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Alternative non-Gaussianity measures for quantum states based on quantum fidelity

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We propose three alternative measures for non-Gaussianity of quantum states: sine distance, Bures angle, and Bures distance, which are based on quantum fidelity introduced by Wang [*Phys. Lett. A* **373** 58 (2008)]. Using them, we evaluate the non-Gaussianity of some relevant single-mode and two-mode non-Gaussian states and find a good consistency of the three examined measures. In addition, we show that such metrics can exactly quantify the degree of Gaussianity of even Schrödinger-cat-like states of small amplitudes that can not be measured by other known non-Gaussianity measures such as the Hilbert–Schmidt metric and the relative entropy metric. We make a comparative study between all existing non-Gaussianity measures according to the metric axioms and point out that the sine distance is the best candidate among them.

Keywords: non-Gaussianity, quantum fidelity, non-Gaussian states, sine metric

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1. Introduction

Non-Gaussianity of quantum states is a useful resource in quantum information schemes with continuous variables since such a property can be exploited to improve their performances, including quantum teleportation,[1–3] quantum cloning,[4] entanglement distillation,[5,6] quantum secure communication,[7] and estimation problem.[8] To quantify non-Gaussianity, some metrics have been introduced over the past ten years. For example, Genoni *et al.* proposed the Hilbert–Schmidt[9] and the relative entropy measures[10] of non-Gaussianity for quantum states. Ivan *et al.* defined another non-Gaussianity measure by means of the Wehrl entropy.[11] Very recently, Fu *et al.* introduced a novel measure for non-Gaussianity of bosonic field states in terms of the Wigner–Yanase skew information.[12] According to the metric axioms,[13] one sees that all such metrics do not satisfy the symmetry and except the Hilbert–Schmidt metric, all the other ones are boundless, thus giving rise to a huge challenge of measuring it experimentally. Therefore, it is highly desirable to look for a plausible measure of non-Gaussianity.

In quantum information theory, non-Gaussianity quantifies a departure of a given quantum state from its reference Gaussian state. When talking about the issue of discriminating two different quantum states, we focus naturally on quantum fidelity, which provides a quantification of the degree of similarity of a pair of quantum states and has many attractive properties.[14] The fidelity was originally defined as Uhlmann's transition probability.[15] When one of two quantum states is pure, the fidelity is easily computable and reduces to their overlap. However, for an arbitrary pair of mixed states, the fidelity involves successive computation of the square roots of Hermitian matrices, which is in general hard to compute analytically. In order to avoid this difficulty, Wang *et al.* introduced an alternative fidelity for two mixed quantum states,[16] which satisfies Jozsa's four axioms up to a normalization factor[14] and is relatively easy to be calculated. As is well known, quantum fidelity is not a metric, but can easily be used to define many metrics on the space of quantum states such as the sine distance,[13] the Bures angle,[17] and the Bures distance.[18] For convenience, we shall refer these metrics as the fidelity-based ones, which have been widely used to quantify the non-classicality,[19] quantum entanglement,[20] and quantum discord of quantum states.[21] Motivated by these ideas, in this paper, we would like to quantify the non-Gaussianity of quantum states using these fidelity-based metrics together with quantum fidelity introduced by Wang.[16] To the best of our knowledge, there are currently few studies examining this problem.

The paper is organized as follows. In Section 2 we review some known quantum fidelities as well as fidelity-based metrics, and elucidate their fundamental properties. We introduce our measures for non-Gaussianity. Using these measures, we assess the non-Gaussianity of some relevant single-mode and two-mode quantum states in Section 3. At the same time, we analyze the consistency of these fidelity-based measures. In Section 4, we devote attention to a comparison between all the known metrics for non-Gaussianity in the literature and discuss their legitimacy according to the metric axioms. Finally, we summarize our main results in Section 5.

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2. Non-Gaussianity measures based on fidelity

Let us consider the fidelity, which is of importance in quantum information science[17] and other fields such as quantum chaos[22] and quantum phase transitions.[23,24] According to Uhlmann,[15] the fidelity of any two quantum states ρ and σ is defined as

$$F(\rho,\sigma)=\left[\mathrm{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)\right]^2, \tag{1}$$

which is called the Uhlmann–Jozsa fidelity. One sees that when at least one of two quantum states is pure, the fidelity can be simplified to the familiar overlap between the states: $F(\rho,\sigma)=\mathrm{Tr}[\sigma\rho]$. However, for any two mixed quantum states, it is not easily computed due to the square roots of Hermitian matrix. Hence, there is a need to define new fidelities to overcome this difficulty. Mendonca *et al.*[25] and Miszczak *et al.*[26] introduced the superfidelity of form

$$F_{\mathrm{S}}(\rho,\sigma)=\mathrm{Tr}(\rho\sigma)+\sqrt{1-\mu[\rho]}\sqrt{1-\mu[\sigma]}, \tag{2}$$

where the purities $\mu[\rho]=\mathrm{Tr}(\rho^2)$ and $\mu[\sigma]=\mathrm{Tr}(\sigma^2)$.

Another proposal of fidelity measure alternative to the Uhlmann–Jozsa fidelity, introduced by Chen *et al.* in Ref. [27], is defined as

$$F_{\mathrm{C}}(\rho,\sigma)=\frac{1-r}{2}+\frac{1+r}{2}F_{\mathrm{S}}(\rho,\sigma), \tag{3}$$

where $r=\frac{1}{d-1}$ with d being the dimension of the state space of ρ and σ. It is seen from Eqs. (1)–(3) that for a two-dimension case, we have $F(\rho,\sigma)|_{d=2}=F_{\mathrm{C}}(\rho,\sigma)|_{d=2}=F_{\mathrm{S}}(\rho,\sigma)|_{d=2}$. Nevertheless, the situation is more complicated in arbitrarily high-dimensional, multipartite and continuous-variable quantum states, giving rise to the computationally challenging tasks. In addition, one sees that the superfidelity and Chen's fidelity fail to satisfy $F_{\mathrm{S}}(\rho,\sigma)=0$ for two orthogonal quantum states. In response to this point, Wang *et al.* proposed an alternative fidelity measure for mixed quantum states, which is defined as[16]

$$F_{\mathrm{W}}(\rho,\sigma)=\frac{|\mathrm{Tr}(\rho\sigma)|}{\sqrt{\mu[\rho]\mu[\sigma]}}, \tag{4}$$

which has the following appealing properties:[16,25]

(i) **Bounds**: $0\le F_{\mathrm{W}}(\rho,\sigma)\le 1$. Furthermore $F_{\mathrm{W}}(\rho,\sigma)=1$ if and only if $\rho=\sigma$, while $F_{\mathrm{W}}(\rho,\sigma)=0$ if and only if $\mathrm{supp}(\rho)\perp\mathrm{supp}(\sigma)$.

(ii) **Symmetry**: $F_{\mathrm{W}}(\rho,\sigma)=F_{\mathrm{W}}(\sigma,\rho)$.

(iii) **Monotonicity under quantum operation** $\mathcal{G}$: $F_{\mathrm{W}}(\mathcal{G}(\rho),\mathcal{G}(\sigma))\ge F_{\mathrm{W}}(\sigma,\rho)$. When quantum operator is a unitary transformation U, we have $F_{\mathrm{W}}(U\rho U^{+},U\sigma U^{+})=F_{\mathrm{W}}(\sigma,\rho)$.

(iv) **Multiplicativity under tensor products**: $F_{\mathrm{W}}(\rho_1\otimes\rho_2,\sigma_1\otimes\sigma_2)=F_{\mathrm{W}}(\rho_1\otimes\sigma_1)F(\rho_2\otimes\sigma_2)$.

(v) **Separate concavity**: $F_{\mathrm{W}}(p_1\rho_1+p_2\rho_2,\sigma)\ge p_1F_{\mathrm{W}}(\rho_1,\sigma)+p_2F_{\mathrm{W}}(\rho_2,\sigma)$ with $p_1,p_2\ge 0$ and $p_1+p_2=1$.

As stated earlier, the fidelity itself is not a metric, but it can be used to define some metrics, such as Bures angle, Bures distance, and sine distance, which are listed in the following.

For Bures angle, it is defined by[17]

$$A(\rho,\sigma)=\arccos\sqrt{F(\rho,\sigma)}. \tag{5}$$

For Bures distance, it can be written as[18]

$$B(\rho,\sigma)=\sqrt{2-2\sqrt{F(\rho,\sigma)}}. \tag{6}$$

The sine distance takes the form[13]

$$C(\rho,\sigma)=\sqrt{1-F(\rho,\sigma)}. \tag{7}$$

As a metric $\delta(\rho,\sigma)$ defined on quantum states, it should satisfy the basic properties: (i) non-negativity: $\delta(\rho,\sigma)\ge 0$, and $\delta(\rho,\sigma)=0$ if and only if $\rho=\sigma$, (ii) symmetry under swapping of the two states: $\delta(\rho,\sigma)=\delta(\sigma,\rho)$, and triangle inequality: $\delta(\rho,\sigma)+\delta(\tau,\sigma)\ge\delta(\rho,\tau)$ for all states ρ, σ and τ. It has been shown in Ref. [25] that for the Uhlmann–Jozsa fidelity, these three metrics are genuine ones. For the super-fidelity, $C(\rho,\sigma)$ preserves the metric properties, while both $A(\rho,\sigma)$ and $B(\rho,\sigma)$ do not always satisfy the triangle inequality. Hence, one needs to study whether or not these measures obey the above-mentioned metric conditions with Wang's quantum fidelity $F_{\mathrm{W}}(\rho,\sigma)$ given by Eq. (4). It can be easily seen that the above-mentioned metrics share naturally the first two properties. Thus, we shall prove separately their triangle inequalities.

Theorem 1 Let ρ, σ, and τ be the density matrices of any three quantum states. The following triangle inequality for Bures angle holds:

$$A(\rho,\sigma)\le A(\rho,\tau)+A(\tau,\sigma), \tag{8}$$

with Wang's quantum fidelity $F_{\mathrm{W}}(\rho,\sigma)$.

Proof Let $|r\rangle$, $|s\rangle$, and $|t\rangle$ be any purifications of ρ, σ, and τ, respectively. According to Wotters's metric for pure states, the triangle inequality is given by[28]

$$\arccos(|\langle r|t\rangle|)+\arccos(|\langle s|t\rangle|)\ge\arccos(|\langle r|s\rangle|). \tag{9}$$

For the Uhlmann–Jozsa fidelity, its equivalent definition in terms of purifications of the states ρ and σ is[4]

$$F(\rho,\sigma)=(\mathrm{Tr}\sqrt{\rho\sigma})^2=\max_{|r\rangle,|s\rangle}|\langle r|s\rangle|^2, \tag{10}$$

where the maximization is taken over all purifications $|r\rangle$ of ρ and $|s\rangle$ of σ. Similarly, Wang's fidelity can be written in terms of purifications of the states ρ and σ as

$$F_{\mathrm{W}}(\rho,\sigma)=\max_{|r\rangle,|s\rangle}\frac{|\langle r|s\rangle|^2}{\sqrt{\mu[\rho]\mu[\sigma]}}=\max_{|s'\rangle}|\langle r'|s'\rangle|^2, \tag{11}$$

where $|r'\rangle = \frac{|r\rangle}{\sqrt[4]{\mu[\rho]}}$ and $|s'\rangle = \frac{|s\rangle}{\sqrt[4]{\mu[\sigma]}}$. The first maximization in Eq. (11) is performed over all purifications $|r\rangle$ of ρ and $|s\rangle$ of σ, while its second maximization is taken over all purification $|s'\rangle$ of σ for a fixed purification $|r'\rangle$ of ρ.

Hence, when choosing adequately the purifications corresponding to ρ and σ, the left-hand-side terms in Eq. (9) can be written in terms of the fidelity between the states ρ (or σ) and τ. To be more specific, for any given purifications of $|t\rangle$ and τ, we can seek for those purifications of ρ and σ that maximize the overlap $|\langle r'|t'\rangle|$ and $|\langle s'|t'\rangle|$, which are thus simplified to computing $F_{\rm W}(\rho,\tau)$ and $F_{\rm W}(\tau,\sigma)$, i.e., $F_{\rm W}(\rho,\tau) = \max\limits_{|r'\rangle} |\langle r'|t'\rangle|^2$ and $F_{\rm W}(\tau,\sigma) = \max\limits_{|s'\rangle} |\langle s'|t'\rangle|^2$. Furthermore, we have

$$\begin{aligned} &\arccos(\sqrt{F_{\rm W}(\rho,\tau)}) + \arccos(\sqrt{F_{\rm W}(\tau,\sigma)}) \\ \geq\ &\arccos(|\langle \tilde{r}'|\tilde{s}'\rangle|), \end{aligned} \tag{12}$$

where $|\tilde{r}'\rangle$ and $|\tilde{s}'\rangle$ indicate the purifications that maximize the overlaps.

Taking into account that

$$F_{\rm W}(\rho,\sigma) = \max_{|r'\rangle,|s'\rangle} |\langle r'|s'\rangle|^2 \geq |\langle \tilde{r}'|\tilde{s}'\rangle|^2, \tag{13}$$

together with the arccosine function decreasing nature, we obtain

$$\begin{aligned} &\arccos(\sqrt{F_{\rm W}(\rho,\tau)}) + \arccos(\sqrt{F_{\rm W}(\tau,\sigma)}) \\ \geq\ &\arccos(\sqrt{F_{\rm W}(\rho,\sigma)}). \end{aligned} \tag{14}$$

This concludes our proof.

Theorem 2 Let ρ, σ, and τ be the density matrices of any three quantum states. With Wang's quantum fidelity $F_{\rm W}(\rho,\sigma)$, the Bures distance and sine distance obey the following triangle inequality:

$$X(\rho,\sigma) \leq X(\rho,\tau) + X(\tau,\sigma), \tag{15}$$

where X refers to either B or C.

Proof To prove this inequality, we define $X(\rho,\sigma) := \alpha\sqrt{1-F_{\rm W}^n(\rho,\sigma)} = \alpha H(\rho,\sigma)$ with (α,n) being a nonnegative number. We can see $X(\rho,\sigma) = B(\rho,\sigma)$ for the case of $\alpha = \sqrt{2}$ and $n = 1/2$, while it reduces to $C(\rho,\sigma)$ for $\alpha = 1$ and $n = 1$. Thus, inequality (15) reduces to proving

$$H(\rho,\sigma) \leq H(\rho,\tau) + H(\tau,\sigma). \tag{16}$$

Since $0 \leq F_{\rm W}(\rho,\sigma) \leq 1$, we define $F_{\rm W}(\rho,\sigma) = \sqrt[n]{\cos^2(\theta_{\rho\sigma})}$ with $\theta_{\rho\sigma}$ being an angle in the range from 0 to $\pi/2$, Define $\theta_{\rho\tau}$ and $\theta_{\tau\sigma}$ similarly. With these definitions we have $H(\rho,\sigma) = \sin(\theta_{\rho\sigma})$, $H(\rho,\tau) = \sin(\theta_{\rho\tau})$, and $H(\tau,\sigma) = \sin(\theta_{\tau\sigma})$.

According to the elementary geometry, the spherical triangle inequality of angle between two quantum states holds[29]

$$\theta_{\rho\sigma} \leq \theta_{\rho\tau} + \theta_{\tau\sigma}. \tag{17}$$

To prove inequality (15), we consider two cases:

$$0 \leq \theta_{\rho\tau} + \theta_{\tau\sigma} \leq \frac{\pi}{2}, \tag{18a}$$

$$\frac{\pi}{2} \leq \theta_{\rho\tau} + \theta_{\tau\sigma} \leq \pi. \tag{18b}$$

In the case of Eq. (18a) and using the fact that $\sin(x)$ is a non-decreasing function in the range $x \in [0, \frac{\pi}{2}]$, we obtain

$$\begin{aligned} H(\rho,\sigma) &= \sin(\theta_{\rho\sigma}) \leq \sin(\theta_{\rho\tau} + \theta_{\tau\sigma}) \\ &= \sin(\theta_{\rho\tau})\cos(\theta_{\tau\sigma}) + \cos(\theta_{\rho\tau})\sin(\theta_{\tau\sigma}) \\ &\leq \sin(\theta_{\rho\tau}) + \sin(\theta_{\tau\sigma}) = H(\rho,\tau) + H(\tau,\sigma). \end{aligned} \tag{19}$$

In the case of Eq. (18b) and thanks to $\theta_{\rho\tau}$ and $\theta_{\tau\sigma} \in [0, \frac{\pi}{2}]$, we have

$$\begin{aligned} H(\rho,\tau) + H(\tau,\sigma) &= \sin(\theta_{\rho\tau}) + \sin(\theta_{\tau\sigma}) \\ &\geq \sin^2(\theta_{\rho\tau}) + \sin^2(\theta_{\tau\sigma}) \\ &= 1 - \cos(\theta_{\rho\tau} - \theta_{\tau\sigma})\cos(\theta_{\rho\tau} + \theta_{\tau\sigma}) \\ &\geq 1 \geq H(\rho,\sigma). \end{aligned} \tag{20}$$

Here, we have used $\cos(\theta_{\rho\tau} + \theta_{\tau\sigma}) \leq 0$ for $\frac{\pi}{2} \leq \theta_{\rho\tau} + \theta_{\tau\sigma} \leq \pi$.

We show that the above three metrics can be regarded as non-classicality measure when one of two quantum states is chosen as statistical mixture of coherent states,[19] entanglement measure when choosing a separable state,[20] and geometric quantum discord when choosing a zero-discord state.[21] Similarly, if one state (i.e., σ) is appointed as a reference Gaussian state, then we can employ these fidelity-based measures to assess how much another non-Gaussianity state contains. We note that recently, Ghiu *et al.* defined a fidelity-based degree of non-Gaussianity, $\delta_F = 1 - \sqrt{F(\rho,\sigma)}$.[30] Obviously, it can be seen as a modified version of the square of $B(\rho,\sigma)$. On the other hand, in calculating the non-Gaussianity of quantum states, one encounters an intractable problem of how to choose a reference Gaussian state. As was stated in Ref. [9], the reference Gaussian state can be designed to be the most general Gaussian thermal one with the same first moments and the same covariance matrix as a given state ρ. Yet, such a treatment is not much easily applicable to an n-mode case since the corresponding reference Gaussian state is required to optimize the $n(2n+3)$ independent parameters,[10,31] which significantly increases the computation works. To overcome the obstacle, we provide a simple and direct scheme for constructing the reference Gaussian state in the phase space.

Let $\chi(\rho,\eta)$ and $\chi_{rG}(\rho_{rG},\eta)$ be the characteristic functions of the considered state ρ and its corresponding reference Gaussian state ρ_{rG}, respectively. Their difference is denoted by $f(\eta) = \chi(\rho,\eta) - \chi_{rG}(\rho_{rG},\eta)$. In order to obtain the same first moments and covariance matrix as ρ, we set $\frac{\partial^m}{\partial\eta^m} f(\eta)|_{\eta=0} = 0$ with $m \leq 2$, yielding

$$\boldsymbol{d}_{\rho_{rG}} = \boldsymbol{d}_{\rho}, \tag{21a}$$

$$\langle \hat{a}_l^{+n} \hat{a}_j^m \rangle_{\rho_{rG}} = \langle \hat{a}_l^{+n} \hat{a}_j^m \rangle_{\rho}, \tag{21b}$$

where the mean value is given by $\boldsymbol{d}_\rho \equiv \langle \hat{a}_1^+, \hat{a}_1, \ldots, \hat{a}_n^+, \hat{a}_n \rangle_\rho$. Moreover, we can construct the characteristic function of reference Gaussian state in the form

$$\chi(\boldsymbol{\eta}, \rho_{rG}) = \exp\left[-\frac{1}{2}\boldsymbol{\eta}\boldsymbol{V}_{\rho_{rG}}\boldsymbol{\eta}^{\mathrm{T}} + \mathrm{i}\sqrt{2}\boldsymbol{d}_{\rho_{rG}}\boldsymbol{\Omega}\boldsymbol{\eta}^{\mathrm{T}}\right], \tag{22}$$

where the symplectic matrix $\boldsymbol{\Omega} = \mathrm{i}\bigotimes_{l=1}^{n}\sigma_2$, σ_2 being the y-Pauli matrix and the covariance matrix is given by

$$\boldsymbol{V}_{\rho_{rG}} = \begin{pmatrix} \langle \hat{a}_1^+\hat{a}_1 + \frac{1}{2}\rangle & -\langle \hat{a}_1^2\rangle & \cdots & \langle \hat{a}_1\hat{a}_n^+\rangle & -\langle \hat{a}_1\hat{a}_n\rangle \\ -\langle \hat{a}_1^{+2}\rangle & \langle \hat{a}_1^+\hat{a}_1 + \frac{1}{2}\rangle & \cdots & -\langle \hat{a}_1^+\hat{a}_n^+\rangle & \langle \hat{a}_1^+\hat{a}_n\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle \hat{a}_1^+\hat{a}_n\rangle & -\langle \hat{a}_1\hat{a}_n\rangle & \cdots & \langle \hat{a}_n^+\hat{a}_n + \frac{1}{2}\rangle & -\langle \hat{a}_n^2\rangle \\ -\langle \hat{a}_1^+\hat{a}_n^+\rangle & \langle \hat{a}_1\hat{a}_n^+\rangle & \cdots & \langle \hat{a}_n^{+2}\rangle & \langle \hat{a}_n^+\hat{a}_n + \frac{1}{2}\rangle \end{pmatrix}. \tag{23}$$

Therefore, we can obtain the associated Wigner function as $W(\boldsymbol{\alpha}, \rho_{rG}) = \frac{1}{\pi^n}\int \exp(\boldsymbol{\eta}^*\boldsymbol{\alpha} - \boldsymbol{\eta}\boldsymbol{\alpha}^*)\chi(\boldsymbol{\eta}, \rho_{rG})\mathrm{d}^{2n}\boldsymbol{\eta}$.

In what follows, we study the basic properties of all the above metrics as non-Gaussianity measure δ_{NG}, which are summarized by the following lemmas.

Lemma 1 If ρ is a Gaussian state, then $\delta_{\mathrm{NG}}(\rho) = 0$; otherwise, $0 < A(\rho) \leq \frac{\pi}{2}$, $0 < B(\rho) \leq \sqrt{2}$, and $0 < C(\rho) \leq 1$.

Proof If $\delta_{\mathrm{NG}}(\rho) = 0$, then $F_{\mathrm{W}}(\rho, \rho_{rG}) = 1$ and thus $\rho = \rho_{rG}$, showing that it is a Gaussian state. If ρ is a Gaussian state, then it is solely determined by its first moment and covariance matrix and thus the reference Gaussian state ρ_{rG} is given by $\rho_{rG} = \rho$, which, in turn, leads to $F_{\mathrm{W}}(\rho, \rho_{rG}) = 1$, and then $\delta_{\mathrm{NG}}(\rho) = 0$.

On the other hand, according to the Cauchy–Schwarz inequality and for a non-Gaussian state, we easily obtain

$$\mathrm{Tr}[\rho_{rG}\rho] < \sqrt{\mu[\rho]\mu[\rho_{rG}]}, \tag{24}$$

so we have

$$0 \leq F_{\mathrm{W}}(\rho_{rG}) < 1. \tag{25}$$

A straightforward computation shows that $0 < A(\rho) \leq \frac{\pi}{2}$, $0 < B(\rho) \leq \sqrt{2}$, and $0 < C(\rho) \leq 1$.

Lemma 2 If $\hat{U}$ is a unitary operator corresponding to a symplectic transformation in the phase space, then $\delta_{\mathrm{NG}}(\hat{U}\rho\hat{U}^+) = \delta_{\mathrm{NG}}(\rho)$.

Proof Under unitary transformation $\hat{U}$, a quantum state ρ maps as $\rho' = \hat{U}\rho\hat{U}^+$ and its reference Gaussian state ρ_{rG} as $\rho'_{rG} = \hat{U}\rho_{rG}\hat{U}^+$. Therefore, we can obtain the purities of the two quantum states ρ' and ρ'_{rG} as

$$\mu[\rho'] = \mathrm{Tr}[\rho'^2] = \mathrm{Tr}[\hat{U}\rho^2\hat{U}^+] = \mu[\rho], \tag{26a}$$

$$\mu[\rho'_{rG}] = \mathrm{Tr}[\rho'^2_{rG}] = \mathrm{Tr}[\hat{U}\rho^2_{rG}\hat{U}^+] = \mu[\rho_{rG}], \tag{26b}$$

and their overlap as

$$\mathrm{Tr}[\rho'\rho'_{rG}] = \mathrm{Tr}[\hat{U}\rho\hat{U}^+\hat{U}\rho_{rG}\hat{U}^+] = \mathrm{Tr}[\rho\rho_{rG}], \tag{27}$$

where we have used the the cyclic property of matrix. Thus $F_{\mathrm{W}}(\rho, \rho_{rG})$ is invariant under unitary transformation, accordingly we can obtain the identity $\delta_{\mathrm{NG}}(\hat{U}\rho\hat{U}^+) = \delta_{\mathrm{NG}}(\rho)$.

Note that for an arbitrary quantum operation described by a completely positive trace-preserving (CPTP) map $\varepsilon : \rho \mapsto \varepsilon(\rho)$, we have $F(\varepsilon(\rho), \varepsilon(\sigma)) \geq F(\rho, \sigma)$.[17] Hence, our introduced non-Gaussianities monotonically decrease under Gaussian quantum channels, that is, $\delta_{\mathrm{NG}}(\varepsilon_{\mathrm{G}}\rho) \leq \delta_{\mathrm{NG}}(\rho)$.

Lemma 3 Consider a bipartite factorized state $\rho = \rho_1 \otimes \rho_2$. If ρ_2 is a Gaussian state, then $\delta_{\mathrm{NG}}(\rho_1 \otimes \rho_2) = \delta_{\mathrm{NG}}(\rho_1)$.

Proof For a bipartite factorized state $\rho_{12} = \rho_1 \otimes \rho_2$, its reference Gaussian state can be written as $\rho_{12,rG} = \rho_{1,rG} \otimes \rho_{2,rG}$. According to the definition of the purity of a quantum state, we have

$$\mu[\rho_{12}] = \mu[\rho_1]\mu[\rho_2], \tag{28a}$$

$$\mu[\rho_{12,rG}] = \mu[\rho_{1,rG}]\mu[\rho_{2,rG}], \tag{28b}$$

$$\begin{aligned} \mathrm{Tr}[\rho_{12}, \rho_{12,rG}] &= \mathrm{Tr}[(\rho_1 \otimes \rho_2)(\rho_{1,rG} \otimes \rho_{2,rG})] \\ &= \mathrm{Tr}[(\rho_1\rho_{1,rG}) \otimes (\rho_2\rho_{2,rG})] \\ &= \mathrm{Tr}[(\rho_1\rho_{1,rG})]\mathrm{Tr}[(\rho_2\rho_{2,rG})], \end{aligned} \tag{28c}$$

where we have used $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = (\boldsymbol{AC} \otimes \boldsymbol{BD})$ and $\mathrm{Tr}[\boldsymbol{A} \otimes \boldsymbol{B}] = \mathrm{Tr}[\boldsymbol{A}]\mathrm{Tr}[\boldsymbol{B}]$. Since ρ_2 is a Gaussian state, we have $\mathrm{Tr}[(\rho_2\rho_{2,rG})] = \mu[\rho_{2,rG}] = \mu[\rho_2]$.

Explicitly,

$$\begin{aligned} F_{\mathrm{W}}(\rho_{12}, \rho_{12,rG}) &= \frac{|\mathrm{Tr}[(\rho_1 \otimes \rho_2)(\rho_{1,rG} \otimes \rho_{2rG})]|}{\sqrt{\mu[(\rho_1 \otimes \rho_2)]\mu[(\rho_{1,rG} \otimes \rho_{2,rG})]}} \\ &= \frac{|\mathrm{Tr}[\rho_1\rho_{1,rG}]|}{\sqrt{\mu[\rho_1]\mu[\rho_{1,rG}]}} = F_{\mathrm{W}}(\rho_1, \rho_{1,rG}). \end{aligned} \tag{29}$$

Finally, we obtain

$$\delta_{\mathrm{NG}}(\rho_1 \otimes \rho_2) = \delta_{\mathrm{NG}}(\rho_1), \tag{30}$$

which concludes the proof.

Lemma 4 Our non-Gaussianities satisfy the symmetry under the swapping of arbitrary two quantum states ρ and σ: $\delta_{\mathrm{NG}}(\rho, \sigma) = \delta_{\mathrm{NG}}(\sigma, \rho)$.

Proof As a matter of fact, the proof follows easily from the definition of $F_{\mathrm{W}}(\sigma, \rho)$, which also has the exchange symmetry.

3. Examples

In this section, by means of the quantum fidelity given by Eq. (4), we use the mentioned-above metrics to evaluate the non-Gaussianity of some relevant single-mode and two-mode non-Gaussian states, e.g., single-mode mixture of Fock states, Schrödinger-cat-like states (SCLSs), and two-mode Bell-like states, by which we investigate the consistency between these non-Gaussianity measures and discuss their success.

3.1. Single-mode mixture of Fock states

We begin by considering a special family of single-mode non-Gaussian mixed states: statistical mixtures of the vacuum state and n-photon Fock state defined by

$$\rho = (1-p)|0\rangle\langle 0| + p|n\rangle\langle n|, \tag{31}$$

where the parameters $p \in [0,1]$. When $np = 0$, the state (31) reduces to vacuum state and then the non-Gaussianity naturally vanishes, regardless of choosing which non-Gaussianity measure to quantify it.

It is easily seen that all the mean values of this state vanish, but its covariance matrix is given by

$$\boldsymbol{V} = \begin{pmatrix} np+\frac{1}{2} & 0 \\ 0 & np+\frac{1}{2} \end{pmatrix}. \tag{32}$$

According to our proposed method for choosing reference Gaussian state, we obtain the purity of reference Gaussian state as

$$\mu[\rho_{rG}] = \frac{1}{1+2np}, \tag{33}$$

whereas the overlap between the state (31) and its reference Gaussian state is given by

$$\mathrm{Tr}(\rho\rho_{rG}) = \frac{1}{1+np}\left[1-p+p\left(\frac{np}{1+np}\right)^n\right]. \tag{34}$$

Furthermore, we can obtain the fidelity F_{W} as

$$F_{\mathrm{W}} = \sqrt{\frac{1+2np}{(1+np)^2(1-2p+2p^2)}} \times \left[1-p+p\left(\frac{np}{1+np}\right)^n\right]. \tag{35}$$

Substituting F_{W} into Eqs. (5)–(7), we finally evaluate the non-Gaussianity of single-mode mixture of Fock states. The results are reported in Fig. 1 as a function of p and for different values of photon number n. From the plots, all of the above three non-Gaussianity metrics δ_{NG} are naturally null for $np = 0$, corresponding to vacuum state, as is expected. One can also see that the three non-Gaussianity measures exhibit a monotonically increasing behavior with the number of photons n and the parameters p. For a case of $np \ll 1$, they present the same behavior, while for $np > 1$, they exhibit different behaviors. Particularly, we can obtain their limiting values as $A = \frac{\pi}{2}$, $B = \sqrt{2}$, and $C = 1$ for $np \gg 1$, as shown in Fig. 1(b). It is worth noting that the measure $g = \frac{\mathrm{Tr}(\rho\rho_{\mathrm{G}})}{\mu[\rho_{\mathrm{G}}]}$ recently introduced in Refs. [32–34] is not an ideal measure for quantifying the Gaussianity of quantum state because it is equal to unity not only for Gaussian states but also for some non-Gaussian states. For example, for mixtures of the vacuum state and n-photon Fock state with $n = 2$ and $p = \frac{1}{2\sqrt{2}}$, we have $g = 1$[33] and then its non-Gaussianity is given by $\delta_{\mathrm{NG}} = 1 - g = 0$. Fortunately our measures overcome this drawback.

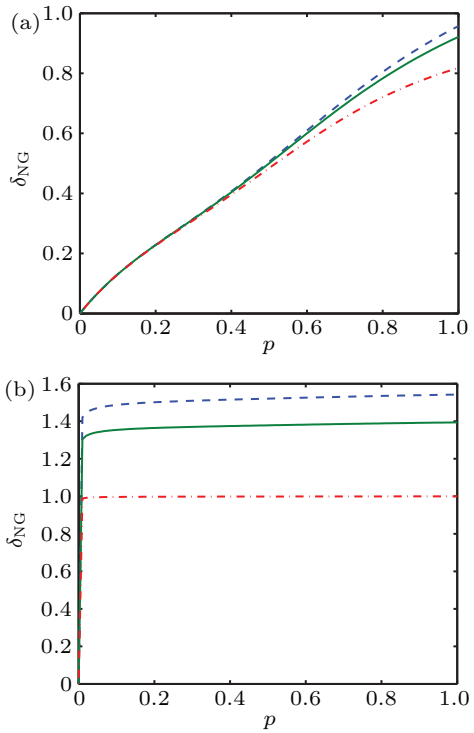


Fig. 1. Non-Gaussianity δ_{NG} of mixtures of the vacuum and n-photon Fock states as a function of n and p: (a) $n = 2$, (b) $n = 400000$. The dashed line corresponds to $A(\rho,\sigma)$, the solid line to $B(\rho,\sigma)$, and the dash-dotted line to $C(\rho,\sigma)$.

3.2. Single-mode photon-added thermal states

The photon addition and subtraction are an important technique for generating quantum non-Gaussian state. Here we consider another family of single-mode non-Gaussian mixed states: m-photon-added thermal states (mPATSs), which are defined as[11,30]

$$\rho_{\mathrm{ad}} \propto \hat{a}^{+m}\rho_{\mathrm{th}}\hat{a}^m, \tag{36}$$

where m is the number of added photons and ρ_{th} is the density operator of thermal state with mean number of photon $\bar{n}$, which can be expressed, in the Fock state basis, as

$$\rho_{\mathrm{th}} = (1-x)\sum_{n=0}^{\infty} x^n|n\rangle\langle n|, \tag{37}$$

where the thermal parameter $x = \frac{\bar{n}}{1+\bar{n}}$; $0 \le x \le 1$. Thus, we see that mPATSs are parameterized by two parameters: x and m.

The limit $x \to 0$ corresponds to Fock state $|m\rangle$, while the limit $m \to 0$ corresponds to the thermal state.

It is easily calculated that the states (36) have zero-mean values but possess the covariance matrix given by

$$\boldsymbol{V} = \begin{pmatrix} \bar{N}+\frac{1}{2} & 0 \\ 0 & \bar{N}+\frac{1}{2} \end{pmatrix}, \tag{38}$$

where $\bar{N} = \bar{n}(m+1)+m$.

Repeating the above procedure, we can calculate the fidelity $F_{\rm W}$ as

$$F_{\rm W} = \frac{\bar{N}^m}{(1+\bar{n}+\bar{N})^{m+1}} \sqrt{\frac{(1+2\bar{N})(1+2\bar{n})^{m+1}}{(-1)^m P_m(z)}}, \tag{39}$$

where $z = -\frac{2\bar{n}^2+2\bar{n}+1}{1+2\bar{n}}$, and $P_m(z)$ is the Legendre polynomial of m order.

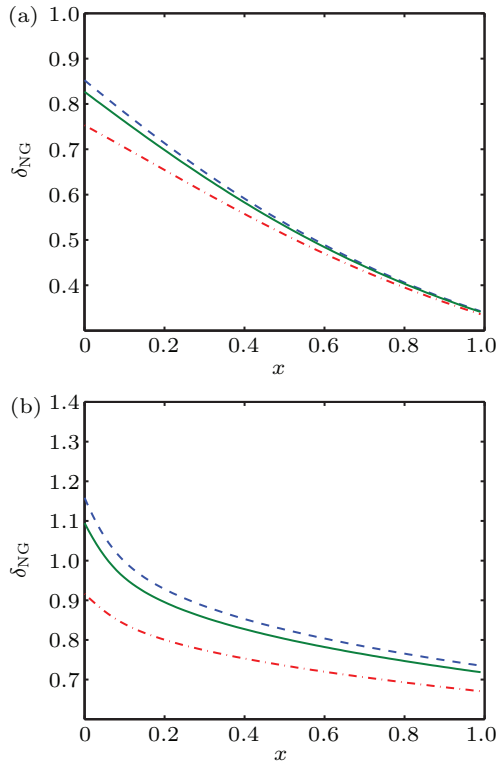


Fig. 2. Non-Gaussianity $\delta_{\rm NG}$ of mPATSs as a function of x and for different values of the added photons m: (a) $m = 1$, and (b) $m = 10$. Curve types are the same as in Fig. 1.

By means of Eqs. (5)–(7), we shall evaluate the non-Gaussianity of mPATSs. In Fig. 2, we plot the non-Gaussianity of mPATSs as a function of the thermal parameter x for different numbers of added photons m, where the meaning of different curve types is the same as in Fig. 1. From Fig. 2, in the case of $m \neq 0$, the three measures of non-Gaussianity $A(\rho,\sigma)$, $B(\rho,\sigma)$, and $C(\rho,\sigma)$ decrease with the increase of the parameter x. This is in agreement with the results obtained by other non-Gaussianity measures such as the Hilbert–Schmidt measure and quantum relative entropy measure.[30] However, it is completely different from the result reported in Ref. [11], where the non-Gaussianity of such states was quantified by the Wehrl-entropy measure and was found to be the same as that of the Fock state $|m\rangle\langle m|$, i.e., $\delta_{\rm NG}(\rho_{ad}) = \delta_{\rm NG}(\rho_{|m\rangle\langle m|})$, independent of the thermal mean occupancy $\bar{n}$. That is because their non-Gaussianity measure was based on the quasi-probability $Q(\alpha)$, $\alpha \in C$ and the Q function of mPATSs was a scaled version of the Q function of the Fock state: $Q^{(m,x)}_{\rm PATS}(\alpha) = \lambda^2 Q_{|m\rangle}(\lambda\alpha)$, $\lambda = \sqrt{1-x}$. In addition, we observe carefully from Fig. 2 that there exists a consistent relation between the three non-Gaussianity measures involved, thus revealing the same ordering of non-Gaussianity for mPATSs.

3.3. Schrödinger-cat-like states

Let us consider single-mode Schrödinger-cat-like states, which are represented by a coherent superposition of two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$, as follows:[19]

$$|\Psi\rangle_{\rm scs} = \frac{1}{N}\left[\cos(\phi)|\alpha\rangle + \sin(\phi)|-\alpha\rangle\right], \tag{40}$$

where $|\alpha\rangle$ denotes a coherent state with amplitude $\alpha \in \mathbb{R}$, $N = \sqrt{1+\mathrm{e}^{-2|\alpha|^2}\sin(2\phi)}$ is a normalization factor, and ϕ is the relative phase. When $\phi = \pm\frac{\pi}{4}$, the states (40) reduce to the so-called even and odd Schrödinger cat states. For convenience later, we shall refer to Schrödinger-cat-like states with $\tan(\phi) > 0$ as even Schrödinger-cat-like ones and Schrödinger-cat-like states with $\tan(\phi) < 0$ as odd Schrödinger-cat-like ones, respectively.

The mean vector of such states is given by

$$\bar{d} = \frac{2\alpha^2}{N^2}\begin{pmatrix} \cos(2\phi) \\ \cos(2\phi) \end{pmatrix}, \tag{41}$$

and the covariance matrix is calculated to be

$$\boldsymbol{V} = \frac{\alpha^2}{N^2}\begin{pmatrix} \left[1-\sin(2\phi)\,\mathrm{e}^{-2\alpha^2}\right]+\frac{N^2}{2\alpha^2} & 1+\sin(2\phi)\,\mathrm{e}^{-2\alpha^2} \\ 1+\sin(2\phi)\,\mathrm{e}^{-2\alpha^2} & \left[1-\sin(2\phi)\,\mathrm{e}^{-2\alpha^2}\right]+\frac{N^2}{2\alpha^2} \end{pmatrix}. \tag{42}$$

After some tedious but straightforward calculations, one can obtain the fidelity $F_{\rm W}$ as follows:

$$\begin{aligned} F_{\rm W} = {} & \frac{(G_1G_2)^{\frac{1}{4}}}{N^3}\left\{\cos^2(\phi)\exp\left\{-\frac{\alpha^2[N^2-\cos(2\phi)]}{2N^4(A-2B)}\right\}+\sin^2(\phi)\exp\left\{-\frac{\alpha^2[N^2+\cos(2\phi)]}{2N^4(A-2B)}\right\}\right. \\ & \left.+\frac{1}{2}\mathrm{e}^{-2\alpha^2}\sin(2\phi)\exp\left\{-\frac{\alpha^2[N^4-\cos^2(2\phi)]A-2\alpha^2[N^4+\cos^2(2\phi)]B}{2N^4(A^2-4B^2)}\right\}\right\}, \end{aligned} \tag{43}$$

with

$$G_1 = 2\alpha^2\cos^2(2\phi) - 2\alpha^2 N^2 - N^4, \tag{44a}$$

$$G_2 = 2\alpha^2 \mathrm{e}^{-2\alpha^2}\sin(2\phi) - N^2, \tag{44b}$$

$$A = 1 + \frac{\alpha^2[1-\sin(2\phi)\mathrm{e}^{-2\alpha^2}]}{N^2} - \frac{\alpha^2\cos^2(2\phi)}{N^4}, \tag{44c}$$

$$B = \frac{\alpha^2[1+\sin(2\phi)]\mathrm{e}^{-2\alpha^2}}{2N^2} - \frac{\alpha^2\cos^2(2\phi)}{2N^4}. \tag{44d}$$

The non-Gaussianities of Schrödinger cat-like states are evaluated by means of Eqs. (5)–(7) and are reported in Fig. 3 as a function of ϕ for different values of α. It can be clearly seen that when $\phi = \pm n\pi$ or $\pm(n+\frac{1}{2})\pi$, $n = 0,1,2,\ldots$, then the non-Gaussianities $\delta_{\mathrm{NG}} = 0$ and Schrödinger cat-like states just become a coherent state, belonging to Gaussian state. We also observe that upon varying the value of the parameters ϕ and α, all such measures can exhibit similar non-Gaussian qualitative behavior. For example, the three measures become even function of ϕ for higher values of the amplitude. The non-Gaussianity of even Schrödinger cat-like states is the same as that of odd Schrödinger cat-like ones. This is in good agreement with the result of Ref. [35]. On the other hand, it is interesting to see that for low values of the amplitude (e.g., $\alpha = 0.2$) and $n\pi \le \phi \le (n+\frac{1}{2})\pi$, $n = 0,\pm1,\pm2,\ldots$, these fidelity-based non-Gaussianities are not equal to zero and non-Gaussian characteristics of such states are strongly pronounced (see Figs. 3(a)–3(b)). This is not the same as reported in Ref. [35] where the non-Gaussianity was almost zero. This can be explained by the fact that even for the small amplitude, even Schrödinger-cat-like ones still have tiny non-Gaussian character, deviating from Gaussian counterparts. Of course, the smaller the amplitude is, the weaker the non-Gaussian character of even Schrödinger-cat-like states becomes, and eventually, it tends to fade away when $\alpha = 0$. Therefore, our example indicates again that the fidelity-based measures can effectively capture weak intrinsic non-Gaussian behavior of quantum states.

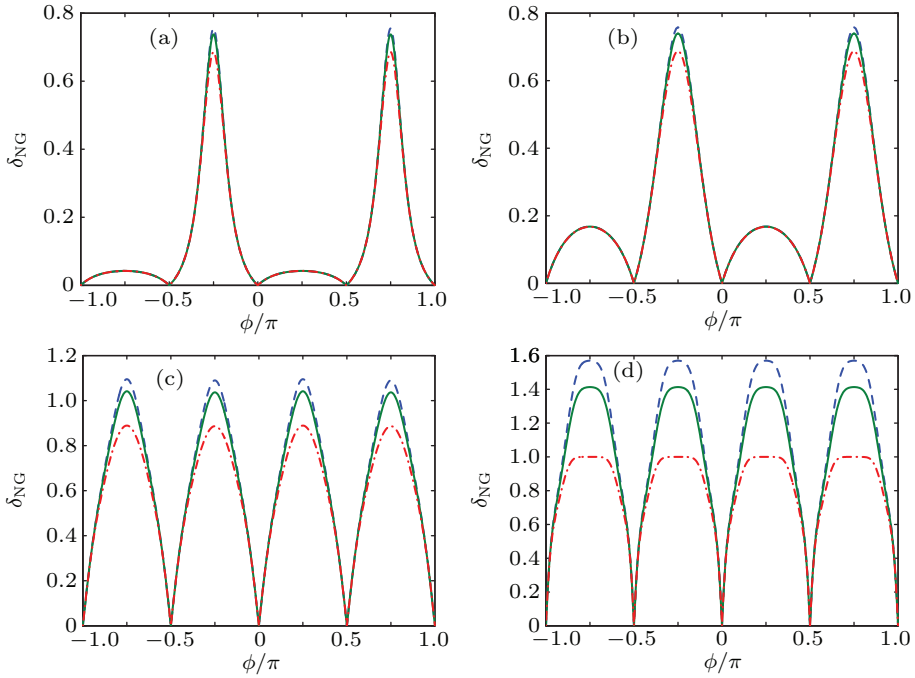


Fig. 3. Non-Gaussianity for single-mode SCLSs as a function of the parameter ϕ and for different values of the amplitude α: (a)–(d) $\alpha = \{0.2, 0.4, 1.5, 5\}$. Curve types are the same as in Fig. 1.

3.4. Two-mode Bell-like states

As our final example, we focus on two-mode Bell-like states, which are written as

$$|\Phi\rangle_{AB} = \cos(\theta)|0,0\rangle_{AB} + \sin(\theta)|1,1\rangle_{AB}, \tag{45a}$$

$$|\Psi\rangle_{AB} = \cos(\theta)|0,1\rangle_{AB} + \sin(\theta)|1,0\rangle_{AB}, \tag{45b}$$

where $|n,m\rangle_{AB}$ denotes the n photons in mode A and the m photons in mode B. When $\theta = \pm\frac{\pi}{4}$, they reduce to four Bell states

$$|\Phi\rangle^{\pm}_{AB} = \frac{1}{\sqrt{2}}(|0,0\rangle_{AB} \pm |1,1\rangle_{AB}), \tag{46a}$$

$$|\Psi\rangle^{\pm}_{AB} = \frac{1}{\sqrt{2}}(|0,1\rangle_{AB} \pm |1,0\rangle_{AB}). \tag{46b}$$

We can obtain that all the first moments of two-mode Bell-like states are zero but their covariance matrices are different from each other. For the states (45a), the covariance matrix is given by

$$\boldsymbol{V}_{|\Phi\rangle_{AB}} = \begin{pmatrix} \left[1-\frac{1}{2}\cos(2\theta)\right]\boldsymbol{I}_2 & \frac{\boldsymbol{\sigma}_3}{2}\sin(2\theta) \\ \frac{\boldsymbol{\sigma}_3}{2}\sin(2\theta) & \left[1-\frac{1}{2}\cos(2\theta)\right]\boldsymbol{I}_2 \end{pmatrix}, \tag{47}$$

whereas for the states (45b), the covariance matrix is given by

$$\boldsymbol{V}_{|\Psi\rangle_{AB}} = \begin{pmatrix} \left[1-\frac{1}{2}\cos(2\theta)\right]\boldsymbol{I}_2 & \frac{1}{2}\sin(2\theta)\boldsymbol{I}_2 \\ \frac{1}{2}\sin(2\theta)\boldsymbol{I}_2 & \left[1-\frac{1}{2}\cos(2\theta)\right]\boldsymbol{I}_2 \end{pmatrix}, \tag{48}$$

where $\boldsymbol{I}_2$ is a 2×2 unity matrix and $\boldsymbol{\sigma}_3 = \mathrm{Diag}(1,-1)$.

We proceed to calculate the fidelity $F_{\rm W}$ according to the calculation strategy as pervious subsections, obtaining the result

$$F_{{\rm W}-|\Phi\rangle_{AB}} = \frac{\left\{2[\cos^2(2\theta)+2]^2 - 11\cos(2\theta) - \cos(6\theta) + 2\right\}[8\sin^4(\theta)+1]^{\frac{1}{2}}}{[\cos^2(2\theta) - 3\cos(2\theta) + 4]^3}, \tag{49a}$$

$$F_{{\rm W}-|\Psi\rangle_{AB}} = \frac{\sqrt{3}}{4}\cos^2(2\theta). \tag{49b}$$

Furthermore, the non-Gaussianities of such two-mode Bell-like states are evaluated by means of Eqs. (5)–(7) and are reported in Fig. 4 as a function of the parameter θ. As it can be seen from Fig. 4 that the three fidelity-based measures have the same qualitative behavior and are even function of θ, but having different periods. For the state $|\Phi\rangle_{AB}$, its period is π, while it is given by $\frac{\pi}{2}$ for the state $|\Psi\rangle_{AB}$. In addition, we observe that the maxima of non-Gaussianity for different measures and for different Bell-like states are different. For example, the maxima of non-Gaussianity of the states $|\Phi\rangle_{AB}$ are, respectively, given by $A = 1.1230$, $B = 1.0649$, and $C = 0.9014$ for $\theta = \pm\frac{\pi}{2}$, while for the state $|\Psi\rangle_{AB}$, they are $A = \frac{\pi}{2}$, $B = \sqrt{2}$, and $C = 1$ for $\theta = \pm\frac{(2n+1)\pi}{4}$, $n = 0, 1$. On the other hand, two-mode Bell-like states are entangled ones and their concurrence is calculate to be $C_{\rm en} = |\sin(2\theta)|$.[36] One would like to establish a connection between non-separability and non-Gaussianity of quantum states. Unfortunately, in our work, the non-Gaussianity does not provide an entanglement detection of two-mode non-Gaussian states.

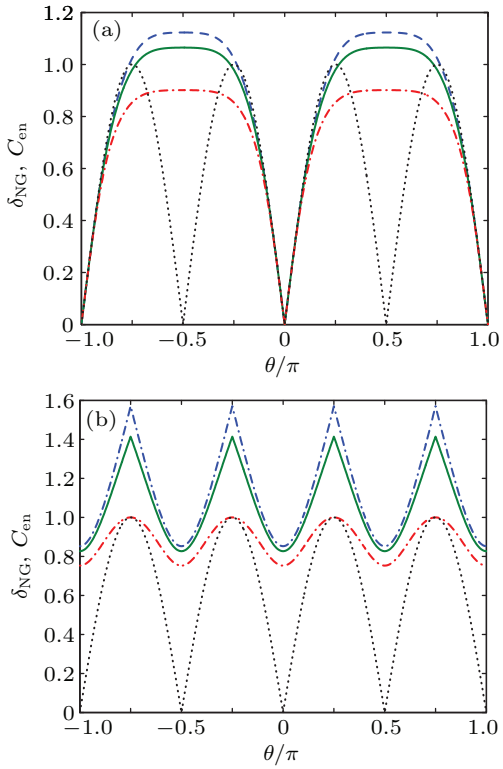


Fig. 4. Non-Gaussianity $\delta_{\rm NG}$ and concurrence $C_{\rm en}$ of two-mode Bell-like states (a) $|\Phi\rangle_{AB}$ and (b) $|\Psi\rangle_{AB}$ as a function of θ for $\theta \in [-\pi, \pi]$. The dotted line stands for $C_{\rm en}$, and other curve types are the same as in Fig. 1.

4. Comparison of non-Gaussianity measures

Before ending our work, we compare our fidelity-based non-Gaussianity measures with several known measures which have been introduced in the last ten years. Based on the Hilbert–Schmidt distance, Genoni *et al.* first proposed the following measure of non-Gaussianity of a state ρ:[9]

$$\delta_A(\rho) = \frac{\mu[\rho] + \mu[\rho_{rG}] - 2\kappa[\rho, \rho_{rG}]}{2\mu[\rho]}. \tag{50}$$

Soon afterwards they defined another measure of non-Gaussianity in terms of quantum relative entropy, which is given by[10]

$$\delta_{\rm B}(\rho) = S(\rho_{rG}) - S(\rho), \tag{51}$$

where $S(\rho)$ is the von Neumann entropy of a quantum state ρ.

Ivan *et al.* proposed a non-Gaussianity of quantum state by virtue of the Wehrl entropy of the Husimi-probability $Q(\alpha) = \langle\alpha|\rho|\alpha\rangle$, which is given by[11]

$$N_{\rm W}(\rho) = H_{\rm W}(\rho_{rG}) - H_{\rm W}(\rho), \tag{52}$$

where $H_{\rm W}(\rho)$ is the Wehrl entropy of a quantum state ρ.

Very recently, Fu *et al.* introduced a non-Gaussianity measure of quantum state via an uncertainty relation[12]

$$N_{\rm G}(\rho) = \min_{\theta\in[0,\pi)} \sqrt{U(\rho, Z_\theta)U(\rho, Z_{\theta+\frac{\pi}{2}})} - \frac{1}{2}, \tag{53}$$

where $Z_\theta = \frac{1}{\sqrt{2}}(a\,{\rm e}^{-i\theta}a^+\,{\rm e}^{i\theta})$ is the rotated quadrature operator and $U(\rho, Z_\theta)$ is a quantity that quantifies quantum uncertainty of the observable X in the state ρ:

$$U(\rho, Z_\theta) = \sqrt{V^2(\rho, X) - C^2(\rho, X)}, \tag{54}$$

where $V(\rho, X) = {\rm Tr}(\rho X^2) - ({\rm Tr}X)^2$ denotes the total uncertainty and $C(\rho, X) = V(\rho, X) - I(\rho, X)$ is the classical uncertainty with $I(\rho, X) = -\frac{1}{2}{\rm Tr}[\sqrt{\rho}, X]^2$ being the Wigner–Yanase skew information.

For comparison, we estimate the non-Gaussianity of Fock states using several significant non-Gaussianity measures available in the literature, which are summarized in Table 1, where $\psi(n) = \sum_{k=1}^{n-1}\frac{1}{k} - \gamma$ is the Digamma function with γ being the Euler constant and the fidelity $F_{\rm W} = \sqrt{\frac{1+2n}{(1+n)^2}}\left(\frac{n}{1+n}\right)^n$. We can see from Table 1 that such measures induce the same ordering of non-Gaussianity for Fock states. That is, the inequality $\delta_{\rm NG}[\rho_1] < \delta_{\rm NG}[\rho_2]$ would be obeyed for a pair of states ρ_1

and ρ_2, regardless of which measure we choose. This is because Fock state is a one-parameter categories of states and its non-Gaussianity has a monotonic behavior with respect to the photon numbers n. However, for a case of multi-parameter quantum states, the conclusion may not be correct. On the other hand, it is apparent from Table 1 that different measures have different limiting values of non-Gaussianity, in particular, the measures $\delta_{\rm B}$, $N_{\rm W}(\rho)$, and $N_{\rm G}(\rho)$ are boundless. As is well-known, it is highly desirable that the chosen metric should be a normalized one, that is, its maximum value is equal to one. According to this point, we think that among these measures, $C(\rho,\sigma)$ is an optimal non-Gaussianity measure for quantum states and then expect that it can be applied to investigate the non-Gaussianity-bounded uncertainty relation[32] and quantum communication.[7,37]

Table 1. Comparison of non-Gaussianity measures for Fock states.

	$\lvert n\rangle\langle n\rvert$	$\lvert 1\rangle\langle 1\rvert$	$\lvert 4\rangle\langle 4\rvert$	Limiting values
$\delta_{\rm A}(\rho)$	$\frac{1+n}{1+2n}-\frac{n^n}{(n+1)^{1+n}}$	0.4167	0.4736	1/2
$\delta_{\rm B}(\rho)$	$(1+n)\ln(1+n)-n\ln n$	1.3863	2.5020	∞
$N_{\rm W}(\rho)$	$\ln(1+n)-n-\log n!+n\psi(n+1)$	0.1159	0.4559	∞
$N_{\rm G}(\rho)$	n	1.0000	4.000	∞
$A(\rho)$	$\arccos(F_{\rm W})$	0.8526	1.0521	$\pi/2$
$B(\rho)$	$\sqrt{2-2\sqrt{F_{\rm W}}}$	0.8270	1.0042	$\sqrt{2}$
$C(\rho)$	$\sqrt{1-F_{\rm W}}$	0.7530	0.8685	1

5. Conclusion

With the quantum fidelity introduced by Wang, we have presented three alternative measures of non-Gaussianity: the sine distance, the Bures angle, and the Bures distance. We have further evaluated the non-Gaussianity of some relevant single-mode and two-mode non-Gaussian states using these fidelity-based measures. We have shown that such measures are capable of precisely evaluating the non-Gaussianity of a wide range of non-Gaussian quantum states, particularly even Schrödinger-cat-like states, and have a very good consistency in displaying non-Gaussian behaviors. Finally, we have focused on the comparison between our introduced measures and the celebrated non-Gaussianity ones according to the metric axioms. We have also shown that the sine distance can be regarded as the best "legitimate" measure for non-Gaussianity of quantum states among all.

Although much work has been carried out, the problem of quantifying non-Gaussianity of quantum states remains completely unsolved to date. This is because the popular measures are hard to meet with all the six requirement of metric. In particular, they are required to have a straightforward physical interpretation and be feasible in an experiment. Therefore, our result may shed certain light on the quantitative aspect of non-Gaussianity of quantum states and we hope that our introduced measures have the potential applications in quantum information technologies.

Acknowledgements

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