3\omega_{1k}^2\cos(\omega_{1k}\tau_{1k}^{(j)})]\\
&\quad+\mathrm{i}[2\omega_{1k}(\alpha_1-1)\cos(\omega_{1k}\tau_{1k}^{(j)})-3\omega_{1k}^2\sin(\omega_{1k}\tau_{1k}^{(j)})],\\
&\left[\lambda(\lambda^2+\alpha_2\lambda+\alpha_3)\right]_{\tau_1=\tau_{1k}^{(j)}}\\
&=-\alpha_2\omega_{1k}^2+\mathrm{i}(\alpha_3\omega_{1k}-\omega_{1k}^3).
\end{aligned} \tag{12}$$

According to Eqs. (10)–(12), we have

$$\begin{aligned}
&\left[\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\tau_1}\right]^{-1}_{\tau_1=\tau_{1k}^{(j)}}\\
&=\mathrm{Re}\left[\frac{(3\lambda^2+2(\alpha_1-1)\lambda)\,\mathrm{e}^{\lambda\tau_1}}{\lambda(\lambda^2+\alpha_2\lambda+\alpha_3)}\right]_{\tau_1=\tau_{1k}^{(j)}}\\
&\quad+\mathrm{Re}\left[\frac{2\lambda+\alpha_2}{\lambda(\lambda^2+\alpha_2\lambda+\alpha_3)}\right]_{\tau_1=\tau_{1k}^{(j)}}\\
&=\frac{1}{\Lambda}\{\left[3\alpha_2\omega_{1k}^2+2(\alpha_1-1)(\alpha_3-\omega_{1k}^2)\right]\omega_{1k}^2\cos(\omega_{1k}\tau_{1k}^{(j)})\\
&\quad+\left[2\alpha_2(\alpha_1-1)-3(\alpha_3-\omega_{1k}^2)\right]\omega_{1k}^3\\
&\quad\times\sin(\omega_{1k}\tau_{1k}^{(j)})+\omega_{1k}^2(2\alpha_3-2\omega_{1k}^2-\alpha_2^2)\}\\
&=\frac{1}{\Lambda}\{3\omega_k^6+2(\alpha_1^2-2\alpha_2)\omega_k^4+(2\alpha_3-\alpha_2^2)\omega_k^2\}\\
&=\left[\frac{z_k(3z_k^2+2e_{11}z_k+e_{12})}{\Lambda}\right]^{-1}_{\tau=\tau_k^{(j)}}=\frac{z_k h'(z_k)}{\Lambda},
\end{aligned}$$

where $\Lambda=(\alpha_2\omega_{1k}^2)^2+(\alpha_3\omega_{1k}-\omega_{1k}^3)^2>0$. Therefore, $\frac{\mathrm{Re}\lambda(\tau_k^{(j)})}{\mathrm{d}\tau}$ has the same sign with $h'(z_k)$.

According to the above derivation, we can obtain the following theorems.

Theorem 1 Suppose that (H_1) holds,

(i) when $\tau_1\in[0,\tau_{10})$, then E_1^* of system (3) is locally asymptotically stable,

(ii) when $\tau_1>\tau_{10}$, E_1^* of system (3) is unstable,

(iii) when $\tau_1=\tau_{10}$, the system (3) exhibits Hopf bifurcation.

Case 3 $\tau_2>0$, $\tau_1=0$.

The calculation is similar to case 2, so we have

Theorem 2 Suppose that (H_1) holds,

(i) when $\tau_2\in[0,\tau_{20})$, then E_1^* of system (3) is locally asymptotically stable,

(ii) when $\tau_2>\tau_{20}$, E_1^* of system (3) is unstable,

(iii) when $\tau_2=\tau_{20}$, the system (3) exhibits Hopf bifurcation, where τ_{20} is the minimum critical value of τ_2 for the occurrence of Hopf bifurcation when $\tau_1=0$.

Case 4 $\tau_1=\tau_2\neq0$.

Theorem 3 Suppose that (H_1) holds,

(i) when $\tau_1\in[0,\tau_{30})$, E_1^* of system (3) is locally asymptotically stable,

(ii) when $\tau_1>\tau_{30}$, E_1^* of system (3) is unstable,

(iii) when $\tau_1=\tau_{30}$, the system (3) exhibits Hopf bifurcation.

Case 5 $\tau_1>0,\tau_2\in[0,\tau_{20})$ and $\tau_1\neq\tau_2$.

We consider Eq. (5) with $\tau_2\in[0,\tau_{20})$, and τ_1 is regarded as the parameter. Let $\mathrm{i}\omega_5(\omega_5>0)$ be the root of Eq. (5), then we can obtain

$$\begin{aligned}
&E_{11}\cos(\omega_5\tau_1)+E_{12}\sin(\omega_5\tau_1)=E_{13},\\
&E_{11}\sin(\omega_5\tau_1)-E_{12}\cos(\omega_5\tau_1)=E_{14},
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
&E_{11}=(\alpha_2-b)\omega_5+b\omega_5\cos(\omega_5\tau_2)-\alpha_3\sin(\omega_5\tau_2),\\
&E_{12}=\omega_5^2-b\omega_5\sin(\omega_5\tau_2)-\alpha_3\cos(\omega_5\tau_2),\\
&E_{13}=\omega_5^3,\\
&E_{14}=(\alpha_1-1)\omega_5^2.
\end{aligned}$$

From Eq. (13), we can reach

$$\begin{aligned}
&\omega_5^6+e_{21}\omega_5^4+e_{22}\omega_5^2+e_{23}+(e_{24}\omega_5^3+e_{25}\omega_5)\sin(\omega_5\tau_2)\\
&+e_{26}\omega_5^2\cos(\omega_5\tau_2)=0,
\end{aligned} \tag{14}$$

where $e_{21}=\alpha_1^2-2\alpha_1$, $e_{22}=-(\alpha_2-b)^2-b^2$, $e_{23}=-\alpha_3^2$, $e_{24}=2b$, $e_{25}=2\alpha_3(\alpha_2-b)$, $e_{26}=2\alpha_3-2b(\alpha_2-b)$.

Now we make the following assumption and get the main conclusions.

(H_2): Eq. (14) has at least finite positive root.

Suppose that (H_2) holds, we denote the positive roots of Eq. (14) as ω_{5k} $(k=1,2,3,4,5,6)$. For every ω_{5k}, the corresponding critical value of $\tau_{5k}^{(j)}$ $(j=1,2,3)$ is

$$\begin{aligned}
&\tau_{5k}^{(j)}=\frac{1}{\omega_{5k}}\left\{\arccos\frac{E_{21}E_{23}-E_{22}E_{24}}{E_{21}^2+E_{22}^2}+2\pi j\right\},\\
&k=1,2,3,4,5,6;\ j=0,1,2\ldots.
\end{aligned}$$

Let $\tau_{50}=\tau_{5k_0}^{(0)}=\min\limits_{k=1,2,\ldots6}\{\tau_{5k}^{(0)}\}$, $\omega_{50}=\omega_{5k_0}$ denotes the root of Eq. (14) with τ_{50}.

Then we differentiate two sides of Eq. (5) with respect to τ_1 to verify the transversality condition. Taking the derivative of λ of τ_1 in Eq. (5) and substituting $\lambda=\mathrm{i}\omega_{50}$, we have

$$\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1}\right]^{-1}=\frac{3\lambda^2+2(\alpha_1-1)\lambda+[2\lambda+(\alpha_2-b)]\,\mathrm{e}^{-\lambda\tau_1}+b\,\mathrm{e}^{-\lambda(\tau_1+\tau_2)}-[\tau_1(\lambda^2+(\alpha_2-b)\lambda)\,\mathrm{e}^{-\lambda\tau_1}+(\tau_1+\tau_2)(b\lambda+\alpha_3)\,\mathrm{e}^{-\lambda(\tau_1+\tau_2)}]}{[\lambda^3+(\alpha_2-b)\lambda^2]\,\mathrm{e}^{-\lambda\tau_1}+(b\lambda^2+\alpha_3\lambda)\,\mathrm{e}^{-\lambda(\tau_1+\tau_2)}},$$

which leads to

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1}\right)^{-1}_{\lambda=\mathrm{i}\omega_{50}} = \mathrm{Re}\left(\frac{A+B\mathrm{i}}{C+D\mathrm{i}}\right) = \frac{AC+BD}{C^2+D^2},$$

where

$$\begin{aligned}
A &= 2\omega_{50}\sin\omega_{50}\tau_{50} + (\alpha_2-b)\cos\omega_{50}\tau_{50} \\
&\quad + b\cos\omega_{50}(\tau_{50}+\tau_2) \\
&\quad - [\tau_{50}(-\omega_{50}^2\cos\omega_{50}\tau_{50} + (\alpha_2-b)\omega_{50}\sin\omega_{50}\tau_{50}) \\
&\quad + (\tau_{50}+\tau_2)(b\omega_{50}\sin\omega_{50}(\tau_{50}+\tau_2) \\
&\quad + \alpha_3\cos\omega_{50}(\tau_{50}+\tau_2))] - 3\omega_{50}^2, \\
B &= 2\omega_{50}\cos\omega_{50}\tau_{50} \\
&\quad - (\alpha_2-b)\sin\omega_{50}\tau_{50} - b\sin\omega_{50}(\tau_{50}+\tau_2) \\
&\quad - [\tau_{50}(\omega_{50}^2\sin\omega_{50}\tau_{50} + (\alpha_2-b)\omega_{50}\cos\omega_{50}\tau_{50}) \\
&\quad + (\tau_{50}+\tau_2)(b\omega_{50}\cos\omega_{50}(\tau_{50}+\tau_2) \\
&\quad - \alpha_3\sin\omega_{50}(\tau_{50}+\tau_2))] + 2(\alpha_1-1)\omega_{50}, \\
C &= -\omega_{50}^3\sin\omega_{50}\tau_{50} - (\alpha_2-b)\omega_{50}^2\cos\omega_{50}\tau_{50} \\
&\quad + \alpha_3\omega_{50}\sin\omega_{50}(\tau_{50}+\tau_2) - b\omega_{50}^2\cos\omega_{50}(\tau_{50}+\tau_2), \\
D &= -\omega_{50}^3\cos\omega_{50}\tau_{50} + (\alpha_2-b)\omega_{50}^2\sin\omega_{50}\tau_{50} \\
&\quad + \alpha_3\omega_{50}\cos\omega_{50}(\tau_{50}+\tau_2) + b\omega_{50}^2\sin\omega_{50}(\tau_{50}+\tau_2).
\end{aligned}$$

Therefore, if the following condition holds

(H_3) $AC+BD\neq 0$,

then, we have $\{\frac{\mathrm{d}(\mathrm{Re}\,\lambda)}{\mathrm{d}\tau_1}\}_{\lambda=\mathrm{i}\omega_{50}}\neq 0$, and

Theorem 4 For system (3), $\tau_1>0$, $\tau_2\in[0,\tau_{20})$ and $\tau_1\neq\tau_2$. Suppose that the conditions (H_2) and (H_3) hold, then

(i) when $\tau_1\in[0,\tau_{50})$, E_1^* of system (3) is locally asymptotically stable,

(ii) when $\tau_1>\tau_{50}$, E_1^* of system (3) is unstable,

(iii) when $\tau_1=\tau_{50}$, the system (3) exhibits Hopf bifurcation.

Case 6 $\tau_2>0$, $\tau_1\in[0,\tau_{10})$ and $\tau_1\neq\tau_2$.

We consider Eq. (5) with $\tau_1\in[0,\tau_{10})$, and take τ_2 as the parameter. The calculation is similar to case 5, so we can get the following results.

Theorem 5 For system (3), $\tau_2>0$, $\tau_1\in[0,\tau_{10})$ and $\tau_2\neq\tau_1$.

(i) when $\tau_2\in[0,\tau_{60})$, E_1^* of system (3) is locally asymptotically stable,

(ii) when $\tau_2>\tau_{60}$, E_1^* of system (3) is unstable,

(iii) when $\tau_2=\tau_{60}$, the system (3) exhibits Hopf bifurcation, where τ_{60} is the minimum critical value of τ_2 for the occurrence of Hopf bifurcation when $\tau_1\in[0,\tau_{10})$.

II. $E^*=E_2^*$, **or** E_3^*.

When $E^*=E_2^*$, or E_3^*, the coefficients of Eq. (5) are $\alpha_1=1-2ac$, $\alpha_2=-2ac-a$, $\beta_1=-2ac$. Under the condition (H_1), Eq. (5) has at least one positive real root, so E_2^* and E_3^* are unstable equilibria.

3. Hopf bifurcation and rank-1 chaos of the delayed system

According to the above analysis, system (3) undergoes Hopf bifurcation for different τ_1,τ_2. Now, we apply Hassard's method,[24] to study Hopf bifurcation of system (3) with respect to $\varepsilon P_T x(t)$, τ_1, $\tau_2\in[0,\tau_{20})$. Then we apply the rank-1 chaos theory to the delayed system,[12] to research the Chua model (3) with two delays and periodically kick $\varepsilon P_{\mathrm{T}}$. Let $\tau_1=\tau_{50}+\mu,\mu\in R, t=s\tau_1$, $x(s\tau_1)=\hat{x}(s), y(s\tau_1)=\hat{y}(s), z(s\tau_1)=\hat{z}(s)$, denote $x=\hat{x}, y=\hat{y}, z=\hat{z}$ and $t=s$, then system (3) with $\varepsilon P_T x(t)$ (i.e., Eq. (2)) can be written as a functional differential equation (FDE) in $C=C([-1,0],R^3)$:

$$\dot{u}(t)=L_\mu u_t+f_\mu(u_t)+\varepsilon P_T\Phi(u_t), \tag{15}$$

where $u(t)=(x(t),y(t),z(t))^{\mathrm{T}}\in C$, and $u_t(\theta)=u(t+\theta)=(x(t+\theta),y(t+\theta),z(t+\theta))^{\mathrm{T}}\in C$ and $L_\mu: C\to R^3$, $F: R\times C\to R^3$ are given by

$$\begin{aligned}
L_\mu\phi &= (\tau_{50}+\mu)A\begin{bmatrix}\phi_1(0)\\ \phi_2(0)\\ \phi_3(0)\end{bmatrix} + (\tau_{50}+\mu)B_1\begin{bmatrix}\phi_1(-\frac{\tau_2}{\tau_{50}})\\ \phi_2(-\frac{\tau_2}{\tau_{50}})\\ \phi_3(-\frac{\tau_2}{\tau_{50}})\end{bmatrix} \\
&\quad + (\tau_{50}+\mu)B_2\begin{bmatrix}\phi_1(-1)\\ \phi_2(-1)\\ \phi_3(-1)\end{bmatrix}, \\
f_\mu(\phi) &= (\tau_{50}+\mu)(-a\mathrm{d}\phi_1^3(0),0,0)^{\mathrm{T}},
\end{aligned}$$

where A, B_1, B_2 are as in Eq. (4), and by the Riesz representation theorem, there is a 3×3 matrix function $\eta(\theta,\mu)$ of bounded variation for $\theta\in[-1,0]$, such that $L_\mu\phi$ as the form

$$L_\mu\phi=\int_{-1}^{0}\mathrm{d}\eta(\theta,\mu)\phi(\theta),\quad \text{for } \phi\in C.$$

In fact, we can choose

$$\eta(\theta,\mu)=\begin{cases}(\tau_{50}+\mu)(A+B_1+B_2), & \theta=0,\\ (\tau_{50}+\mu)(B_1+B_2), & \theta\in[-\frac{\tau_2}{\tau_{50}},0),\\ (\tau_{50}+\mu)B_2, & \theta\in(-1,-\frac{\tau_2}{\tau_{50}}),\\ 0, & \theta=-1.\end{cases}$$

Then write Eq. (2) as

$$\dot{u}_t=A(\mu)u_t+Ru_t,$$

where

$$A(\mu)\phi=\begin{cases}\frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta\in[-1,0),\\ \int_{-1}^{0}\mathrm{d}\eta(s,\mu)\phi(s), & \theta=0,\end{cases}$$

and

$$R\phi=\begin{cases}0, & \theta\in[-1,0),\\ f_\mu(\phi)+\varepsilon P_T\Phi(\phi), & \theta=0.\end{cases}$$

For $\varphi\in C([-1,0],(R^3)^*)$, where $(R^3)^*$ is the 3-dimensional space of row vectors, and A^* is adjoint operator of $A(0)$:

$$A^*\varphi(s)=\begin{cases}-\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s}, & s\in(0,1],\\ \int_{-1}^{0}\mathrm{d}\eta^{\mathrm{T}}(t,0)\varphi(-t), & s=0.\end{cases}$$

For $\phi \in C([-1,0],R^3), \varphi \in C([-1,0],(R^3)^*)$, and the bilinear form

$$\begin{aligned}&\langle \varphi(s), \phi(s)\rangle\\ &= \bar{\varphi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{\varphi}(\xi-\theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi,\end{aligned}$$

where $\eta(\theta)=\eta(\theta,0), A=A(0)$ and A^* are adjoint operators. We have $\pm \mathrm{i}\omega_{50}\tau_{50}$ that are eigenvalues of $A(0)$ and A^*.

Assume that $q(\theta)$ is the eigenvector of $A(0)$ corresponding to $\mathrm{i}\omega_{50}\tau_{50}$, then $A(0)q(\theta)=i\omega_{50}\tau_{50}q(\theta)$. According to the definition of $A(0)$, we can reach

$$\begin{pmatrix} \mathrm{i}\omega_{50}+ac+3adx^{*2} & -a\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_{50}} & 0\\ -1 & \mathrm{i}\omega_{50}+\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_{50}} & -\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_2}\\ 0 & b\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_{50}} & \mathrm{i}\omega_{50}\end{pmatrix} q(\theta) = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Thus, we have $q(\theta)=(1,\alpha,\beta)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\omega_{50}\tau_{50}\theta}$, where

$$\alpha = \frac{\mathrm{e}^{\mathrm{i}\omega_{50}\tau_{50}}(\mathrm{i}\omega_{50}+ac+3adx^{*2})}{a}, \quad \beta = \frac{b(\mathrm{i}\omega_{50}+ac+3adx^{*2})}{-\mathrm{i}a\omega_{50}}.$$

Similarly, we have $q^*(s)=D(1,\alpha^*,\beta^*)\mathrm{e}^{\mathrm{i}\omega_{50}\tau_{50}s}$ that is the eigenvector to the eigenvalue $-\mathrm{i}\omega_{50}\tau_{50}$ of A^*, which is the adjoint operator of $A(0)$, where

$$\alpha^* = \mathrm{i}\omega_{50}+ac+3adx^{*2}, \quad \beta^* = \frac{\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_2}(\mathrm{i}\omega_{50}+ac+3adx^{*2})}{\mathrm{i}\omega_{50}}.$$

Therefore, we have

$$\begin{aligned}\langle q^*(s), q(\theta)\rangle &= \bar{D}(1,\bar{\alpha}^*,\bar{\beta}^*)(1,\alpha,\beta)^{\mathrm{T}} - \int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{D}(1,\bar{\alpha}^*,\bar{\beta}^*)\mathrm{e}^{-\mathrm{i}(\xi-\theta)\omega_{50}\tau_{50}}\mathrm{d}\eta(\theta)(1,\alpha,\beta)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\xi\omega_{50}\tau_{50}}\mathrm{d}\xi\\ &= \bar{D}(1+\alpha\bar{\alpha}^*+\beta\bar{\beta}^* - \int_{-1}^{0}(1,\bar{\alpha}^*,\bar{\beta}^*)\theta\mathrm{e}^{\mathrm{i}\theta\omega_{50}\tau_{50}}\mathrm{d}\eta(\theta)(1,\alpha,\beta)^{\mathrm{T}})\\ &= \bar{D}(1+\alpha\bar{\alpha}^*+\beta\bar{\beta}^*+a\alpha\tau_{50}\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_{50}}+\bar{\alpha}^*(\beta\tau_2\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_2}-\alpha\tau_{50}\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_{50}})+b\alpha\bar{\beta}^*\tau_{50}\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_{50}}).\end{aligned}$$

Then we can choose $\bar{D}$ as

$$\bar{D} = [1+\alpha\bar{\alpha}^*+\beta\bar{\beta}^*+a\alpha\tau_{50}\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_{50}}+\bar{\alpha}^*(\beta\tau_2\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_2}-\alpha\tau_{50}\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_{50}})+b\alpha\bar{\beta}^*\tau_{50}\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_{50}}]^{-1} = \frac{1}{\mathrm{Re}(D^{-1})+\mathrm{i}\,\mathrm{Im}(D^{-1})},$$

where

$$\begin{aligned}\mathrm{Re}(D^{-1}) &= 1+\tau_{50}(ac+3adx^{*2})+\frac{|\alpha^*|^2}{a}(\cos\omega_{50}\tau_{50}-\tau_{50})+\frac{b|\alpha^*|^2}{a\omega_{50}}\Big(-\frac{1}{\omega_{50}}\cos\omega_{50}\tau_2-\tau_{50}\sin\omega_{50}\tau_2+\tau_2\sin\omega_{50}\tau_2\Big),\\ \mathrm{Im}(D^{-1}) &= \tau_{50}\omega_{50}+\frac{|\alpha^*|^2}{a}\sin\omega_{50}\tau_{50}+\frac{b|\alpha^*|^2}{a\omega_{50}}\Big(-\frac{1}{\omega_{50}}\sin\omega_{50}\tau_2+\tau_{50}\cos\omega_{50}\tau_2+\tau_2\cos\omega_{50}\tau_2\Big),\end{aligned}$$

such that $< q^*(s), q(\theta) >= 1, < q^*(s), \bar{q}(\theta) >= 0$.

Then we have the coordinates to describe the center manifold C_0 at $\tau=\tau_{50}$.

Notice that $u_t(\theta)=(x_t(\theta),y_t(\theta),z_t(\theta))^{\mathrm{T}} = zq(\theta)+\bar{z}\bar{q}(\theta)+W(t,\theta)$, then we can obtain

$$\begin{aligned}x_t(0) &= z+\bar{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\bar{z}+W_{02}^{(1)}(0)\frac{\bar{z}^2}{2}+W_{30}^{(1)}(0)\frac{z^3}{6}+W_{21}^{(1)}(0)\frac{z^2\bar{z}}{2}+\cdots,\\ y_t(0) &= z\alpha+\bar{z}\bar{\alpha}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\bar{z}+W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}+W_{30}^{(2)}(0)\frac{z^3}{6}+W_{21}^{(2)}(0)\frac{z^2\bar{z}}{2}+\cdots,\\ z_t(0) &= z\beta+\bar{z}\bar{\beta}+W_{20}^{(3)}(0)\frac{z^2}{2}+W_{11}^{(3)}(0)z\bar{z}+W_{02}^{(3)}(0)\frac{\bar{z}^2}{2}+W_{30}^{(3)}(0)\frac{z^3}{6}+W_{21}^{(3)}(0)\frac{z^2\bar{z}}{2}+\cdots,\\ x_t(-1) &= z\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_{50}}+\bar{z}\mathrm{e}^{\mathrm{i}\omega_{50}\tau_{50}}+W_{20}^{(1)}(-1)\frac{z^2}{2}+W_{11}^{(1)}(-1)z\bar{z}+W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2}+W_{30}^{(1)}(-1)\frac{z^3}{6}+W_{21}^{(1)}(-1)\frac{z^2\bar{z}}{2}+\cdots,\\ y_t(-1) &= z\alpha\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_{50}}+\bar{z}\bar{\alpha}\mathrm{e}^{\mathrm{i}\omega_{50}\tau_{50}}+W_{20}^{(2)}(-1)\frac{z^2}{2}+W_{11}^{(2)}(-1)z\bar{z}+W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2}+W_{30}^{(2)}(-1)\frac{z^3}{6}+W_{21}^{(2)}(-1)\frac{z^2\bar{z}}{2}+\cdots,\\ z_t(-1) &= z\beta\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_{50}}+\bar{z}\bar{\beta}\mathrm{e}^{\mathrm{i}\omega_{50}\tau_{50}}+W_{20}^{(3)}(-1)\frac{z^2}{2}+W_{11}^{(3)}(-1)z\bar{z}+W_{02}^{(3)}(-1)\frac{\bar{z}^2}{2}+W_{30}^{(3)}(-1)\frac{z^3}{6}+W_{21}^{(3)}(-1)\frac{z^2\bar{z}}{2}+\cdots,\end{aligned}$$

$$
\begin{aligned}
x_t\left(-\frac{\tau_2}{\tau_{50}}\right) &= z\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_2}+\bar{z}\mathrm{e}^{\mathrm{i}\omega_{50}\tau_2}+W_{20}^{(1)}\left(-\frac{\tau_2}{\tau_{50}}\right)\frac{z^2}{2}+W_{11}^{(1)}\left(-\frac{\tau_2'}{\tau_{50}}\right)z\bar{z}+W_{02}^{(1)}\left(-\frac{\tau_2'}{\tau_{50}}\right)\frac{\bar{z}^2}{2}\\
&\quad+W_{30}^{(1)}\left(-\frac{\tau_2'}{\tau_{50}}\right)\frac{z^3}{6}+W_{21}^{(1)}(-\frac{\tau_2}{\tau_{50}})\frac{z^2\bar{z}}{2}+\cdots,\\
y_t\left(-\frac{\tau_2}{\tau_{50}}\right) &= z\alpha\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_2}+\bar{z}\bar{\alpha}\mathrm{e}^{\mathrm{i}\omega_{50}\tau_2}+W_{20}^{(2)}\left(-\frac{\tau_2}{\tau_{50}}\right)\frac{z^2}{2}+W_{11}^{(2)}\left(-\frac{\tau_2}{\tau_{50}}\right)z\bar{z}+W_{02}^{(2)}\left(-\frac{\tau_2}{\tau_{50}}\right)\frac{\bar{z}^2}{2}\\
&\quad+W_{30}^{(2)}\left(-\frac{\tau_2}{\tau_{50}}\right)\frac{z^3}{6}+W_{21}^{(2)}\left(-\frac{\tau_2}{\tau_{50}}\right)\frac{z^2\bar{z}}{2}+\cdots,\\
z_t\left(-\frac{\tau_2}{\tau_{50}}\right) &= z\beta\mathrm{e}^{-\mathrm{i}\omega_{50}\tau_2}+\bar{z}\bar{\beta}\mathrm{e}^{\mathrm{i}\omega_{50}\tau_2}+W_{20}^{(3)}\left(-\frac{\tau_2}{\tau_{50}}\right)\frac{z^2}{2}+W_{11}^{(3)}\left(-\frac{\tau_2}{\tau_{50}}\right)z\bar{z}+W_{02}^{(3)}\left(-\frac{\tau_2}{\tau_{50}}\right)\frac{\bar{z}^2}{2}\\
&\quad+W_{30}^{(3)}\left(-\frac{\tau_2}{\tau_{50}}\right)\frac{z^3}{6}+W_{21}^{(3)}\left(-\frac{\tau_2}{\tau_{50}}\right)\frac{z^2\bar{z}}{2}+\cdots.
\end{aligned}
$$

Thus, from Eq. (15) we have

$$
\begin{aligned}
g(z,\bar{z}) &= \bar{q}^*(0)f_0(z,\bar{z})=\tau_{50}\bar{D}(1,\bar{\alpha}^*,\bar{\beta}^*)\begin{pmatrix}-ad\phi_1^3(0)\\0\\0\end{pmatrix}\\
&= -ad\phi_1^3(0)\tau_{50}\bar{D}.
\end{aligned}
$$

We can obtain

$$
g_{20}=g_{11}=g_{02}=0,\quad g_{21}=-6ad\tau_{50}\bar{D},
$$

and

$$
k_1(0)=\frac{g_{21}}{2},\quad \mu_2=-\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_{50})\}},\quad \beta_2=2\mathrm{Re}\{k_1(0)\},
$$

which determine the stability and bifurcation direction of periodic solutions at the critical value τ_{50}, i.e., μ_2 determines the directions of a Hopf bifurcation: if $\mathrm{Re}\{\lambda'(\tau_{50})\}>0$, $\mu_2>0$ (resp. $\mu_2<0$), then the Hopf bifurcation is supercritical (resp. subcritical) and the periodic solution exist for $\tau>\tau_{50}$ ($\tau<\tau_{50}$). Here, β_2 determines the stability of the bifurcation periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2<0$ ($\beta_2>0$). We can also obtain $\mathrm{Re}(k_1(0))$ and $\mathrm{Im}(k_1(0))$ as follows:

$$
\begin{aligned}
\Psi_x(x,y) &= \mathrm{Re}\{\bar{q}^*(0)\Phi(2\mathrm{Re}\,(x+\mathrm{i}y)q(0))\}\\
&= \mathrm{Re}\left\{\tau_{50}\bar{D}(1,\alpha^*,\beta^*)\begin{pmatrix}(x+\mathrm{i}y)\alpha+(x-\mathrm{i}y)\bar{\alpha}\\0\\0\end{pmatrix}\right\}\\
&= \mathrm{Re}\{\tau_{50}\bar{D}((x+\mathrm{i}y)\alpha+(x-\mathrm{i}y)\bar{\alpha})\}\\
&= 2\tau_{50}(x\mathrm{Re}(\alpha)-y\mathrm{Im}(\alpha))\mathrm{Re}(\bar{D}),
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_y(x,y) &= \mathrm{Im}\{\bar{q}^*(0)\Phi(2\mathrm{Re}\,zq(0))\}\\
&= 2\tau_{50}(x\mathrm{Re}(\alpha)-y\mathrm{Im}(\alpha))\mathrm{Im}(\bar{D}).
\end{aligned}
$$

Let $x=\cos\theta$, $y=\sin\theta$, then

$$
\begin{aligned}
\varphi(\theta) &= \cos\theta\Psi_x(\hat{s}_0)+\sin\theta\Psi_y(\hat{s}_0)\\
&= 2\tau_{50}[\mathrm{Re}(\bar{D})(\cos\theta\,\mathrm{Re}(\alpha)-\sin\theta\,\mathrm{Im}(\alpha))\cos\theta\\
&\quad+\mathrm{Im}(\bar{D})((\cos\theta\,\mathrm{Re}(\alpha)-\sin\theta\,\mathrm{Im}(\alpha)))\sin\theta].
\end{aligned}
$$

According to Theorem 2,[12] from Yang and Lin, if $\varphi(\theta)$ is a Morse function and $\mathrm{Re}(k_1(0))<0$, then rank-1 chaos will occur when $\left|\varepsilon\frac{\mathrm{Im}(k_1(0))}{\mathrm{Re}(k_1(0))}\right|>K_2$.

4. Numerical simulations

In this section, we consider the following particular example of system (2):

$$
\begin{aligned}
\dot{x}(t) &= 0.84[y(t-\tau_1)-(x(t)-x^3(t))]+\varepsilon P_{\mathrm{T}}x(t),\\
\dot{y}(t) &= x(t)-y(t-\tau_1)+z(t-\tau_2),\\
\dot{z}(t) &= -1.95y(t-\tau_1).
\end{aligned}
\tag{16}
$$

It is easy to see that (H_1) holds, for $\tau_1>0,\tau_2=0$, we have $\omega_{10}\approx 1.3128,\tau_{10}\approx 0.485$. By Theorem 1, we can obtain that E^* is asymptotically stable when $\tau_1\in[0,\tau_{10})$, when τ_1 passes through τ_{10}, E^* will lose its stability, and Hopf bifurcation occurs. We can also obtain $k_1(0)\approx -0.0457+2.3334\mathrm{i},\mu_2\approx 0.0431,\beta_2\approx -0.0914$, so the bifurcation is supercritical and the stable periodic solution emerges from E^*, see Figs. 1 and 2.

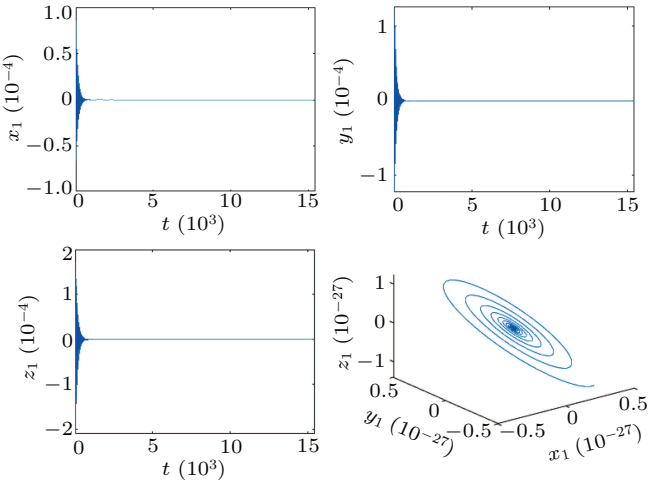


Fig. 1. The trajectories and phase graphs of system (16) with $\varepsilon=0$, $\tau_1=0.4$, $\tau_2=0$. E^* becomes unstable, and a stable periodic solution bifurcates from the equilibrium E^*.

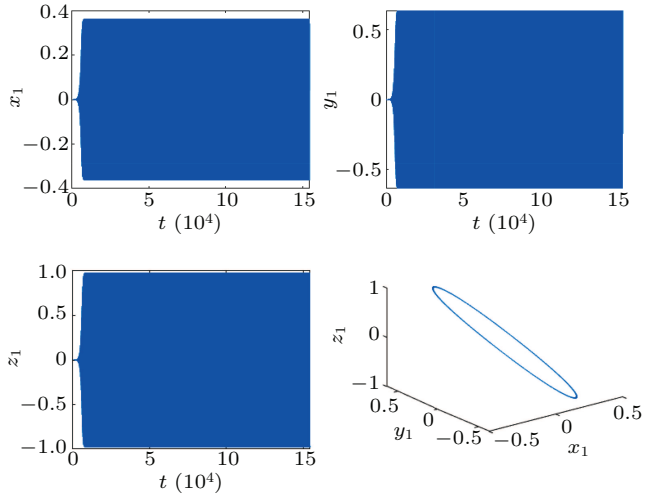


Fig. 2. The trajectories and phase graphs of system (16) with $\varepsilon = 0$, $\tau_1 = 0.485$, $\tau_2 = 0$. E^* become unstable, and a stable periodic solution bifurcates from the equilibrium E^*.

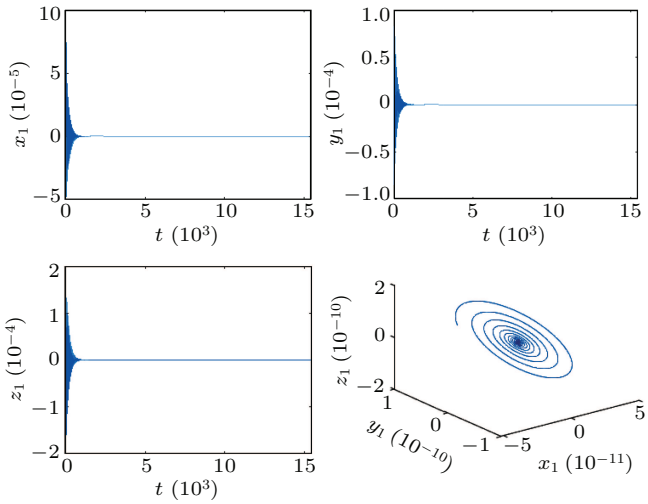


Fig. 3. The trajectories and phase graphs of system (16) with $\varepsilon = 0$, $\tau_1 = 0$, $\tau_2 = 0.3$. E^* becomes unstable, and a stable periodic solution bifurcates from the equilibrium E^*.

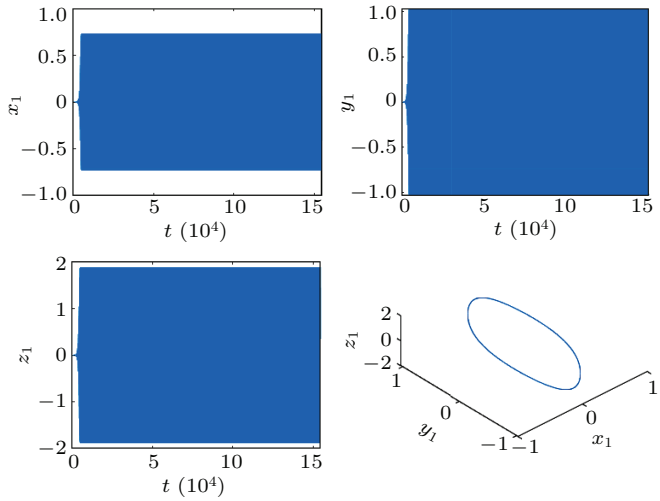


Fig. 4. The trajectories and phase graphs of system (16) with $\varepsilon = 0$, $\tau_1 = 0$, $\tau_2 = 0.36$. E^* becomes unstable, and a stable periodic solution bifurcates from the equilibrium E^*.

Similarly, for $\tau_1 = 0, \tau_2 > 0$, we have $\omega_{20} \approx 1.1271, \tau_{20} \approx 0.3619$, from Theorem 2, we can realize that E^* is asymptotically stable when $\tau_2 \in [0, \tau_{20})$. When τ_2 passes through τ_{20}, E^* will lose its stability and Hopf bifurcation emerges, see Figs. 3 and 4. For $\tau_1 = \tau_2 \triangleq \tau_3 \neq 0$, we can obtain $\omega_{30} \approx 1.6186$ and $\tau_{30} \approx 1.8446$.

For $\tau_1 > 0, \tau_2 = 0.33 \in [0, \tau_{20})$, we have obtain $\omega_{50} \approx 1.1554, \tau_{50} \approx 0.02$. By Theorem 4, E^* is asymptotically stable when $\tau_1 \in [0, \tau_{50})$. When $\tau_1 > \tau_{50}$ the steady state becomes unstable and the sustained oscillations occur. The bifurcation diagram of the system (16) for x is shown in Fig. 5, where the control parameter is time delay τ_1. At low values of τ_1, the system reaches a stable steady state. With the increase of τ_1, bifurcation occurs $\tau_1 \geq 0.02$. We choose $\tau_1 = 0.02$, after the computation, we have $k_1(0) \approx -0.0249 - 0.0042\mathrm{i}, \mu_2 \approx 0.0125, \beta_2 \approx -0.0498$, we know that the bifurcation is supercritical and the stable periodic solution emerges from E^*, see Figs. 6 and 7.

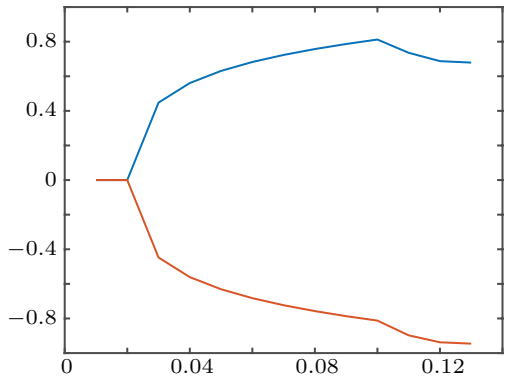


Fig. 5. The bifurcation diagram of the system (16) for x.

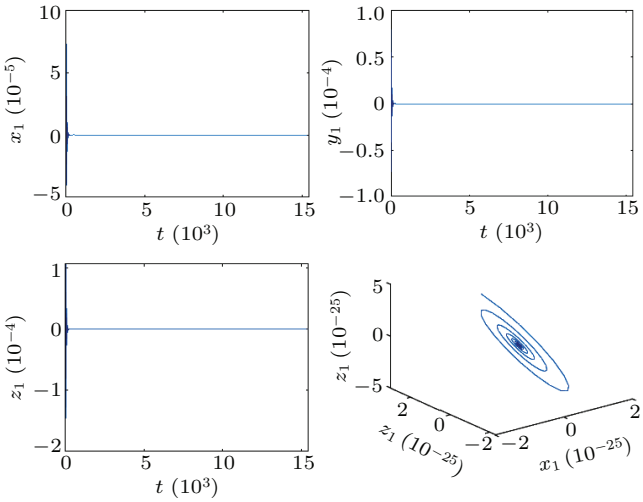


Fig. 6. The trajectories and phase graphs of system (16) with $\varepsilon = 0$, $\tau_1 = 0.011$, $\tau_2 = 0.33$. E^* becomes unstable, and a stable periodic solution bifurcates from the equilibrium E^*.

For all θ such that $\varphi'(\theta) = 0, \varphi''(\theta) \neq 0$, we can realize that $\varphi(\theta)$ is a Morse function (see Fig. 8), meanwhile, $E(0) \doteq 0.0249 > 0, |F(0)/E(0)| \doteq 0.1686$. Then we choose $T = 6897$, which is quite "large" to afford us a long relaxation period between consecutive kicks of the external force. In Figs. 9–11 we present a rank-1 attractor at $(\varepsilon, T) =$

$(0.73,6897),(0.73,7183),(0.93,7183)$ when $\tau_1 = 0.02, \tau_2 = 0.33$. The results of our numerical simulations well confirm the previous rank-1 theory.

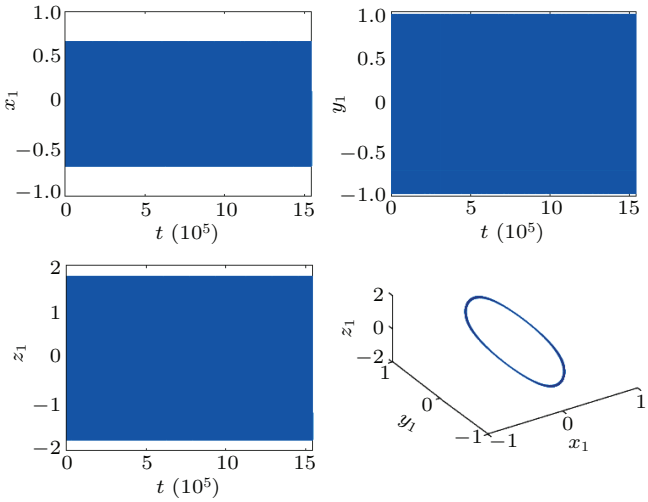


Fig. 7. The trajectories and phase graphs of system (16) with $\varepsilon = 0$, $\tau_1 = 0.02$, $\tau_2 = 0.33$. E^* becomes unstable, and a stable periodic solution bifurcates from the equilibrium E^*.

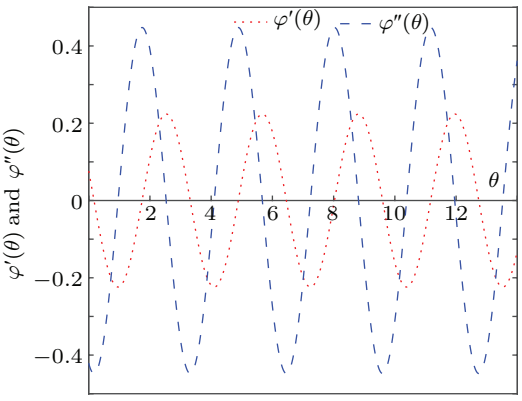


Fig. 8. The graphs of $\varphi'(\theta)$ and $\varphi''(\theta)$.

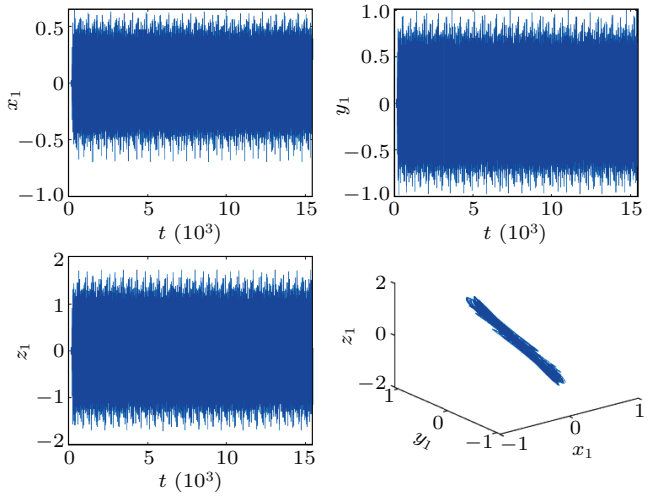


Fig. 9. The trajectories and phase graphs of system (16) with $T = 6897$, $\varepsilon = 0.73$, $\tau_1 = 0.02$, $\tau_2 = 0.33$. A rank-1 strange attractor occurs.

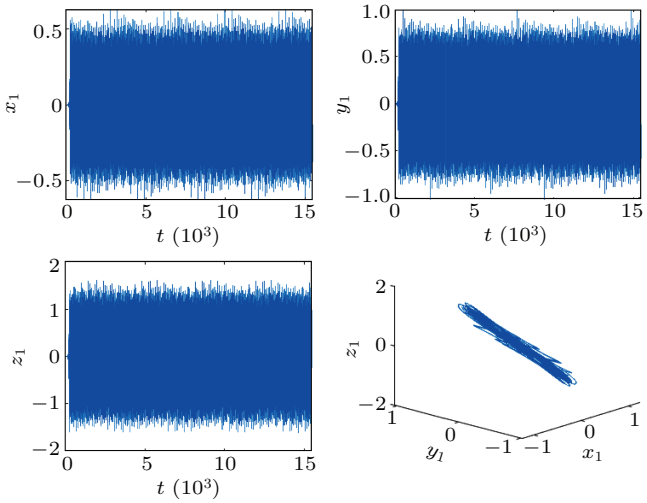


Fig. 10. The trajectories and phase graphs of system (16) with $T = 7183$, $\varepsilon = 0.73$, $\tau_1 = 0.02$, $\tau_2 = 0.33$. A rank-1 strange attractor occurs.

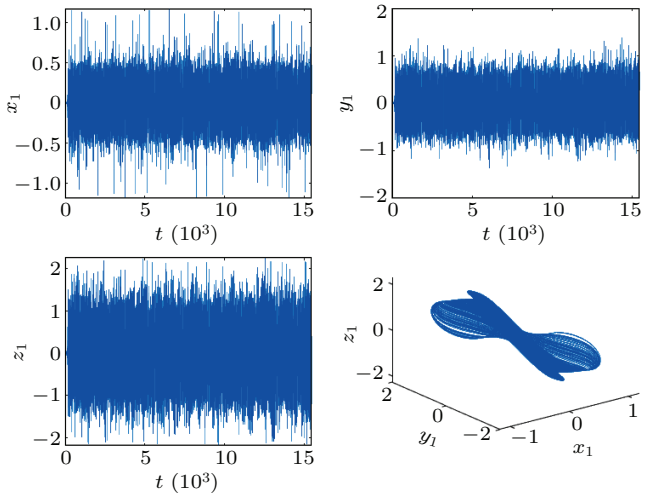


Fig. 11. The trajectories and phase graphs of system (16) with $T = 7183$, $\varepsilon = 0.93$, $\tau_1 = 0.02$, $\tau_2 = 0.33$. A rank-1 strange attractor occurs.

5. Conclusion

In summary, we have studied two-delayed Chua models. We first analyze the local stability of the system equilibria and the existence of Hopf bifurcations, and obtain the conditions that locally asymptotically stable, unstable, Hopf bifurcations occur on bases of τ_1 and τ_2. Then we obtain some explicit formulas about Hopf bifurcation, find that μ_2 determines the directions of Hopf bifurcation, and β_2 determines the stability of the bifurcation periodic solutions. Also, we show that rank-1 chaos occurs when our Chua model with two delays undergoes a supercritical Hopf bifurcation and encounters an external periodic force, namely, suppose that $\varphi(\theta)$ is a Morse function and $\mathrm{Re}(k_1(0)) < 0$, rank-1 chaos occurs when $\left|\varepsilon \frac{\mathrm{Im}(k_1(0))}{\mathrm{Re}(k_1(0))}\right| > K_2$. Finally, we illustrate the theoretical analysis by numerical simulations. Our theoretical analysis is verified by numerical simulations, which may be a novel phenomenon to explain mechanisms of some circuits.

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