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Explosive synchronization in a mobile network in the presence of a positive feedback mechanism

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Synchronization is a process that describes the coherent dynamics of a large ensemble of interacting units. The study of explosive synchronization transition attracts considerable attention. Here, I report the explosive transition within the framework of a mobile network, while each oscillator is controlled by global-order parameters of the system. Using numerical simulation, I find that the explosive synchronization (ES) transition behavior can be controlled by simply adjusting the fraction of controlled oscillators. The influences of some parameters on explosive synchronization are studied. Moreover, due to the presence of the positive feedback mechanism, I prevent the occurrence of the synchronization of continuous-phase transition and make phase transition of the system a first-order phase transition accompanied by a hysteresis loop.

Keywords: complex network, explosive synchronization, positive feedback, mobile agent

PACS: 05.45.Xt, 89.75.–k, 89.75.Hc, 89.75.Fb **DOI:** 10.1088/1674-1056/ac16d3

1. Introduction

Synchronization is a collective phenomenon of a population of dynamically interacting units. Synchronization processes can be observed in many fields, such as biology,[1,2] neuroscience,[3,4] engineering,[5,6] or even in social sciences.[7] In the past, the Kuramoto model has been widely used in the study of synchronization[8–11] because it captures the essence of collective dynamics. The continuous and second-order phase transition from the incoherent state to the synchronous state has been widely studied using the standard Kuramoto model and its generalized forms.[12–15] However, phase transition is first order, abrupt and irreversible under certain conditions.[9,16–20] Jesus Gomez-Gardenes *et al.*[21] discovered this phenomenon firstly in heterogeneous networks by setting the natural frequency of the oscillator to be positively correlated with its degree. They named it explosive synchronization (ES) by referring to explosive percolation. Explosive synchronization has rapidly become a hot topic in the research frontiers of complex systems. Many mechanisms have been found to induce explosive synchronization transition, including frequency-weight,[16,22,23] time delay,[20,24] inertia,[25] and so on.[9,18,19,26,27]

Recently, positive feedback mechanisms that induce explosive synchronization have attracted increasing attention. Zhang *et al.*[17] utilized the local-order parameter to adjust the actual coupling strength of each oscillator. Using this method, they reported the explosive synchronization transition both in a single network and a multiplex network. Dai *et al.*[28,29] introduced a positive feedback mechanism between the coupling and the global-order parameter in the D-dimensional Kuramoto model and showed that the synchronization transition is explosive in all even dimensions, while it is mediated by a time-dependent rhythmic state in all odd dimensions.[30] Their studies provide inspiration for the understanding of the mechanism between the coupling of the elementary constituents and the level of coherence in the system. For example, if two neurons or neuronal systems are always excited at the same time, a "combination" will be formed. The excitation of one neuron will promote the excitation of the other.[31]

In this work, I study the synchronization phenomenon within the framework of a dynamic network. Although the synchronization of moving oscillators has been studied recently,[12,32–35] previous studies mostly explored the role of mobile networks in promoting synchronization. The method involving using the average property (global-order parameter in our work) of the whole system to adjust a single oscillator is rarely used in mobile networks.[36] In this paper, I first propose a Kuramoto model within the framework of dynamic networks, in which the coupling strength of the oscillator is controlled by positive feedback of the global-order parameters of the whole system. In addition, I study the influence of the symmetry of node frequency distribution on the synchronization phenomenon within the framework of a mobile network, and also discuss the impact of the communication radius on the phase transition mode. In Section 2, I propose the Kuramoto model in the presence of a positive feedback mechanism. In Section 3, numerical simulations are provided to explore the impact of parameters. Finally, I present a discussion and conclusions in Section 4.

2. Model

The N Kuramoto oscillators move in an $L \times L$ square area with the periodic boundary conditions. And each oscillator has

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its phase θ_i ($i = 1,2,\ldots,N$, $\theta_i \in [-\pi, \pi]$). The time evolution of the phase of the oscillator i is defined as

$$\dot{\theta}_i = \omega_i + \lambda \alpha_i \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i), \tag{1}$$

where ω_i is the natural frequency of the oscillator i, λ is the overall coupling strength, and A_{ij} are the elements of the adjacency matrix A; $A_{ij} = 1$ if the i-th and j-th nodes are coupled, otherwise $A_{ij} = 0$. Hence, the degree of oscillators can be calculated and $k_i = \sum_{j=0}^{N} A_{ij}$. In our study, natural frequency is independent of the degree of the node.

Compared with previous studies,[32,33] our contribution lies in the coupling strength being tuned by the global-order parameter R through α_i. R is used to measure the coherence of the collective motion and can be defined by $Re^{I\Psi} = (1/N)\sum_{j=1}^{N} e^{I\theta_j}$, where $0 \le R \le 1$. The system is in an incoherent state when $R = 0$. The value of R increases gradually while the oscillators perform synchronization. Finally, the system is fully synchronized when the value of R reaches 1. Here, Ψ represents the average phase of all oscillators and I is the imaginary unit. I divide the oscillator in the system into two parts: one is f fraction controlled oscillators, the other is $1-f$ fraction uncontrolled oscillators. In the controlled oscillators, $\alpha_i = R$, and $\alpha_i = 1$ in the others. I make the global-order parameter positive feedback act on the coupling strength of the oscillators. In contrast to previous research,[17,37] I apply the global-order parameter of the whole system to each oscillator uniformly and positively. By utilizing this positive feedback control method, it is difficult for the system to generate a synchronized core in the network. Therefore, it is difficult for local synchronous communities to appear.

I use the dynamic network to describe the coupling relationship between oscillators. In an $L \times L$ square area, each moving agent denotes an oscillator. In the initial state, all agents are randomly distributed in the area and move with angle $\varphi_i(t)$ and velocity $v_i(t)$. Here, $\varphi_i(t)$ ($\varphi_i(t) \in [-\pi, \pi]$) denotes the direction of the movement of oscillator i at time t, while $v_i(t)$ denotes the speed of movement. Thus, the position of agent i evolves in time as

$$\begin{aligned} x_i(t+1) &= x_i(t) + v_i(t)\cos\varphi_i(t), \\ y_i(t+1) &= y_i(t) + v_i(t)\sin\varphi_i(t), \\ \varphi_i(t+1) &= \varphi_i(t) + \psi_i(t), \end{aligned} \tag{2}$$

where $x_i(t)$ and $y_i(t)$ denote the horizontal ordinate and vertical ordinate of node i at time t, respectively. And $\psi_i(t)$ is the change in the angle between time t and $t+1$. To meet the reality, I use Euclidean distance to judge whether the oscillators are coupled or not. Euclidean distance is defined as

$$D_{ij}(t) = \sqrt{[x_i(t) - x_j(t)]^2 + [y_i(t) - y_j(t)]^2}.$$

Only when $D_{ij}(t) \le r$, the oscillator i and j can be coupled and perform synchronization at discrete time t, namely, $A_{ij} = 1$ when $D_{ij}(t) \le r$ otherwise $A_{ij} = 0$. Here, r is the communication radius. In wireless sensor networks, the communication radius refers to the wireless transmission and reception distance of each agent.[36] In our study, r affects the number of coupled neighbors of the oscillator. For the sake of simplicity, I set the communication radius r of all nodes to be the same without additional explanation.

3. Simulation results

The numerical simulation results refer to a network of N Kuramoto oscillators. Without affecting the conclusion and properties of the simulation, I set $N = 500$ for all plots. Initially, phase $\theta_i(t)$ of all oscillators and the natural frequency ω_i are drawn from a uniform probability distribution in $[-\pi, \pi]$, unless otherwise specified. To calculate the phase $\theta_i(t)$ of oscillator i in Eq. (1), I solve the differential equations using the 4th-Runge–Kutta method with the integration time step $dh = 0.01$. In addition, to compute a stationary value of the order parameter, the system is allowed to integrate for a long-enough time period (10^5 time step). To observe and track the phase transition, I change the coupling strength, which includes adiabatically increasing from the minimum value to the maximum value, and then adiabatically decreasing from the maximum value to the minimum value in small steps of $\delta\lambda$. Furthermore, I take the steady value of the phase obtained from the last calculation as the initial state for the next iteration.

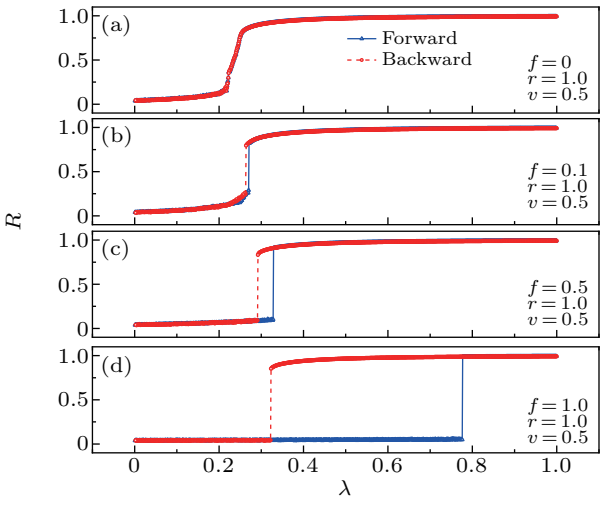


Fig. 1. The transition to synchronization. R versus λ for different values of f; I fix $N = 500$, $\delta\lambda = 0.001$ and natural frequency follows random distribution in $[-\pi, \pi]$. The values used for the other parameters are reported in four plots. In all plots, the blue triangles and solid lines represent forward transition, while the red circles and dashed lines represent backward transition.

The first step is to study the effect of the positive feedback of R on the phase transition. I divided the oscillators into the original oscillators and the oscillators regulated by the positive feedback mechanism and studied the effect of the positive

feedback mechanism on the synchronization mode. The simulations are reported in Fig. 1 and reveal that controlling the actual coupling strength of the oscillator (also known as the effective coupling strength) using the global-order parameter can indeed lead to explosive synchronization (namely, phase transition is abrupt and discontinuous, accompanied by the hysteresis loop).

In Fig. 1, the curves of the dependences of the global-order parameter R and global coupling strength λ for forward transition (FW) and backward transition (BW) are reported for four values of f. I set $L = 10$ (I fixed the value of L in all simulations of this study, for the sake of simplicity), $v = 0.5$, $r = 1$, $\delta\lambda = 0.001$ in Figs. 1(a)–1(d). And the natural frequencies ω of all the oscillators are uniformly distributed in $[-\pi, \pi]$. It is clearly shown that the continuous transitions (second order) occur when PF is absent (i.e., $f = 0$). The two curves of FW and BW coincide completely. By contrast, the explosive synchronization emerges when I add the fraction of controlled oscillators in Figs. 1(b)–1(d). When some oscillators in the system are controlled by the positive feedback mechanism, the phase transition is first order and discontinuous, and a hysteresis loop also appears between the forward-phase transition and backward-phase transition. In fact, when the positive feedback effect does not exist, the phase transition is a second-order and continuous-phase transition. By comparing with Figs. 1(a)–1(d), I conclude that the positive feedback can induce explosive synchronization with a hysteresis loop, and increasing the value of f leads to the increase in the width of hysteresis loop d. However, the critical coupling strength of the forward transition (i.e., the state changes from the incoherent state to the synchronized state) λ_{cF} increases with the increase in f. I infer that this is because when oscillators are controlled by the global-order parameter of the system, it is difficult to form a local and small synchronous community in the system. Thus, the positive feedback is also a kind of inhibition. Only when the coupling strength increases until the critical value is exceeded, the explosive transition suddenly occurs. Therefore, the larger the value of f is, the greater the coupling strength that is needed for transition.

To further investigate microscopic details behind the dynamic of ES, I monitor each oscillator's effective natural frequency in forward transition in Fig. 2, which is defined by $\omega_i^{\text{eff}} = (1/T)\int_{t_0}^{t_0+T} \dot{\theta}_i(t)\,\mathrm{d}(t)$, where T is the total time, $T \gg 1$, and t_0 is the initial transients. I set $\delta\lambda = 0.01$, and the other parameters are the same as those in Fig. 1. At $f = 0.1$, just 10 percent of oscillators are under the control of R. As depicted in Fig. 2(a), the onset of discontinuous and abrupt transition occurs when $R \approx 0.4$. This transition behavior can be known as weak explosive synchronization transition.[20] I can well understand this phenomenon by studying the ω_i^{eff} versus λ curve in Fig. 2(c). As λ reaches the critical coupling strength, not all the oscillators get synchronized abruptly and a few oscillators are still disconnected away from the synchronous cluster. They become part of the largest synchronous cluster only when λ increases beyond the critical coupling strength. In Fig. 2(c), some ω_i^{eff} of the oscillators jump to a value which nears the average value of all ω_i^{eff} and they gradually reach an average value when $\lambda > \lambda_c$. At $f = 1$, the transition behavior is strong ES transition, as depicted in Fig. 2(b). In Fig. 2(d), ω_i^{eff} of all the oscillators suddenly jumps to the average value at once, $\lambda = \lambda_c$. It can be deduced that the positive feedback in the coupling strength of the oscillators is significant for the appearance of ES. It only needs a small proportion of the oscillators to be regulated, and the transition behavior can be abrupt, discontinuous and irreversible.

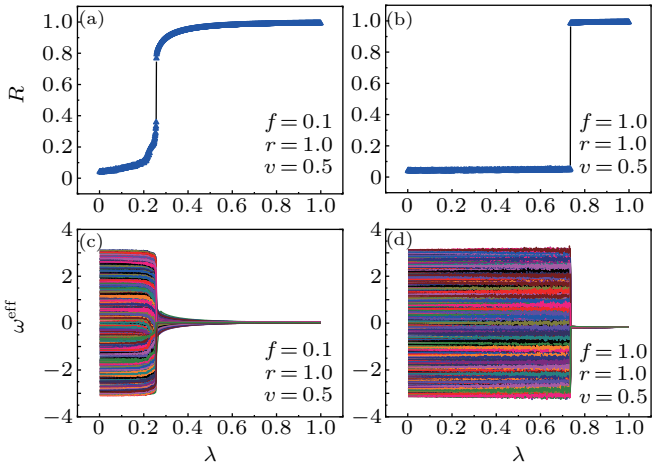


Fig. 2. The forward transition to synchronization for two values of f ($f = 0.1$ for (a), (c); $f = 1.0$ for (b), (d)). The top row panels and bottom panels display the curves of R versus λ and effective frequency ω_i^{eff} versus λ, respectively. The other parameters are the same as those in Fig. 1.

The next interesting question is whether or not the probability density distribution of natural frequency may affect the feature of phase transition. I compare different distributions of natural frequency, and focus on two symmetric distributions $g(\omega)$, such as normal distribution and Lorentzian distribution, and two asymmetric $g(\omega)$, such as power-law distribution and Rayleigh distribution. I set $\delta\lambda = 0.01$ and the other parameters are the same as those in Fig. 1(d). In our study, the Lorentzian distribution has the expression $g(\omega) = (1/\pi)\left(\left(\gamma/\left(\omega-\omega_0\right)^2\right)+\gamma^2\right)$, with $\omega_0 = 0$ and $\gamma = 0.5$, while the normal distribution follows $g(\omega) = (1/\sqrt{2\pi}\sigma)\mathrm{e}^{-(\omega-\mu)^2/2\sigma^2}$, with $\omega = 0$ and $\sigma^2 = 1$. For the asymmetric distribution, I replace random distribution with the power-law distribution, which follows $g(\omega) \sim \omega^{-\alpha}$, with $\omega > 0$ and $\alpha = 3$, and the Rayleigh distribution following $g(\omega) = (\omega/\beta^2)\mathrm{e}^{-\omega^2/2\beta^2}$, with $\omega > 0$ and $\beta = 2$. In Fig. 3, I draw not only the curve of R versus λ, but also the curve of ω_i^{eff} versus λ. As illustrated in Fig. 3, explosive synchronization takes place in four simulations with different natural frequency distributions, although the curves in Figs. 3(a1)–3(d1) are not quite the same. Therefore, in our study, the distribution

of natural frequency does not affect the occurrence of synchronization. All the effective frequencies of the oscillators, whose natural frequency distributions follow the normal or Rayleigh distribution in Figs. 3(b2) and 3(d2), get locked to the average value. However, for the Lorentzian and Rayleigh distributions, a few of the oscillators cannot synchronize with the others, while the effective frequencies of most of the oscillators get locked. I consider that this is because of the wide range of probability density function (PDF) of the Lorentzian and Rayleigh distributions. Some oscillators' natural frequencies are far from the value of the average natural frequency of all oscillators. Thus, it is hard for them to synchronize with the other oscillators. Going back to Figs. 3(a1)–3(d1), the loop sizes of the hysteresis in Figs. 3(b1) and 3(d1) are larger than those in Figs. 3(a1) and 3(c1). The narrower distribution range may lead to a wider hysteresis loop, which means it is easier for ES transition to occur in the system.

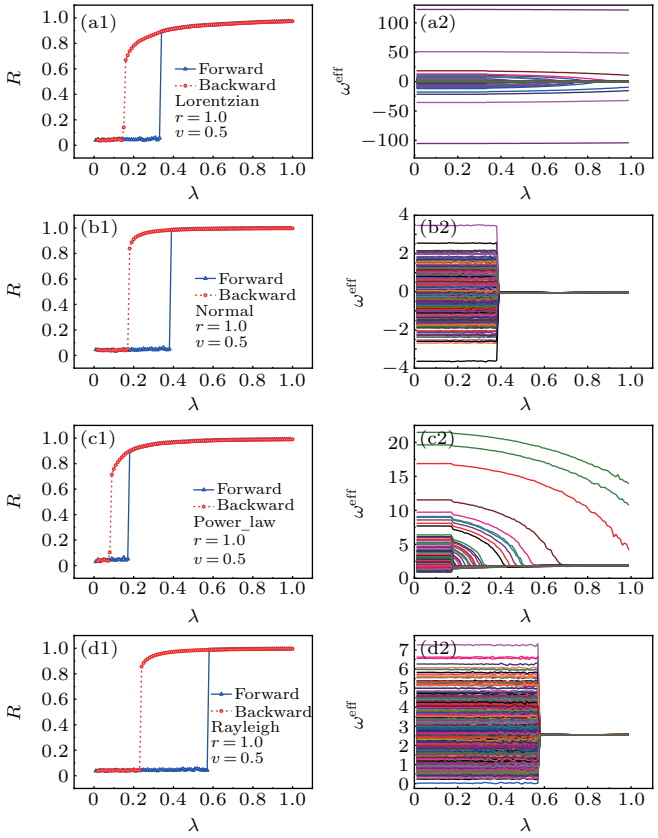


Fig. 3. The panels in the left column and right column display the relationships of R versus λ and ω_i^{eff} versus λ, respectively. We focus on two symmetric distributions $g(\omega)$, such as normal distribution and Lorentzian distribution, and two asymmetric $g(\omega)$, such as power-law distribution and Rayleigh distribution (see text for detail parameters).

In many natural instances, the structures of networks always change constantly. Previous studies have shown that explosive synchronization is closely related to the network structure. Thus, I study the communication radii of the mobile oscillators, which are the inherent parameters of parameters and affect the topology of the network. To understand the role that different values of the communication radius r play in changing transition behavior, I alter the value of the communication radius r and plot curves of R versus λ for selected $r(0.4, 0.6, 0.8, 1.2)$ in Fig. 4. I fix $\delta\lambda = 0.01$, and the other parameters are the same as those in Fig. 1(d). For $r = 0.4$, synchronization does not occur when $R \approx 0$, although I increase the coupling strength to 2. I deduce that the positive feedback mechanism of the order parameter prevents the synchronization in our model. I set $f = 0$ again, and the other parameters remain unchanged. The insert depicts the relation of the global-order parameter and the coupling strength. The system reaches synchronization when $f = 0$; however, the phase transition is continuous. In the dynamic network model, r can determine the critical connectivity of the network. For $N = 500$, $L = 10$, the critical communication radius $r_c \approx 0.54$ ($(N-1)\pi r_c/L^2 \approx 4.51$[38]) if all nodes own neighbors in the single time step t. Due to the oscillators' motion, the system can also reach synchronization when PF is absent. However, it is hard for the system to synchronize under the positive feedback and inhibition of the global-order parameter R. When the communication radius r exceeds r_c, I see that ES transition emerges in Figs. 4(b)–4(d). I also notice that the critical coupling strength λ_{cF} and loop size of hysteresis d both decrease with the increase in r. This phenomenon may come from the fact that a larger value of r leads to the oscillator having more neighbors, so it is easier for each oscillator to synchronize with their own neighbors and the system needs less coupling strength to synchronize.

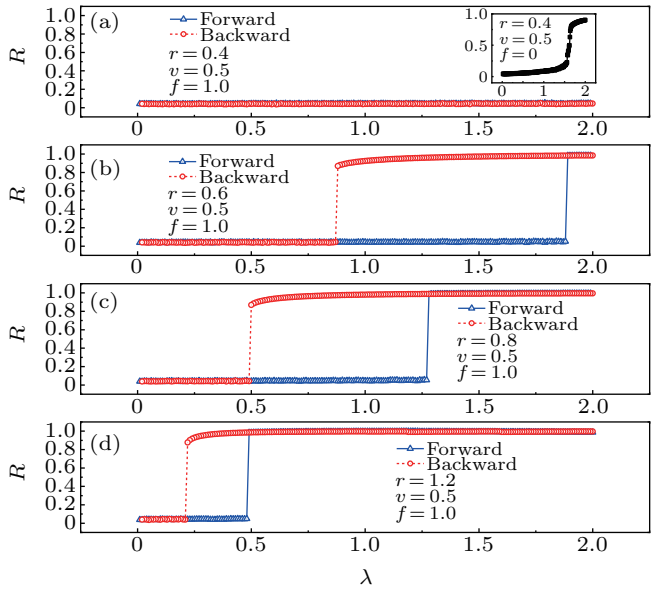


Fig. 4. Dependence of the global-order parameter R versus coupling strength λ in forward transition (blue open triangle) and backward transition (red open circle), for different values of the communication radius r ((a) $r = 0.4$; (b) $r = 0.6$; (c) $r = 0.8$; (d) $r = 1.2$), when $N = 500$ and the natural frequency follows random distribution. The insert in panel (a) reports the dependence of the global-order parameter and coupling strength when $f = 0$.

4. Conclusion

In this article, I have presented the Kuramoto model within a dynamic network framework, wherein the oscilla-

tors' coupling strength is controlled by positive feedback of the global-order parameter of the whole system. The explosive synchronization transition behavior can be controlled by simply adjusting the fraction of controlled oscillators. And the loop size of the hysteresis increases with the increase in the fraction of controlled oscillators. However, the transition is continuous (second-order transition without a hysteresis loop) when positive feedback of the global-order parameter is completely absent. By focusing on different probability density distributions of natural frequency, I further find that, whether the frequency distribution of nodes is symmetrical or not, this does not affect the synchronization behavior of the system. The effective frequencies of all oscillators get locked to the average value of the natural frequencies when the system reaches full synchronization. Lastly, I explore the role that the communication radius plays in ES transition. I note that a larger communication radius contributes a smaller loop size and transition cannot occur if the value of r is less than the critical communication radius in our model. Moreover, the system cannot reach synchronization by second-order phase transition, in our model, when $r < r_c$ and all oscillators are controlled by global-order parameters. The model with Kuramoto oscillators in the presence of a positive feedback mechanism of the global state within a dynamic network framework will be of interest in neuro science[31] and social sciences.[7,39] In addition, there is little doubt that research can be verified and promoted in a more traditional experimental environment. For example, driverless vehicles that receive surrounding information through laser radar can synchronize driving behavior in traffic flow through global information regulation that reduces conflicts between other vehicles. Therefore, this paper provides cues for further research on synchronization in more specific dynamic networks.

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