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Quantifying coherence with dynamical discord*

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Quantum coherence and discord are two kinds of manifestations of nonclassicality. By calculating the coherence and discord in the specific bipartite quantum systems, we show quantitative connections between the coherence and the discord in the bipartite quantum systems created from local systems with the help of incoherent operations. We show that the coherence bounds the dynamical discord, and under particular conditions of the initial quantum states, the coherence of single systems is equal to the dynamical discord. We extend these results to the multipartite quantum systems.

Keywords: quantum coherence, dynamical discord, relative entropy

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1. Introduction

Quantum coherence is an important feature of quantum mechanics. In recent years, quantum coherence was considered to be a physical resource in quantum information processing[1,2] and gained lots of attentions.[3,4] Quantum discord as a kind of quantum correlations, which can exists in unentangled systems, was regarded as an important quantum resource.[5–9] Quantum discord was used to implement some tasks, such as remote state preparation,[10] information encoding,[11] and quantum computation.[12] These two kinds of resources were considered to be the specific manifestation of the superposition principle in quantum mechanics. Previous studies have showed some connections between the quantum coherence and discord.[13–16]

Resources can be converted into each other. Local coherence can be converted into entanglement via incoherent operations,[17–19] and converting coherence into discord was studied in Ref. [20]. Other resource conversions among nonclassicality and quantum correlations were also studied in Refs. [21–26], and some related experiments were made in the optics and superconducting systems.[27–30] It is benefit for systematic study of the various notions of quantumness with resource theories,[31–35] due to it can quantify the quantumness in an objective manner, and this quantification satisfies the constraints of monotonicity under incoherent operations.

For better understanding the relations among the various notions of nonclassicality, it is meaningful to investigate the resource conversion from quantum coherence to correlations. Ma *et al.* showed that quantum coherence can be converted into discord via bipartite incoherent operations, and the amount of coherence bounds the discord. Based on the measures of the relative entropy, these two quantities are identical.[20] The similar result is provided for the created entanglement in Ref. [19]. Recently, Theurer *et al.* studied the relations between dynamical coherence and dynamical entanglement, and they provided a quantitative connection between the dynamical coherence and the dynamical entanglement.[36] According to the results of Refs. [19,20], the coherence of single quantum systems is equal to the dynamical entanglement and discord, respectively, and note that these two formulas are identical. We know that entanglement and discord are different notions, so there exist some conditions in the initial quantum state to distinguish it. We focus on the discord in this paper.

For quantifying the coherence with the dynamical discord, it is necessary to calculate the created discord and the coherence of quantum systems. Although the theory of quantum discord is well developed, calculating the discord of generic quantum systems is complicated. We adopt two specific quantum systems that the discord and coherence can be calculated, and these two specific quantum systems are created by incoherent operations. Is the created discord equal to the coherence displaying in the initial quantum systems? What the conditions should be satisfied in the initial states that the coherence is equal to the created discord? We resolve these questions in this work.

In this paper, we study the operational resource conversion between quantum coherence and discord via two specific quantum systems. In the conversion from coherence to discord, we restrict the operations to the bipartite and multipartite controlled-not gate operations. We find that the coherence of initial quantum systems bounds the bipartite dynamical discord and the inequalities are saturated under the particular conditions of the initial state. The similar result is provided for the multipartite quantum systems. We also show the advantage of

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characterizing coherence with dynamical discord than dynamical entanglement.

2. Preliminaries

We review the definitions of coherence and discord from resource theories. The relative entropy of resource has an operational interpretation in resource theories, which was firstly defined for entanglement theory in Ref. [31]. The relative entropy is a unique measure in reversible quantum resource theories,[37] and the unique asymptotic rate of interconversion among static resources under operations.[38] The relative entropy of resource is also the optimal rate in the resource erasure.[39,40] Due to the properties above, the relative entropy is an important measure in quantum resource theories. For the two quantum states ρ and ω, the relative entropy is defined as $S(\rho||\omega) \equiv -\mathrm{tr}(\rho\log\omega) - S(\rho)$, where $S(\rho) = -\mathrm{tr}(\rho\log\rho)$ is the von Neumann entropy of ρ.[41]

Discord is defined as the disturbance of multipartite states by local measurements.[5,9] Quantum discord reduces to entanglement for pure states, and separable mixed states can have nonzero discord. The zero discord states are called classical states which contain mixtures of locally distinguishable states, $\varphi = \sum_{k_n} p_{k_1\cdots k_N} |k_1\cdots k_N\rangle\langle k_1\cdots k_N|$, where $p_{k_1\cdots k_N}$ is a joint probability distribution and local states $|k_n\rangle$ span an orthogonal basis.[8] For a quantum state ρ, the relative entropy of discord is defined as $D(\rho) = \min_{\omega\in J} S(\rho||\omega)$ where J denotes the set of classical states. Let χ be the closest classical state to the quantum state ρ, and the relative entropy of discord is

$$D(\rho) = S(\rho||\chi) = -\mathrm{Tr}(\rho\log\chi) - S(\rho). \tag{1}$$

Given a quantum state ρ the relative entropy of coherence is defined as $C(\rho) = \min_{\omega\in\tau} S(\rho||\omega)$, where τ denotes the set of incoherent states the closest incoherent state ρ_{d} is the matrix containing only the leading diagonal elements of ρ in the reference basis,[1] and the relative entropy of coherence is

$$C(\rho) = S(\rho||\rho_{\mathrm{d}}) = S(\rho_{\mathrm{d}}) - S(\rho). \tag{2}$$

The controlled-not gate (CNOT) is a specific case of the generalized controlled-not gate for two qubits, and has the form

$$U_{\mathrm{CN}} = \sum_{i=1}^{2}\sum_{j=1}^{2} |i\rangle\langle i|^A \otimes |\bmod(i+j,2)\rangle\langle j|^B, \tag{3}$$

where $\{|i\rangle\}$ and $\{|j\rangle\}$ are the reference basis of subsystems A and B respectively, with A as the control qubit and B as the target qubit.[19]

Recently, the notions of the dynamical coherence and dynamical entanglement were proposed in Ref. [36], that is, when a quantum state be generated from local states with the help of quantum operations, the coherence and entanglement of the generated state are called dynamical coherence and dynamical entanglement, respectively. In a similar manner, in this paper, the dynamical discord is the discord presenting in the states which be generated from local states with the help of quantum operations.

3. Characterizing coherence with dynamical discord in bipartite pure states

We now analyze the conditions of quantifying coherence with dynamical discord when a product pure state is converted to an correlated state via the CNOT gate operation.

Theorem 1 For the initial product state $\rho_{AB} = (\alpha|0\rangle + \beta|1\rangle)\otimes(a|0\rangle + b|1\rangle)$ the evolved state under the CNOT gate is $U_{\mathrm{CN}}(\rho_{AB}) = \alpha a|00\rangle + \alpha b|01\rangle + \beta a|11\rangle + \beta b|10\rangle$, and if $|a|^2 = 0$ or $|a|^2 = 1$ the coherence of the coherent control states is equal to the dynamical discord of the evolved states, $C(\rho_A) = D(U_{\mathrm{CN}}(\rho_{AB}))$.

Proof: Performing the CNOT gate on the control qubit and the target qubit $\rho_{AB} = (\alpha|0\rangle + \beta|1\rangle)\otimes(a|0\rangle + b|1\rangle)$, the product state evolves into

$$U_{\mathrm{CN}}(\rho_{AB}) = \alpha a|00\rangle + \alpha b|01\rangle + \beta a|11\rangle + \beta b|10\rangle, \tag{4}$$

where $|\alpha|^2 + |\beta|^2 = 1$ and $|a|^2 + |b|^2 = 1$. We have known that quantum discord reduces to the entanglement for pure states, for a bipartite pure state, the relative entropy of discord is equal to the entropy of the subsystem, $D(\rho_{AB}) = S(\rho_A)$, where $\rho_A = \mathrm{tr}_B(\rho_{AB})$.[5] We calculate the dynamical discord of the evolved state,

$$D(U_{\mathrm{CN}}(\rho_{AB})) = 1 + H(x)/2, \tag{5}$$

where $H(x) = -(1+x)\log(1+x) - (1-x)\log(1-x)$ and $x = \sqrt{1 - 4|\alpha|^2|\beta|^2(|a|^2 - |b|^2)^2}$. The function $H(x)$ is a symmetric concave function of x with $|x| \le 1$ the symmetric point $x = 0, H(0) = 0$ is the maximal value, such that $H(x) \le 0$ and $H(x) \in [-2, 0]$. When $0 \le x \le 1$, $H(x)$ is a monotonically decreasing function of x with $H(x) \in [0, -2]$. Hence the formula $D(U_{\mathrm{CN}}(\rho_{AB}))$ is also a monotonically decreasing function of x with $D(U_{\mathrm{CN}}(\rho_{AB})) \in [0, 1]$. For the condition $0 \le x \le 1$, i.e., $4|\alpha|^2|\beta|^2(|a|^2 - |b|^2)^2 \le 1$ using the properties $2|\alpha||\beta| \le (|\alpha|^2 + |\beta|^2)$ and the conditions $|\alpha|^2 + |\beta|^2 = 1$ and $|a|^2 + |b|^2 = 1$, we can prove $4|\alpha|^2|\beta|^2(|a|^2 - |b|^2)^2 \le (|\alpha|^2 + |\beta|^2)^2(|a|^2 - |b|^2)^2 = (2|a|^2 - 1)^2 \le 1$.

According to the Eq. (2), we calculate the relative entropy of coherence of the evolved state

$$C(\rho_{AB}) = C(U_{\mathrm{CN}}(\rho_{AB})) = h(x_1) + h(x_2), \tag{6}$$

where $h(x_1) = -x_1\log x_1 - (1-x_1)\log(1-x_1)$ is the binary Shannon entropy, the parameters are $x_1 = |\alpha|^2$ with $0 \le x_1 \le 1$, and $x_2 = |a|^2$ with $0 \le x_2 \le 1$. As well known that $h(x_1)$ is a symmetric concave function of x_1 with $0 \le x_1 \le 1$ the symmetric point is $1/2$, $h(1/2) = 1$ is the maximal value, hence $h(x_1) \ge 0$ and $h(x_1) \in [0, 1]$. Similarly, we obtain $h(x_2) \in [0, 1]$ and the function of coherence $C(\rho_{AB}) \in [0, 2]$.

The coherence can be converted into discord under the CNOT gate if the control qubit is the coherent states.[20] If the control state is given, i.e., $|\alpha|^2 = c$, where c is a constant, then $h(x_1)$ is also a constant. From the concavity of the functions $H(x)$ and $h(x_2)$ and the majorization theory, when $x_2 = |a|^2 = 1/2$, namely, $x = 1$, we find $D(U_{\rm CN}(\rho_{AB}))$ is the minimal value, and simultaneously, $C(U_{\rm CN}(\rho_{AB}))$ is the maximal value. When x_2 varies from $1/2$ approaching to zero, or x_2 varies from $1/2$ approaching to 1 namely, the variable x varies from 1 to $x = |\beta|^2 - |\alpha|^2$, we get to know that the quantities $D(U_{\rm CN}(\rho_{AB}))$ becomes larger, and the quantities $C(U_{\rm CN}(\rho_{AB}))$ becomes smaller. When $x_2 = |a|^2 = 0$ or $x_2 = |a|^2 = 1$, namely, $x = |\beta|^2 - |\alpha|^2$ we find the maximal value of $D(U_{\rm CN}(\rho_{AB}))$ is arrived, and simultaneously, the minimal value of $C(U_{\rm CN}(\rho_{AB}))$ is arrived, too. Furthermore, these two extreme values are equivalent,

$$\begin{aligned} C(U_{\rm CN}(\rho_{AB})) &= D(U_{\rm CN}(\rho_{AB})) \\ &= -|\alpha|^2 \log|\alpha|^2 - |\beta|^2 \log|\beta|^2 . \end{aligned} \tag{7}$$

Unifying the analysis above, we obtain the difference between the coherence and the dynamical discord

$$C(U_{\rm CN}(\rho_{AB})) - D(U_{\rm CN}(\rho_{AB})) \geq 0, \tag{8}$$

which holds when $|a|^2 = 0$ or $|a|^2 = 1$.

The target state is pure incoherent state with the conditions $|a|^2 = 0$ or $|a|^2 = 1$. Due to the additive properties of the coherence,[42] for the product states $\rho_A \otimes \rho_B$ we have $C(\rho_A \otimes \rho_B) = C(\rho_A) + C(\rho_B)$. So the coherence of the control state is equal to the dynamical discord of the evolved state under CNOT gate,

$$C(\rho_A) = C(\rho_{AB}) = C(U_{\rm CN}(\rho_{AB})) = D(U_{\rm CN}(\rho_{AB})) . \tag{9}$$

In other words, by using the CNOT gate as an incoherent operation on the product states ρ_{AB}, when the coherent control state is $\alpha|0\rangle + \beta|1\rangle$ and the incoherent target states is $|0\rangle$ or $|1\rangle$, the coherence of the coherent control states is equal to the dynamical discord of the evolved states. The evolved state are $\alpha|00\rangle + \beta|11\rangle$ or $\alpha|01\rangle + \beta|10\rangle$. The coherence of the coherent control state is

$$C(\rho_A) = D(U_{\rm CN}(\rho_{AB})) = -|\alpha|^2 \log|\alpha|^2 - |\beta|^2 \log|\beta|^2 . \tag{10}$$

This completes the proof.

4. Characterizing coherence with dynamical discord in bipartite mixed states

We now analyze the conditions of quantifying coherence with dynamical discord when a product mixed state is converted to an correlated state via the CNOT gate operation. Perform the CNOT gate on the product state $\rho_A \otimes \rho_B$, and the evolved states is

$$\rho_{\rm CN} = U_{\rm CN}(\rho_A \otimes \rho_B) U_{\rm CN}^{\dagger}. \tag{11}$$

Consider the control qubit in the quantum state $\rho_A = \left(I + \sum_j n_j \sigma_j\right)/2$ and the target qubit in the quantum state $\rho_B = \left(I + \sum_j n'_j \sigma_j\right)/2$, where $j \in \{1,2,3\}$ and σ_j being the jth Pauli matrix with $n_1^2 + n_2^2 + n_3^2 \leq 1$ and $(n'_1)^2 + (n'_2)^2 + (n'_3)^2 \leq 1$. For convenience, we assume the Bloch vectors are $\boldsymbol{n} = \{n_1, n_2 = 0, n_3\}$ and $\boldsymbol{n}' = \{n'_1, n'_2 = 0, n'_3\}$. In the reference basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the product state is

$$\rho_A \otimes \rho_B = \frac{1}{4} \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{bmatrix}, \tag{12}$$

where $\rho_{11} = (1+n_3)(1+n'_3)$, $\rho_{12} = \rho_{21} = (1+n_3)n'_1$, $\rho_{13} = \rho_{31} = n_1(1+n'_3)$, $\rho_{14} = \rho_{41} = n_1 n'_1$, $\rho_{23} = \rho_{32} = n_1 n'_1$, $\rho_{33} = (1-n_3)(1+n'_3)$, $\rho_{24} = \rho_{42} = n_1(1-n'_3)$, $\rho_{22} = (1+n_3)(1-n'_3)$, $\rho_{34} = \rho_{43} = (1-n_3)n'_1$, and $\rho_{44} = (1-n_3)(1-n'_3)$.

We perform the CNOT gate on the product state, and the initial state evolves into the state

$$\rho_{\rm CN} = U_{\rm CN}(\rho_A \otimes \rho_B) U_{\rm CN}^{\dagger} = \frac{1}{4} \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{14} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{24} & \rho_{23} \\ \rho_{41} & \rho_{42} & \rho_{44} & \rho_{43} \\ \rho_{31} & \rho_{32} & \rho_{34} & \rho_{33} \end{bmatrix}. \tag{13}$$

Calculating these elements of the density matrix, we find when $n'_1 = 0$, the evolved state becomes a two-qubit X-state

$$\rho'_{\rm CN} = \frac{1}{4} \begin{bmatrix} \rho_{11} & 0 & 0 & \rho_{13} \\ 0 & \rho_{22} & \rho_{24} & 0 \\ 0 & \rho_{42} & \rho_{44} & 0 \\ \rho_{31} & 0 & 0 & \rho_{33} \end{bmatrix}. \tag{14}$$

The Bell-diagonal (BD) states are simple states which keep high relevance to the theoretical and experimental research in quantum information. The BD states also named T-states or two-qubit states with maximally mixed marginals,[43] which include the well-known Bell and Werner states[44] and often be employed as resources for performing tasks such as entanglement activation and distribution via discord-type correlations.[45–47] The BD state is defined as

$$\rho = \sum_{i=1}^{4} \lambda_i |\psi_i\rangle\langle\psi_i| , \tag{15}$$

where $\{|\psi_i\rangle\}$ are the four Bell states $|\Phi^{\pm}\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ $|\Psi^{\pm}\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$, and $\{\lambda_i\} \geq 0$, $\sum_{i=1}^{4} \lambda_i = 1$.

Let the density operators $\delta_{1,2} = |\Phi^{\pm}\rangle\langle\Phi^{\pm}|$ and $\delta_{3,4} = |\Psi^{\pm}\rangle\langle\Psi^{\pm}|$, where $|\Phi^{\pm}\rangle$ and $|\Psi^{\pm}\rangle$ are the four Bell bases, and we define a particular BD state in the form

$$\rho = \lambda_1 \delta_1 + \lambda_2 \delta_2 + \lambda_3 \delta_3 + \lambda_4 \delta_4, \tag{16}$$

where $\{\lambda_i\} \geq 0$ and $\sum_{i=1}^{4} \lambda_i = 1$.

Theorem 2 For the particular BD state $\rho = \lambda_1\delta_1 + \lambda_2\delta_2 + \lambda_3\delta_3 + \lambda_4\delta_4$, if $\{\lambda_1,\lambda_2\} \geq \{\lambda_3,\lambda_4\}$ or $\{\lambda_1,\lambda_2\} \leq \{\lambda_3,\lambda_4\}$, then $C(\rho) = D(\rho)$, otherwise, $C(\rho) > D(\rho)$.

Proof: For any BD states $\rho = \sum_{i=1}^{4}\lambda_i|\psi_i\rangle\langle\psi_i|$ with $\{\lambda_i\}$ are ordered in nonincreasing size, i.e., $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$, the closest classical state is given by $\chi = \frac{q}{2}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|) + \frac{1-q}{2}(|\psi_3\rangle\langle\psi_3| + |\psi_4\rangle\langle\psi_4|)$ with $q = \lambda_1 + \lambda_2$.[8] For the BD state $\rho = \lambda_1\delta_1 + \lambda_2\delta_2 + \lambda_3\delta_3 + \lambda_4\delta_4$, we list all $\{\lambda_i\}$ in nonincreasing size, i.e., $\lambda_{\max} \geq \lambda_{\sec} \geq \lambda_{\text{thi}} \geq \lambda_{\min}$. According to Eq. (1), we obtain the relative entropy of discord

$$\begin{aligned} D(\rho) &= -\mathrm{Tr}(\rho\log\chi) - S(\rho) \\ &= 1 + \sum_{i=1}^{4}\lambda_i\log\lambda_i - (\lambda_{\max}+\lambda_{\sec})\log(\lambda_{\max}+\lambda_{\sec}) \\ &\quad - (\lambda_{\text{thi}}+\lambda_{\min})\log(\lambda_{\text{thi}}+\lambda_{\min}). \end{aligned} \tag{17}$$

For the BD state $\rho = \lambda_1\delta_1 + \lambda_2\delta_2 + \lambda_3\delta_3 + \lambda_4\delta_4$ according to Eq. (2) , we obtain the relative entropy of coherence,

$$\begin{aligned} C(\rho) &= S\left(\rho^d\right) - S(\rho) \\ &= 1 + \sum_{i=1}^{4}\lambda_i\log\lambda_i - (\lambda_1+\lambda_2)\log(\lambda_1+\lambda_2) \\ &\quad - (\lambda_3+\lambda_4)\log(\lambda_3+\lambda_4). \end{aligned} \tag{18}$$

For the BD state $\rho = \lambda_1\delta_1 + \lambda_2\delta_2 + \lambda_3\delta_3 + \lambda_4\delta_4$, if $\{\lambda_1,\lambda_2\} \geq \{\lambda_3,\lambda_4\}$ or $\{\lambda_1,\lambda_2\} \leq \{\lambda_3,\lambda_4\}$, according to Eq. (17), the relative entropy of discord is

$$\begin{aligned} D(\rho) &= 1 + \sum_{i=1}^{4}\lambda_i\log\lambda_i - (\lambda_1+\lambda_2)\log(\lambda_1+\lambda_2) \\ &\quad - (\lambda_3+\lambda_4)\log(\lambda_3+\lambda_4), \end{aligned} \tag{19}$$

which is identical to Eq. (18) of the relative entropy of coherence, and we get

$$C(\rho) = D(\rho). \tag{20}$$

If $\{\lambda_1,\lambda_3\} \geq \{\lambda_2,\lambda_4\}$ or $\{\lambda_1,\lambda_3\} \leq \{\lambda_2,\lambda_4\}$, the relative entropy of discord is

$$\begin{aligned} D(\rho) &= 1 + \sum_{i=1}^{4}\lambda_i\log\lambda_i - (\lambda_1+\lambda_3)\log(\lambda_1+\lambda_3) \\ &\quad - (\lambda_2+\lambda_4)\log(\lambda_2+\lambda_4). \end{aligned} \tag{21}$$

Which can be compared with Eq. (18) of the relative entropy of coherence. From the concavity of von Neumann entropy and the majorization theory[41] we obtain the relation

$$C(\rho) \geq D(\rho). \tag{22}$$

Note that, the equality holds when $\lambda_2 = \lambda_3$. If $\lambda_2 = \lambda_3$, the conditions $\{\lambda_1,\lambda_3\} \geq \{\lambda_2,\lambda_4\}$ or $\{\lambda_1,\lambda_3\} \leq \{\lambda_2,\lambda_4\}$ are identical to $\{\lambda_1,\lambda_2\} \geq \{\lambda_3,\lambda_4\}$ or $\{\lambda_1,\lambda_2\} \leq \{\lambda_3,\lambda_4\}$. Therefore, we obtain the result $C(\rho) > D(\rho)$ if $\{\lambda_1,\lambda_3\} \geq \{\lambda_2,\lambda_4\}$ or $\{\lambda_1,\lambda_3\} \leq \{\lambda_2,\lambda_4\}$.

If $\{\lambda_1,\lambda_4\} \geq \{\lambda_2,\lambda_3\}$ or $\{\lambda_1,\lambda_4\} \leq \{\lambda_2,\lambda_3\}$, the relative entropy of discord is

$$\begin{aligned} D(\rho) &= 1 + \sum_{i=1}^{4}\lambda_i\log\lambda_i - (\lambda_1+\lambda_4)\log(\lambda_1+\lambda_4) \\ &\quad - (\lambda_2+\lambda_3)\log(\lambda_2+\lambda_3). \end{aligned} \tag{23}$$

Using the similar methods above, we obtain the relations

$$C(\rho) \geq D(\rho), \tag{24}$$

which holds when $\lambda_2 = \lambda_4$. We get $C(\rho) > D(\rho)$ if $\{\lambda_1,\lambda_4\} \geq \{\lambda_2,\lambda_3\}$ or $\{\lambda_1,\lambda_4\} \leq \{\lambda_2,\lambda_3\}$. This completes the proof.

Let the BD state $\rho = \lambda_1\delta_1 + \lambda_2\delta_2 + \lambda_3\delta_3 + \lambda_4\delta_4$ be equal to Eq. (14) of the final X-state, i.e., the corresponding elements of the two density matrices are all identical,

$$\begin{aligned} \rho_{11} &= (1+n_3)\left(1+n_3'\right) = 2(\lambda_1+\lambda_2), \\ \rho_{13} &= \rho_{31} = n_1\left(1+n_3'\right) = 2(\lambda_1-\lambda_2), \\ \rho_{33} &= (1-n_3)\left(1+n_3'\right) = 2(\lambda_1+\lambda_2), \\ \rho_{24} &= \rho_{42} = n_1\left(1-n_3'\right) = 2(\lambda_3-\lambda_4), \\ \rho_{22} &= (1+n_3)\left(1-n_3'\right) = 2(\lambda_3+\lambda_4), \\ \rho_{44} &= (1-n_3)\left(1-n_3'\right) = 2(\lambda_3+\lambda_4). \end{aligned} \tag{25}$$

Based on these equations, we obtain the following relations:

$$\begin{aligned} \lambda_1 &= \left(1+n_1+n_3'+n_1n_3'\right)/4, \quad \lambda_2 = \left(1-n_1+n_3'-n_1n_3'\right)/4, \\ \lambda_3 &= \left(1+n_1-n_3'-n_1n_3'\right)/4, \quad \lambda_4 = \left(1-n_1-n_3'+n_1n_3'\right)/4, \\ n_3 &= 0. \end{aligned} \tag{26}$$

That is to say when the control state and the target state are in the forms of $\boldsymbol{n} = \{n_1, n_2 = 0, n_3 = 0\}$ and $\boldsymbol{n}' = \{n_1' = 0, n_2' = 0, n_3'\}$, respectively, the product state of the two qubits evolves into the BD state under the CNOT gate,

$$\rho_{\text{CN}}'' = \lambda_1\delta_1 + \lambda_2\delta_2 + \lambda_3\delta_3 + \lambda_4\delta_4, \tag{27}$$

where $\{\lambda_i\}$ satisfy these equations (26).

Corollary 1 For the initial product state $\rho_A \otimes \rho_B$ where $\rho_A = (I + n_1\sigma_1)/2$ and $\rho_B = (I + n_3'\sigma_3)/2$, the evolved BD state under the CNOT gate is $\rho_{\text{CN}}'' = U_{\text{CN}}(\rho_A\otimes\rho_B)U_{\text{CN}}^{\dagger} = \lambda_1\delta_1 + \lambda_2\delta_2 + \lambda_3\delta_3 + \lambda_4\delta_4$. When $|n_1| \leq |n_3'|$, the coherence of the coherent control states is equal to the dynamical discord of the evolved states $C(\rho_A) = D\left(U_{\text{CN}}(\rho_A\otimes\rho_B)U_{\text{CN}}^{\dagger}\right)$.

Proof The evolved BD state $\rho_{\text{CN}}'' = \lambda_1\delta_1 + \lambda_2\delta_2 + \lambda_3\delta_3 + \lambda_4\delta_4$ has the following relations in terms of Eq. (26):

$$\lambda_1 = \left(1+n_1+n_3'+n_1n_3'\right)/4, \quad \lambda_2 = \left(1-n_1+n_3'-n_1n_3'\right)/4,$$

$$\lambda_3 = \left(1+n_1-n_3'-n_1n_3'\right)/4, \quad \lambda_4 = \left(1-n_1-n_3'+n_1n_3'\right)/4.$$

According to Theorem 1 we know in the BD states $\rho = \lambda_1\delta_1 + \lambda_2\delta_2 + \lambda_3\delta_3 + \lambda_4\delta_4$, if $\{\lambda_1,\lambda_2\} \geq \{\lambda_3,\lambda_4\}$ or $\{\lambda_1,\lambda_2\} \leq \{\lambda_3,\lambda_4\}$, then $C(\rho) = D(\rho)$, otherwise, $C(\rho) > D(\rho)$. Now, for the evolved BD states $\rho_{\text{CN}}'' = \lambda_1\delta_1 + \lambda_2\delta_2 + \lambda_3\delta_3 + \lambda_4\delta_4$, the condition $\{\lambda_1,\lambda_2\} \geq \{\lambda_3,\lambda_4\}$ becomes $\left\{(1+n_1+n_3'+n_1n_3'),(1-n_1+n_3'-n_1n_3')\right\} \geq \left\{(1+n_1-n_3'-n_1n_3'),(1-n_1-n_3'+n_1n_3')\right\}$. After calculation, we obtain

$$-n_3' \leq \left\{n_1, -n_1n_3'\right\} \leq n_3'. \tag{28}$$

Similarly, the condition $\{\lambda_1,\lambda_2\} \leq \{\lambda_3,\lambda_4\}$ becomes $\{(1+n_1+n_3'+n_1n_3'),(1-n_1+n_3'-n_1n_3')\} \leq \{(1+n_1-n_3'-n_1n_3'),(1-n_1-n_3'+n_1n_3')\}$. After calculation, we obtain

$$n_3' \leq \{n_1,-n_1n_3'\} \leq -n_3'. \tag{29}$$

Unifying these two conditions (28) and (29), we have $C(\rho_{\rm CN}'') = D(\rho_{\rm CN}'')$ when $\{|n_1|,|n_1n_3'|\} \leq |n_3'|$ (obviously, $|n_1n_3'| \leq |n_3'|$), such that we have the condition of $C(\rho_{\rm CN}'') = D(\rho_{\rm CN}'')$ in the BD states $\rho_{\rm CN}'' = \lambda_1\delta_1+\lambda_2\delta_2+\lambda_3\delta_3+\lambda_4\delta_4$, namely, $|n_1| \leq |n_3'|$.

Using the additive properties of coherence for the product states $\rho_A\otimes\rho_B$, we have $C(\rho_A\otimes\rho_B) = C(\rho_A)+C(\rho_B)$. As known that the unitary operation cannot change the coherence of a quantum state in the fixed reference basis and the CNOT gate is an unitary operation[26] we have $C(\rho) = C(U_{\rm CN}\rho U_{\rm CN}^\dagger)$. According to Corollary 2, for the coherent control states $\rho_A = (I+n_1\sigma_1)/2$ and the incoherent target states $\rho_B = (I+n_3'\sigma_3)/2$, with $|n_1| \leq |n_3'|$, the evolved state under CNOT gate is the BD states $\rho_{\rm CN}'' = \lambda_1\delta_1+\lambda_2\delta_2+\lambda_3\delta_3+\lambda_4\delta_4$ and we obtain the relations,

$$C(\rho_A) = C(\rho_A\otimes\rho_B) = C(\rho_{\rm CN}'') = D(\rho_{\rm CN}''). \tag{30}$$

That is to say, by using the CNOT gate as an incoherent operation on the product states $\rho_A\otimes\rho_B$ with $\rho_A = (I+n_1\sigma_1)/2$, $\rho_B = (I+n_3'\sigma_3)/2$, and $|n_1| \leq |n_3'|$, the coherence of the coherent control states is equal to the dynamical discord of the evolved states. The coherence of the coherent control state is

$$C(\rho_A) = D\left(U_{\rm CN}(\rho_A\otimes\rho_B)U_{\rm CN}^\dagger\right) = -H(n_1)/2, \tag{31}$$

where $H(n_1) = -(1+n_1)\log(1+n_1)-(1-n_1)\log(1-n_1)$.

With expanding the initial product states $\rho_A\otimes\rho_B$ to general states, its Bloch vectors are $\boldsymbol{n} = \{n_1,n_2,n_3\}$ and $\boldsymbol{n}' = \{n_1',n_2',n_3'\}$, respectively. After calculation, we find if $\boldsymbol{n} = \{n_1,n_2=0,n_3=0\}$ and $\boldsymbol{n}' = \{n_1'=0,n_2'=0,n_3'\}$ with $|n_1| \leq |n_3'|$ the coherence of the coherent control states is equal to the dynamical discord of the evolved states under CNOT gate, $C(\rho_A) = D\left(U_{\rm CN}(\rho_A\otimes\rho_B)U_{\rm CN}^\dagger\right)$. In other words, if the BD states $\rho_{\rm CN}'' = \lambda_1\delta_1+\lambda_2\delta_2+\lambda_3\delta_3+\lambda_4\delta_4$ is obtained by CNOT gate operating on the product states $\rho_A\otimes\rho_B$, with $\{\lambda_i\}$ satisfy Eqs. (26) and $|n_1| \leq |n_3'|$, we have $C(\rho_A) = D\left(U_{\rm CN}(\rho_A\otimes\rho_B)U_{\rm CN}^\dagger\right)$. This completes the proof.

It is already known that the coherence of single-party system is equal to the dynamical entanglement,[19,36] and the discord is a quantum correlation beyond entanglement. A question arises: Is it possible that the coherence be equal to the dynamical discord without generating dynamical entanglement? The answer will be presented in the following.

For any particular Bell-diagonal states $\rho = \sum_{i=1}^4\lambda_i|\psi_i\rangle\langle\psi_i|$, where λ_i are ordered in nonincreasing size, i.e., $\lambda_1\geq\lambda_2\geq\lambda_3\geq\lambda_4$, ρ is entangled when $\lambda_1 > 1/2$.[48] For the evolved BD states $\rho_{\rm CN}'' = \lambda_1\delta_1+\lambda_2\delta_2+\lambda_3\delta_3+\lambda_4\delta_4$, the maximal value of $\{\lambda_i\}$ is $\lambda_{\max} = (1+|n_1|+|n_3'|+|n_1n_3'|)/4$, and when $\lambda_{\max} > 1/2$, the evolved BD states is an entangled state. Let

$$\lambda_{\max} = (1+|n_1|+|n_3'|+|n_1n_3'|) \leq 2, \tag{32}$$

then the entanglement of the evolved BD states $\rho_{\rm CN}''$ is zero. We have known when

$$|n_1| \leq |n_3'|, \tag{33}$$

the coherence is equal to the dynamical discord, $C(\rho_A) = D(\rho_{\rm CN}'')$. Combining these two formulas (32) and (33) , we find if

$$|n_1| \leq |n_3'| = \sqrt{2}-1, \tag{34}$$

the coherence of single-party systems is equal to the amount of the dynamical discord without generating dynamical entanglement via the CNOT gate operations. That is to say, characterizing the coherence with dynamical discord possesses more advantages than dynamical entanglement.

To better investigate the condition of characterizing coherence with dynamical discord, we now use another approach to quantify these two kinds of resources. We quantify the coherence with the l_1 norm of coherence[1] which measures coherence via the off-diagonal elements of a density matrix, $C_{l_1} = \sum_{i\neq j}|\rho_{ij}|$. For the coherent control states $\rho_A = (I+n_1\sigma_1)/2$, the l_1 norm of coherence is $C_{l_1}(\rho_A) = |n_1|$. We quantify the discord with the geometric measure of quantum discord for the BD state, $\rho_{\rm CN}'' = \lambda_1\delta_1+\lambda_2\delta_2+\lambda_3\delta_3+\lambda_4\delta_4$, the geometric discord is in the form of $D(\rho_{\rm CN}'') = \sqrt{(|n_1n_3'|^2+|n_i|^2)/2}$, where n_i is the lower value of $\{|n_1|,|n_3'|\}$.[49] If $|n_1| > |n_3'|$, we have $D(\rho_{\rm CN}'') = \sqrt{(|n_1n_3'|^2+|n_3'|^2)/2} < \sqrt{((1+|n_3'|^2)|n_1|^2)/2} < |n_1|$, namely, $D(\rho_{\rm CN}'') < C_{l_1}(\rho_A)$. If $|n_1| < |n_3'|$, we have $D(\rho_{\rm CN}'') = \sqrt{(|n_1n_3'|^2+|n_1|^2)/2} = \sqrt{((1+|n_3'|^2)|n_1|^2)/2} \leq |n_1|$, which holds when $|n_3'| = 1$, namely, $D(\rho_{\rm CN}'') \leq C_{l_1}(\rho_A)$. That is to say, when the initial quantum states satisfy $|n_3'| = 1$, the coherence of single-party systems is equal to the dynamical discord via the CNOT gate operation.

Through the above analysis, we conclude that the conditions of the initial quantum states for characterizing coherence with dynamical discord are different from different measures.

5. Characterizing coherence with dynamical discord in multipartite systems

The multipartite controlled-not operation (MCNOT) can be written as[50]

$$U_{\rm MCN} = \sum_{i=1}^2\sum_{j_1=1}^2\cdots\sum_{j_N=1}^2 |i\rangle\langle i|^A\otimes|\bmod(i+j_1,2)\rangle\langle j_1|^{B_1} \otimes\cdots\otimes|\bmod(i+j_N,2)\rangle\langle j_N|^{B_N}, \tag{35}$$

which takes any pure incoherent qubit state $|i\rangle\otimes|j_1\rangle\cdots\otimes|j_N\rangle$ to another pure incoherent state, $U_{\rm MCN}(|i\rangle\otimes|j_1\rangle\cdots\otimes|j_1\rangle)=|i\rangle\otimes|\bmod(i+j_1,2)\rangle\cdots\otimes|\bmod(i+j_N,2)\rangle$. For multi-qubit systems $\rho_A\otimes\rho_B^{\otimes N-1}$, what conditions should be satisfied if the coherence of the control state is equal to the dynamical discord of the evolved state under the MCNOT gate?

Extending the above results to the multi-qubit systems we can obtain the following results. For the initial product state with a pure coherent control state and an $N-1$-partite pure incoherent target state $(\alpha|0\rangle+\beta|1\rangle)\otimes|0\rangle^{\otimes N-1}$, the evolved state under MCNOT gate is

$$\rho=U_{\rm MCN}\left((\alpha|0\rangle+\beta|1\rangle)\otimes|0\rangle^{\otimes N-1}\right)=\alpha|0\rangle^{\otimes N}+\beta|1\rangle^{\otimes N}, \tag{36}$$

which is the so-called maximally correlated state.[51] While quantum discord reduce to entanglement for pure states,[5,9] we can calculate entanglement to represent the discord in multipartite quantum systems. For quantifying entanglement in multipartite quantum systems, the genuine multipartite entanglement measures are generalized from the bipartite entanglement measures,[51]

$$E^{\rm GME}(\rho)=\min_{\{\chi\}}E_\chi(\rho), \tag{37}$$

where ρ is an N-partite quantum state and χ denotes all possible bipartite partitions $\chi/\bar{\chi}$ in the composite systems. After calculation, we obtain

$$D(\rho)=E^{\rm GME}(\rho)=-|\alpha|^2\log|\alpha|^2-|\beta|^2\log|\beta|^2. \tag{38}$$

Considering Eq. (10), we obtain the results

$$C(\rho_A)=D\left(U_{\rm MCN}\left((\alpha|0\rangle+\beta|1\rangle)\otimes|0\rangle^{\otimes N-1}\right)\right). \tag{39}$$

For the initial product state with a mixed coherent control state and an $N-1$-partite pure incoherent target state $\frac{1}{2}(I+n_1\sigma_1)\otimes|0\rangle\langle0|^{\otimes N-1}$ (note that, $n_3'=1$ in this formula), the evolved state under MCNOT gate is

$$\begin{aligned}&U_{\rm MCN}\left(\frac{1}{2}(I+n_1\sigma_1)\otimes|0\rangle\langle0|^{\otimes N-1}\right)U_{\rm MCN}^\dagger\\&=\frac{1}{2}\left(|0\rangle\langle0|^{\otimes N}+|1\rangle\langle1|^{\otimes N}+n_1|0\rangle\langle1|^{\otimes N}+n_1|1\rangle\langle0|^{\otimes N}\right).\end{aligned} \tag{40}$$

Compared with the bipartite discord, characterizing multipartite discord is more complicated. Rulli *et al.*[7] proposed the global quantum discord (GQD) in multipartite systems, which is obtained in terms of relative entropy and local von Neumann measurements. The GQD for any multipartite state $\rho_{A_1\cdots A_N}$ under a set of local measurements $\left\{\Pi_{A_1}^{j_1}\otimes\cdots\otimes\Pi_{A_N}^{j_N}\right\}$ is defined as

$$D(\rho_{A_1\cdots A_N})=\min_{\{\Pi_k\}}\left[S(\rho_{A_1\cdots A_N}\|\Phi(\rho_{A_1\cdots A_N}))-\sum_{j=1}^{N}S(\rho_{A_j}\|\Phi_j(\rho_{A_j}))\right], \tag{41}$$

where $\Phi_j(\rho_{A_j})=\sum_j'\Pi_{A_j}^{j'}\rho_{A_j}\Pi_{A_j}^{j'}$ and $\Phi(\rho_{A_1\cdots A_N})=\sum_k\Pi_k\rho_{A_1\cdots A_N}\Pi_k$, with $\Pi_k=\Pi_{A_1}^{j_1}\otimes\cdots\otimes\Pi_{A_N}^{j_N}$ and k denotes the index string $(j_1\cdots j_N)$.

Let us analyze the GQD in the evolved multipartite mixed states. Since all $\{\rho_{A_j}\}$ are proportional to identity operators, we have $S(\rho_{A_j}\|\Phi_j(\rho_{A_j}))=0$. Due to the evolved state under MCNOT gate possess identical structure with $\rho=\lambda_1\delta_1+\lambda_2\delta_2$, the quantum discord is obtained by a systematic extension of the bipartite quantum discord. Hence, the GQD is simply given by

$$D\left(U_{\rm MCN}\left(\frac{1}{2}(I+n_1\sigma_1)\otimes|0\rangle\langle0|^{\otimes N-1}\right)U_{\rm MCN}^\dagger\right)=-H(n_1)/2, \tag{42}$$

where $H(n_1)=-(1+n_1)\log(1+n_1)-(1-n_1)\log(1-n_1)$. Considering Eq. (31), we have

$$C(\rho_A)=D\left(U_{\rm MCN}\left(\frac{1}{2}(I+n_1\sigma_1)\otimes|0\rangle\langle0|^{\otimes N-1}\right)U_{\rm MCN}^\dagger\right). \tag{43}$$

For multi-qubit systems $\rho_A\otimes\rho_B^{\otimes N-1}$, if the initial quantum state with particular conditions as above, the coherence of the control state is equal to the dynamical discord of the evolved states under the MCNOT gate.

6. Conclusions

In summary, we provide a mathematically rigorous approach to quantify coherence with dynamical discord in two specific quantum systems, that is, the bipartite pure states and the BD states. In order to directly compare the quantities between the coherence and the dynamical discord, we quantify these two kinds of resources with the relative entropy of coherence and discord, which can be calculated for these two specific quantum states. In the resource conversion from coherence to discord, we restrict the incoherent operations to the CNOT gates. In our results, we show that any nonzero amount of coherence in the control qubit A can be converted into the discord between A and an initially incoherent target qubit B under CNOT gate. Particularly, we find the conditions of the coherence of the control state are equal to the dynamical discord of the evolved state. Namely, the initial bipartite pure product state consists of any coherent state and a pure incoherent state, , the initial bipartite mixed product state consists of the control state in the form of $\boldsymbol{n}=\{n_1,n_2=0,n_3=0\}$ and the target state in the form of $\boldsymbol{n}'=\{n_1'=0,n_2'=0,n_3'\}$, and the parameters satisfy $|n_1|\leq|n_3'|$. Note that, this results is different from that in Ref. [20]. We also show that quantifying the coherence with the dynamical discord possesses more advantages than the dynamical entanglement. We extend the results of bipartite quantum systems to multipartite quantum systems, for the multipartite correlated state, and the discord is calculated by the global quantum discord. This builds the conditions under which the coherence as a universal resource can be converted into the discord of various quantum systems.

These results demonstrate a fundamental qualitative and quantitative equivalence between the coherence and discord, and provide an specific operational scheme to interchange these two quantum resources for suitable applications in quantum technologies.

Via this process of resource conversion, we show the relations between the coherent states and the genuine correlated states by the coefficients of the initial quantum states. Through revealing the connections between the coherence of single quantum systems and the dynamical discord in particular composite systems, this work provides useful methods for realizing experiment of quantum resources conversion. It is important to realize a unification between the notions of coherence and discord from a quantum informational viewpoint, and we confirm this work makes a practical step in this direction.

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