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SPECIAL TOPIC — Non-Hermitian physics

Geometry of time-dependent $\mathcal{PT}$-symmetric quantum mechanics*

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A new type of quantum theory known as time-dependent $\mathcal{PT}$-symmetric quantum mechanics has received much attention recently. It has a conceptually intriguing feature of equipping the Hilbert space of a $\mathcal{PT}$-symmetric system with a time-varying inner product. In this work, we explore the geometry of time-dependent $\mathcal{PT}$-symmetric quantum mechanics. We find that a geometric phase can emerge naturally from the cyclic evolution of a $\mathcal{PT}$-symmetric system, and further formulate a series of related differential-geometry concepts, including connection, curvature, parallel transport, metric tensor, and quantum geometric tensor. These findings constitute a useful, perhaps indispensible, tool to investigate geometric properties of $\mathcal{PT}$-symmetric systems with time-varying system's parameters. To exemplify the application of our findings, we show that the unconventional geometric phase [*Phys. Rev. Lett.* **91** 187902 (2003)], which is the sum of a geometric phase and a dynamical phase proportional to the geometric phase, can be expressed as a single geometric phase unveiled in this work.

Keywords: time-dependent $\mathcal{PT}$-symmetric quantum mechanics, geometry, time-varying inner product, unconventional geometric phase

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1. Introduction

Standard quantum mechanics is built upon a fixed Hilbert space, with the associated inner product of two complex vectors being defined by the Dirac bra-ket notation. However, such a quantum theory may not consistently treat physical problems with time-varying Hilbert spaces. For example, it is not obvious how to depict the dynamics of a particle in an infinitely deep square-well potential with a moving boundary, of which the instantaneous Hilbert space changes with time. It is thus necessary and motivating to formulate new types of quantum theories that allow the inner product to change along a parameter path.

Inspired by the pioneering work of Bender and Boettcher concerning time-independent parity-time-reversal-symmetric ($\mathcal{PT}$-symmetric) quantum mechanics ($\mathcal{PT}$QM),[1] Gong and Wang proposed a Schrödinger-like equation capable of explicitly accounting for a time-varying inner product.[2] Such a proposal was further enriched and developed into a conceptual framework applicable to generic non-Hermitian systems,[3] in which a series of notions like the time-dependent Hilbert space, the observable, and the measurement postulate are formulated. These results lead to a consistent quantum theory extending the time-independent $\mathcal{PT}$QM into the time-dependent domain, referred to as time-dependent $\mathcal{PT}$QM hereafter.

The advent of time-dependent $\mathcal{PT}$QM has spurred a renewed interest in the fundamentals of quantum mechanics.[4–10] In particular, how to construct the time-varying inner product was demonstrated in Refs. [5,6] and how to define the energy observable for a time-dependent Hilbert space was discussed in Refs. [3,9]. Further developments are witnessed by the reexaminations of some interesting issues in statistics mechanics,[11–15] such as the Jarzynski equality[11,13] and the Crooks fluctuation theorem.[3,12]

In this work, we explore the geometrical aspects of time-dependent $\mathcal{PT}$QM, which is a fascinating topic for the following reasons. First, given the fact that a time-varying inner product is always excluded in standard quantum mechanics, its nontrivial interplay with other concepts in quantum physics, e.g., the Berry phase,[16] is still not well understood in a unified fashion. Second, even in the absence of a time-varying inner product, geometric aspects of standard quantum mechanics are known to be of profound importance in various frontier topics of quantum information science and condensed-matter physics. One thus anticipates that physics arising from a time-varying inner product shall advance our fundamental understanding of the profound role of geometry in time-dependent $\mathcal{PT}$QM. Last but not least, there have been ongoing investigations of physical properties, especially topological proper-

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ties, of $\mathcal{PT}$-symmetric systems recently.[17–38] In view of this, a systematic inspection of the geometry of time-dependent $\mathcal{PT}$QM would be a useful, perhaps indispensible, reference point to tackle physical problems involving time-varying system's parameters.

The purpose of this work is to present comprehensive and rigorous results regarding the geometry of time-dependent $\mathcal{PT}$QM. To this end, we start with the identification of a geometric phase (GP) that emerges naturally from a cyclic evolution of a $\mathcal{PT}$-symmetric system. Then, with the motivation of revealing the geometry underlying the GP, we formulate, in succession, a series of related differential-geometry concepts, including connection, curvature, parallel transport, metric tensor, and quantum geometric tensor. Of particular interest is the metric tensor advocated here, which may be Riemannian or pseudo-Riemannian, depending on the physical context under consideration. Its pseudo-Riemannian feature is absent in standard quantum mechanics. As detailed below, the findings of this work are applicable to a rather general physical context and include the results of Refs. [4,10] as special cases.

To exemplify the application of our findings, we revisit one well-known example displaying the so-called unconventional GP,[39] which is the sum of a GP and a dynamical phase (DP) proportional to the GP, and as such, may not be well understood within the geometric theory in standard quantum mechanics. Here, we show that the unconventional GP, instead of being the sum of a GP and a DP, can be expressed as a single GP found in this paper, with the associated metric tensor categorized as a pseudo-Riemannian metric. This provides an interesting interpretation of the geometric nature of the unconventional GP.

This paper is organized as follows. In Section 2, we recapitulate some fundamentals of $\mathcal{PT}$QM. In Section 3, we identify the GP. In Section 4, we formulate a series of differential geometry concepts, including connection, curvature, parallel transport, metric tensor, and quantum geometric tensor. Section 5 presents our interpretation of the unconventional GP. Section 6 provides a general formula for GP which unifies several real and gauge-invariant GPs in the literature, and Section 7 concludes this work.

2. Fundamentals of $\mathcal{PT}$QM

To present our findings clearly, we recapitulate some notions in time-dependent $\mathcal{PT}$QM,[2,3] such as the metric operator, the physical Hilbert space, the physical observable, the Schrödinger-like equation, and the density operator.

A positive-definite operator W is said to be the metric operator for a $\mathcal{PT}$-symmetric system S with unbroken $\mathcal{PT}$ symmetry if it satisfies

$$WH = H^\dagger W, \tag{1}$$

where H denotes the Hamiltonian of the system S. Usually, H depends on some system's parameters $\lambda := (\lambda^1, \ldots, \lambda^m)$, which belong to a parameter manifold M [$m = \dim(M)$] representing the classical configuration of control fields. In the following, we denote the metric operator and the Hamiltonian by $W(\lambda)$ and $H(\lambda)$, respectively, in order to indicate their dependence on the parameters λ.

The physical Hilbert space of S, denoted as $\mathcal{H}(\lambda)$, is defined to be the Hilbert space endowed with the following inner product:

$$\prec \cdot, \cdot \succ_\lambda := \langle \cdot | W(\lambda) | \cdot \rangle, \tag{2}$$

which is dependent on λ. It is worth noting that two states $|\psi\rangle \in \mathcal{H}(\lambda)$ and $|\phi\rangle \in \mathcal{H}(\lambda')$ are not comparable when $\lambda \neq \lambda'$, as they belong to two different physical Hilbert spaces.

An operator O is said to be a physical observable of S if it is a Hermitian operator under the new inner product in Eq. (2),

$$\prec O\cdot, \cdot \succ_\lambda = \prec \cdot, O\cdot \succ_\lambda, \tag{3}$$

or equivalently,

$$W(\lambda)O = O^\dagger W(\lambda). \tag{4}$$

Apparently, $H(\lambda)$ is a physical observable. It has been argued that $H(\lambda)$ can be regarded as the energy observable of S.[3,9]

An evolution of S is associated with a parameter path $\lambda_t \in M$ over a time interval $[0, \tau]$. For this, the physical Hilbert space moves with time, and the evolving state $|\psi(t)\rangle$ at time t belongs to $\mathcal{H}(\lambda_t)$. The Schrödinger-like equation guaranteeing the unitarity of the evolution is found to be ($\hbar = 1$)[2]

$$\mathrm{i}\partial_t |\psi(t)\rangle = [H(t) + \mathrm{i}K(t)]\,|\psi(t)\rangle, \tag{5}$$

where $H(t) = H(\lambda_t)$ and

$$K(t) = -\frac{1}{2} W^{-1}(\lambda_t)\partial_t W(\lambda_t) \tag{6}$$

is a gauge field arising from the geometric nature of the underlying Hilbert-space bundle.[9] That is, the inner product (2) of two arbitrary initial states is preserved during the evolution.

Let $\mathcal{N}(\lambda)$ denote the set of normalized states in $\mathcal{H}(\lambda)$,

$$\mathcal{N}(\lambda) := \{|\psi\rangle \in \mathcal{H}(\lambda) | \prec \psi, \psi \succ_\lambda = 1\}. \tag{7}$$

For a state $|\psi\rangle \in \mathcal{N}(\lambda)$, the associated density operator ρ is defined to be[3,10]

$$\rho = |\psi\rangle\langle\psi| W(\lambda). \tag{8}$$

It is easy to see that ρ is a positive operator with respect to the inner product (2) and satisfies $\mathrm{tr}(\rho) = 1$ and $\rho^2 = \rho$; that is, it fulfills the conditions of being a density operator for a pure state. Similar to the two states $|\psi\rangle \in \mathcal{N}(\lambda)$ and $|\phi\rangle \in \mathcal{N}(\lambda')$, their associated density operators $\rho = |\psi\rangle\langle\psi| W(\lambda)$ and $\sigma = |\phi\rangle\langle\phi| W(\lambda')$ can be regarded as identical if and only if $\lambda = \lambda'$ and $|\psi\rangle = \mathrm{e}^{\mathrm{i}\vartheta}|\phi\rangle$ for some $\vartheta \in \mathbb{R}$.

3. Geometric phase

Let us consider the situation that the evolving state $|\psi(t)\rangle$ of S returns to its initial physical state, $|\psi(\tau)\rangle = \mathrm{e}^{\mathrm{i}\alpha}|\psi(0)\rangle$, and moreover, the system's parameters return to their initial values, $\lambda_\tau = \lambda_0$. This defines a curve of density operators

$$C : t \in [0,\tau] \mapsto \rho(t), \tag{9}$$

with

$$\rho(t) := |\psi(t)\rangle\langle\widetilde{\psi}(t)|. \tag{10}$$

Here, we introduce the notation

$$|\widetilde{\psi}\rangle := W(\lambda)|\psi\rangle \tag{11}$$

and express ρ as $\rho = |\psi\rangle\langle\widetilde{\psi}|$ for convenience. Since $\rho(0)$ and $\rho(\tau)$ are identical, C in Eq. (9) with Eq. (10) represents a closed curve.

Now, define an auxiliary state

$$|\phi_a(t)\rangle := \mathrm{e}^{-\mathrm{i}f(t)}|\psi(t)\rangle, \tag{12}$$

with $f(\tau) - f(0) = \alpha$. By definition, $|\phi_a(\tau)\rangle = |\phi_a(0)\rangle$. Substituting $|\psi(t)\rangle = \mathrm{e}^{\mathrm{i}f(t)}|\phi_a(t)\rangle$ into Eq. (5) and contracting its both sides with $\langle\widetilde{\phi}_a(t)|$, we have

$$\begin{aligned}\dot{f}(t) \;=\; & -\prec \phi_a(t), [H(t)+\mathrm{i}K(t)]\phi_a(t)\succ_{\lambda_t} \\ & +\mathrm{i}\prec \phi_a(t), \dot{\phi}_a(t)\succ_{\lambda_t},\end{aligned} \tag{13}$$

where the dot denotes the time derivative. Integrating the above equation and simplifying it by noting that

$$\begin{aligned}\prec \phi_a(t), K(t)\phi_a(t)\succ_{\lambda_t} \;=\; & \frac{1}{2}\prec \dot{\phi}_a(t), \phi_a(t)\succ_{\lambda_t} \\ & +\frac{1}{2}\prec \phi_a(t), \dot{\phi}_a(t)\succ_{\lambda_t},\end{aligned}$$

we obtain $\alpha = \beta + \gamma$, with

$$\beta := -\int_0^\tau \mathrm{d}t \prec \phi_a(t), H(t)\phi_a(t)\succ_{\lambda_t}, \tag{14}$$

$$\gamma := -\Im\int_0^\tau \mathrm{d}t \prec \phi_a(t), \dot{\phi}_a(t)\succ_{\lambda_t}. \tag{15}$$

Equation (14) indicates that the phase β depends explicitly on the Hamiltonian and thus represents a DP. On the contrary, the phase γ, as a factor obtained by removing the DP from the total phase, depends solely upon the closed curve C in Eq. (9), as will be proved shortly. Therefore, γ is our GP. It is interesting to note that DP and GP for non-Hermitian systems were usually defined to be complex in the previous works (see, e.g., Refs. [40,41]). By contrast, both the DP and GP defined here are real phase factors. Indeed, since $H(t)$ is a physical observable satisfying Eq. (3), the term $\prec \phi_a(t), H(t)\phi_a(t)\succ_{\lambda_t}$ appearing in Eq. (14) is real, which leads to the fact that β is real. The fact that γ is real follows immediately from the operation of taking the imaginary part of a complex number in Eq. (15).

To prove that γ is uniquely determined by C, we resort to the following gauge-invariant formula of γ:

$$\gamma = \arg\prec \phi(0), \phi(\tau)\succ_{\lambda_0} -\Im\int_0^\tau \mathrm{d}t \prec \phi(t), \dot{\phi}(t)\succ_{\lambda_t}. \tag{16}$$

Here, $|\phi(t)\rangle \in \mathcal{H}(\lambda_t)$ stands for a gauge which satisfies $|\phi(t)\rangle = \mathrm{e}^{\mathrm{i}\vartheta(t)}|\phi_a(t)\rangle$ for some real function $\vartheta(t)$. It is easy to verify that γ in Eq. (16) is independent of specific choices of $|\phi(t)\rangle$. Note that there is a special gauge, denoted by $|\phi_b(t)\rangle$, that satisfies

$$\Im\prec \phi_b(t), \dot{\phi}_b(t)\succ_{\lambda_t} = 0. \tag{17}$$

From Eqs. (16) and (17), it follows immediately that

$$\gamma = \arg\prec \phi_b(0), \phi_b(\tau)\succ_{\lambda_0}. \tag{18}$$

On the other hand, noting that $\rho(t) = |\phi_b(t)\rangle\langle\widetilde{\phi}_b(t) = |\phi_b(t)\rangle\langle\phi_b(t)|W(\lambda_t)$, we have, after direct calculations,

$$\dot{\rho}(t)|\phi_b(t)\rangle = |\dot{\phi}_b(t)\rangle - \prec \phi_b(t), \dot{\phi}_b(t)\succ_{\lambda_t}|\phi_b(t)\rangle. \tag{19}$$

Here, the fact

$$\begin{aligned}\langle\phi_b(t)|\dot{W}(\lambda_t)|\phi_b(t)\rangle \;=\; & -\prec \dot{\phi}_b(t), \phi_b(t)\succ_{\lambda_t} \\ & -\prec \phi_b(t), \dot{\phi}_b(t)\succ_{\lambda_t}\end{aligned}$$

has been used. Rewriting Eq. (19) by using Eq. (17) gives

$$|\dot{\phi}_b(t)\rangle = \left[\dot{\rho}(t) + \mathrm{Re}\prec \phi_b(t), \dot{\phi}_b(t)\succ_{\lambda_t}\right]|\phi_b(t)\rangle. \tag{20}$$

Integrating this differential equation, we have

$$|\phi_b(t)\rangle = T\,\mathrm{e}^{\int_0^t \mathrm{d}s\dot{\rho}(s)}|\phi_b(0)\rangle\,\mathrm{e}^{\int_0^t \mathrm{d}s\mathrm{Re}\prec\phi_b(s),\dot{\phi}_b(s)\succ_{\lambda_s}}, \tag{21}$$

where T denotes the time-ordering operator. Substituting Eq. (21) into Eq. (18), we arrive at an expression for γ exclusively in terms of $\rho(s)$:

$$\gamma = \arg\mathrm{tr}\left[\rho(0)T\,\mathrm{e}^{\int_0^t \mathrm{d}s\dot{\rho}(s)}\right]. \tag{22}$$

Here, the term $\mathrm{e}^{\int_0^t \mathrm{d}s\mathrm{Re}\prec\phi_b(s),\dot{\phi}_b(s)\succ_{\lambda_s}}$ has been neglected, since it is a real and positive number and thus makes no contribution. Equation (22) clearly shows that γ is uniquely determined by C.

It is worth noting that the result just presented includes the results of Refs. [4,10] as special cases. Indeed, the GP obtained in Refs. [4,10] is for the eigenstate of $H(\lambda)$, and therefore, is a counterpart of Berry's phase.[16] In contrast, the GP γ obtained here is for a generic cyclic state which may or may not be the eigenstate of $H(\lambda)$. In this sence, γ is a counterpart of Aharonov–Anandan's phase.[42] Moreover, it has been shown[3] that the dynamics of a generic non-Hermitian system can be described by the Schrödinger-like equation (5) with Eq. (6). This implies that the GP found here is applicable to a much more general physical context compared with Refs. [4,10].

4. Differential geometry concepts

We first set the stage of our analysis by introducing several notions needed. One of the notions needed is the space of rays for time-dependent $\mathcal{PT}$QM. For each fixed λ, we define a subset of the space of rays, denoted as $\mathcal{R}(\lambda)$, which is associated with the physical Hilbert space $\mathcal{H}(\lambda)$. A ray in $\mathcal{R}(\lambda)$, represented by the symbol $[|\psi\rangle]$, is an equivalence class, $[|\psi\rangle] := \{|\phi\rangle \in \mathcal{H}(\lambda) \,|\, |\phi\rangle = c|\psi\rangle \text{ for some non-zero } c \in \mathbb{C}\}$, obtained by identifying states $|\phi\rangle$ in $\mathcal{H}(\lambda)$ which differ from $|\psi\rangle$ only by an overall rescaling. The space of rays itself, denoted by $\mathcal{R}$, is defined to be the disjoint union of $\mathcal{R}(\lambda)$ over $\lambda \in M$, i.e., $\mathcal{R} := \bigsqcup_{\lambda\in M}\mathcal{R}(\lambda)$.

There is a one–one correspondence between rays in $\mathcal{R}(\lambda)$ and density operators over $\mathcal{H}(\lambda)$. Indeed, given a ray $[|\psi\rangle]$ in $\mathcal{R}(\lambda)$, one can assign to it a unique density operator ρ over $\mathcal{H}(\lambda)$, which is $\rho = |\psi\rangle\langle\widetilde{\psi}|/\prec\psi,\psi\succ_\lambda$. Conversely, given a density operator ρ over $\mathcal{H}(\lambda)$, one can express it as $\rho = |\psi\rangle\langle\widetilde{\psi}|$ for some $|\psi\rangle \in \mathcal{N}(\lambda)$. Then, the unique ray associated to ρ is $[|\psi\rangle]$. Under the effect of this one–one correspondence, a curve in $\mathcal{R}$ can be simply understood as a curve of density operators. Now, it becomes clear that C in Eq. (9) is actually a curve in the space of rays $\mathcal{R}$.

To obtain a fiber-bundle structure, we define a projection map, denoted as Π, which maps a state $|\psi\rangle \in \mathcal{N}(\lambda)$ to the ray $[|\psi\rangle] \in \mathcal{R}(\lambda)$, i.e., $\Pi(|\psi\rangle) = [|\psi\rangle]$. This is a map from the disjoint union of $\mathcal{N}(\lambda)$ over $\lambda \in M$, i.e., $\bigsqcup_{\lambda\in M}\mathcal{N}(\lambda)$, to the space of rays $\mathcal{R}$. Under the influence of Π, the disjoint union of $\mathcal{N}(\lambda)$ over $\lambda \in M$ becomes a principle $U(1)$-bundle with $\mathcal{R}$ acting as the base manifold. Indeed, since all states of the kind $\mathrm{e}^{\mathrm{i}\theta}|\psi\rangle \in \mathcal{N}(\lambda)$, θ real, are mapped via Π to the same ray $[|\psi\rangle] \in \mathcal{R}(\lambda)$, the fibers on top of a point of $\mathcal{R}$ constitute a $U(1)$ group. Hereafter, we denote by $P(\mathcal{R},U(1))$ this principle $U(1)$-bundle.

A coordinate system can be established by considering a local patch on $\mathcal{R}$ and the region of $P(\mathcal{R},U(1))$ over the patch. Let $(\lambda^1,\ldots,\lambda^m,\lambda^{m+1},\ldots,\lambda^{m+n})$ be the local coordinates of a point of $\mathcal{R}$ on the patch. Here, λ^μ, $\mu = 1,\ldots,m$, are the system's parameters as before, used to specify which subset, $\mathcal{R}(\lambda)$, the point belongs to. The remainder λ^μ, $\mu = m+1,\ldots,m+n$, are used to represent local coordinates parameterizing the manifold $\mathcal{R}(\lambda)$ [$n = \dim(\mathcal{R}(\lambda))$]. Then, the local coordinates of a point of $P(\mathcal{R},U(1))$ can be expressed as $(\theta,\lambda^1,\ldots,\lambda^m,\lambda^{m+1},\ldots,\lambda^{m+n})$, where $\theta \in \mathbb{R}$ is defined up to an integer multiple of 2π.[43] Using these local coordinates, we can express density operators and states in a coordinate-dependent form. Since a density operator ρ is a point of $\mathcal{R}$, it can be represented as $\rho = \rho(\lambda^1,\ldots,\lambda^{m+n})$. Likewise, a state $|\phi\rangle$, as a point of $P(\mathcal{R},U(1))$, can be written as $|\phi\rangle = |\phi(\theta,\lambda^1,\ldots,\lambda^{m+n})\rangle$.

With these notions, we may now start to formulate differential geometry concepts, including connection, curvature, parallel transport, metric tensor, and quantum geometric tensor.

4.1. Connection

A curve in $P(\mathcal{R},U(1))$ can be expressed as $|\phi(t)\rangle := |\phi(\theta_t,\lambda_t^1,\ldots,\lambda_t^{m+n})\rangle$. Associated with each curve $|\phi(t)\rangle$, there is a tangent vector $\mathrm{d}|\phi(t)\rangle/\mathrm{d}t$. Conversely, every tangent vector can be produced in this way, i.e., by acting the operator $\mathrm{d}/\mathrm{d}t$ on a curve $|\phi(t)\rangle$ in $P(\mathcal{R},U(1))$. A connection on $P(\mathcal{R},U(1))$ is specified by splitting all tangent vectors into vertical and horizontal parts, that is, $|\dot\phi(t)\rangle = |\dot\phi_v(t)\rangle + |\dot\phi_h(t)\rangle$. Like tangent vectors, horizontal parts $|\dot\phi_h(t)\rangle$ are produced by acting the covariant derivative $D/\mathrm{d}t$ on curves $|\phi(t)\rangle$, i.e., $|\dot\phi_h(t)\rangle = D|\phi(t)\rangle/\mathrm{d}t$. The action of the covariant derivative reads $D|\phi(t)\rangle/\mathrm{d}t = \dot\lambda^\mu D_\mu|\phi(t)\rangle$, with $D_\mu := \partial_\mu - A_\mu\partial_\theta$. Here, $\partial_\mu := \partial/\partial\lambda^\mu$, $\partial_\theta := \partial/\partial\theta$, and A_μ is a real coefficient representing the component of a connection one-form $A := A_\mu\,\mathrm{d}\lambda^\mu$, to be specified later on. Throughout, the Einstein summation convention is assumed, and the summation is understood to be taken over all the indices $\mu = 1,\ldots,m+n$.

To specify the connection A, we adopt the following definition:

$$|\dot\phi_h(t)\rangle := |\dot\phi(t)\rangle - \mathrm{i}\Im\prec\phi(t),\dot\phi(t)\succ_{\lambda_t}|\phi(t)\rangle. \tag{23}$$

Equation (23) implies that the horizontal part satisfies

$$\Im\prec\phi(t),\dot\phi_h(t)\succ_{\lambda_t} = 0. \tag{24}$$

Inserting $|\dot\phi_h(t)\rangle = \dot\lambda^\mu D_\mu|\phi(t)\rangle$ into Eq. (24) and noting that $\dot\lambda^\mu$ can take arbitrary values, we have

$$\Im\prec\phi(t),D_\mu\phi(t)\succ_{\lambda_t} = 0. \tag{25}$$

Rewriting this equation by using $D_\mu = \partial_\mu - A_\mu\partial_\theta$ gives

$$A_\mu = \frac{\Im\prec\phi,\partial_\mu\phi\succ_\lambda}{\Im\prec\phi,\partial_\theta\phi\succ_\lambda}. \tag{26}$$

Here, t is omitted for ease of notation. On the other hand, by comparing an infinitesimal $U(1)$ action

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\delta\theta}|\phi(\theta,\lambda^1,\ldots,\lambda^{m+n})\rangle \;=\; &|\phi(\theta,\lambda^1,\ldots,\lambda^{m+n})\rangle\\ &+\mathrm{i}\delta\theta|\phi(\theta,\lambda^1,\ldots,\lambda^{m+n})\rangle\end{aligned}$$

with a Taylor-series expansion

$$\begin{aligned}|\phi(\theta+\delta\theta,\lambda^1,\ldots,\lambda^{m+n})\rangle \;=\; &|\phi(\theta,\lambda^1,\ldots,\lambda^{m+n})\rangle\\ &+\delta\theta\partial_\theta|\phi(\theta,\lambda^1,\ldots,\lambda^{m+n})\rangle,\end{aligned}$$

we deduce that

$$\partial_\theta|\phi(\theta,\lambda^1,\ldots,\lambda^{m+n})\rangle = \mathrm{i}|\phi(\theta,\lambda^1,\ldots,\lambda^{m+n})\rangle, \tag{27}$$

where we have used the fact

$$\mathrm{e}^{\mathrm{i}\delta\theta}|\phi(\theta,\lambda^1,\ldots,\lambda^{m+n})\rangle = |\phi(\theta+\delta\theta,\lambda^1,\ldots,\lambda^{m+n})\rangle.$$

Substituting the above equality, i.e., $\partial_\theta|\phi\rangle = i|\phi\rangle$, into Eq. (26), we arrive at the explicit expression of A_μ,

$$A_\mu = \Im \prec \phi, \partial_\mu \phi \succ_\lambda, \tag{28}$$

defining the connection A with the one-form $A = A_\mu \mathrm{d}\lambda^\mu$. The specific form of A computed from Eq. (28) is gauge-dependent, i.e., depends on the specific choice of $|\phi\rangle$. Indeed, under gauge transformations $|\phi\rangle \to \mathrm{e}^{\mathrm{i}\vartheta}|\phi\rangle$, where $\vartheta = \vartheta(\lambda^1, \ldots, \lambda^{m+n})$, A transforms as $A \to A + \mathrm{d}\vartheta$, i.e., as proper gauge potentials.

Equation (28) represents the connection responsible for the appearance of our GP. To see this, we examine the curve $|\phi_a(t)\rangle$, for which $A_\mu = \Im \prec \phi_a(t), \partial_\mu \phi_a(t) \succ_{\lambda_t}$. Rewriting Eq. (15) by using $\mathrm{d}/\mathrm{d}t = \dot{\theta}\partial_\theta + \dot{\lambda}^\mu \partial_\mu$ and $\partial_\theta|\phi_a(t)\rangle = \mathrm{i}|\phi_a(t)\rangle$ gives

$$\gamma = -\int_0^\tau \left(\mathrm{d}\theta + \Im \prec \phi_a, \partial_\mu \phi_a \succ_\lambda \mathrm{d}\lambda^\mu\right). \tag{29}$$

Here, t is omitted again for ease of notation. Since $|\phi_a(\tau)\rangle = |\phi_a(0)\rangle$, the coordinates of $|\phi_a(t)\rangle$ satisfy $\theta_\tau = \theta_0 \pmod{2\pi}$ and $\lambda_\tau^\mu = \lambda_0^\mu$, $\mu = 1, \ldots, m+n$. Besides, the relation between $\rho(t)$ in the curve C and $|\phi_a(t)\rangle$, i.e., $\rho(t) = |\phi_a(t)\rangle\langle\widetilde{\phi}_a(t)|$, implies that the coordinates of $\rho(t)$ are the λ^μ-components of coordinates of $|\phi_a(t)\rangle$. That is, the coordinates of $\rho(t)$ in the curve C are $(\lambda_t^1, \ldots, \lambda_t^{m+n})$. With the above knowledge, we deduce from Eq. (29) that

$$\gamma = -\oint_C A \pmod{2\pi}, \tag{30}$$

where $A = A_\mu \mathrm{d}\lambda^\mu$, with $A_\mu = \Im \prec \phi_a, \partial_\mu \phi_a \succ_\lambda$. Equation (30) clearly shows that A in Eq. (28) is indeed the connection associated with our GP. So, A can be regarded as a counterpart of the Berry connection.[16]

4.2. Curvature

Using Stokes' theorem, we deduce from Eq. (30) that

$$\gamma = -\int_S \Omega \pmod{2\pi}, \tag{31}$$

where S is any surface enclosed by the curve C, and $\Omega := \mathrm{d}A$ represents the curvature associated to our GP, with d denoting the exterior derivative. Ω is a counterpart of the Berry curvature.[16] To obtain an explicit expression of Ω, we insert $A = A_\mu \mathrm{d}\lambda^\mu$ into $\Omega = \mathrm{d}A$, and obtain $\Omega = \partial_\mu A_\nu \mathrm{d}\lambda^\mu \wedge \mathrm{d}\lambda^\nu$, where $\wedge$ denotes the wedge product. Using the anti-commutativity of the wedge product, i.e., $\mathrm{d}\lambda^\mu \wedge \mathrm{d}\lambda^\nu = -\mathrm{d}\lambda^\nu \wedge \mathrm{d}\lambda^\mu$, we can rewrite the above expression as

$$\Omega = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\mathrm{d}\lambda^\mu \wedge \mathrm{d}\lambda^\nu. \tag{32}$$

So, the components of Ω read

$$\Omega_{\mu\nu} = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu), \tag{33}$$

i.e., $\Omega = \Omega_{\mu\nu}\mathrm{d}\lambda^\mu \wedge \mathrm{d}\lambda^\nu$. Substituting $A_\mu = \Im \prec \phi_a, \partial_\mu \phi_a \succ_\lambda$ into Eq. (33) and noting that $\prec \phi_a, \partial_\mu \phi_a \succ_\lambda = \langle\widetilde{\phi}_a|\partial_\mu \phi_a\rangle$, we obtain

$$\Omega_{\mu\nu} = \frac{1}{2}\Im\left(\langle\partial_\mu \widetilde{\phi}_a|\partial_\nu \phi_a\rangle + \langle\partial_\mu \phi_a|\partial_\nu \widetilde{\phi}_a\rangle\right). \tag{34}$$

Here, we have used the fact $\Im\langle\partial_\nu \widetilde{\phi}_a|\partial_\mu \phi_a\rangle = -\Im\langle\partial_\mu \phi_a|\partial_\nu \widetilde{\phi}_a\rangle$.

It is worth noting that $\Omega_{\mu\nu}$ is gauge-invariant. This point can be verified straightforwardly by plugging $|\phi\rangle = \mathrm{e}^{\mathrm{i}\vartheta}|\phi_a\rangle$ into Eq. (34). As an immediate consequence, $|\phi_a\rangle$ appearing in Eq. (34) can be replaced by any gauge $|\phi\rangle$ with $|\phi\rangle = \mathrm{e}^{\mathrm{i}\vartheta}|\phi_a\rangle$ for some ϑ, that is,

$$\Omega_{\mu\nu} = \frac{1}{2}\Im\left(\langle\partial_\mu \widetilde{\phi}|\partial_\nu \phi\rangle + \langle\partial_\mu \phi|\partial_\nu \widetilde{\phi}\rangle\right). \tag{35}$$

It is also worth noting that like the Berry curvature,[16] $\Omega_{\mu\nu}$ is a real anti-symmetric tensor, i.e., $\Omega_{\mu\nu} = \Omega_{\mu\nu}^*$ and $\Omega_{\mu\nu} = -\Omega_{\nu\mu}$.

4.3. Parallel transport

A choice of connection is equivalent to a notion of parallel transport. By definition, a curve $|\phi(t)\rangle$ is said to be parallel transported along a curve in the base manifold if the vertical part of its tangent vector, i.e., $|\dot{\phi}_v(t)\rangle$, vanishes. Using Eq. (23) and noting the relation $|\dot{\phi}(t)\rangle = |\dot{\phi}_v(t)\rangle + |\dot{\phi}_h(t)\rangle$, we have

$$|\dot{\phi}_v(t)\rangle = \mathrm{i}\Im \prec \phi(t), \dot{\phi}(t) \succ_{\lambda_t} |\phi(t)\rangle. \tag{36}$$

Therefore, $|\dot{\phi}_v(t)\rangle$ vanishes if and only if

$$\Im \prec \phi(t), \dot{\phi}(t) \succ_{\lambda_t} = 0, \tag{37}$$

representing the parallel transport condition associated with the connection A. Equation (37) is a counterpart of the Berry–Simon parallel transport condition.[44] It depicts a parallel way of transporting $|\phi(t)\rangle$ along a curve in $\mathcal{R}$. Evidently, $|\phi_b(t)\rangle$ fulfills Eq. (37) and hence is parallel transported. This transport is along the closed curve C in Eq. (9), since $|\phi_b(t)\rangle\langle\widetilde{\phi}_b(t)| = \rho(t)$. Starting at an initial point $|\phi_b(0)\rangle$, the transport will end at a different point $|\phi_b(\tau)\rangle = \mathrm{e}^{\mathrm{i}\gamma}|\phi_b(0)\rangle$, as can be easily verified by using Eq. (18). The difference, known as holonomy, is precisely our GP γ.

4.4. Metric tensor

To obtain a metric tensor, we resort to the formula for the fidelity between two nearby density operators $\rho(\lambda^1, \ldots, \lambda^{m+n})$ and $\rho(\lambda^1 + \delta\lambda^1, \ldots, \lambda^{m+n} + \delta\lambda^{m+n})$.[10] It reads

$$F(\rho(\lambda^1,\ldots,\lambda^{m+n}),\rho(\lambda^1+\delta\lambda^1,\ldots,\lambda^{m+n}+\delta\lambda^{m+n}))$$
$$:= \mathrm{tr}|\rho^{1/2}(\lambda^1,\ldots,\lambda^{m+n})\rho(\lambda^1+\delta\lambda^1,\ldots,\lambda^{m+n}+\delta\lambda^{m+n})\rho^{1/2}(\lambda^1,\ldots,\lambda^{m+n})|^{1/2}. \quad (38)$$

Here, for an operator X, $|X| := \sqrt{X^{\ddagger}X}$, with $X^{\ddagger}$ being the Hermitian conjugate of X with respect to the inner product (2). This formula is almost of the same form as that in standard quantum mechanics. Note that

$$\rho(\lambda^1,\ldots,\lambda^{m+n}) = |\phi(\theta,\lambda^1,\cdots,\lambda^{m+n})\rangle\langle\widetilde{\phi}(\theta,\lambda^1,\cdots,\lambda^{m+n})|,$$
$$\rho(\lambda^1+\delta\lambda^1,\ldots,\lambda^{m+n}+\delta\lambda^{m+n}) = |\phi(\theta,\lambda^1+\delta\lambda^1,\cdots,\lambda^{m+n}+\delta\lambda^{m+n})\rangle\langle\widetilde{\phi}(\theta,\lambda^1+\delta\lambda^1,\cdots,\lambda^{m+n}+\delta\lambda^{m+n})|.$$

Here, the coordinate θ has no effect, since $|\phi(\theta,\lambda^1,\ldots,\lambda^{m+n})\rangle = \mathrm{e}^{\mathrm{i}\theta}|\phi(0,\lambda^1,\ldots,\lambda^{m+n})\rangle$.[45] Inserting these two expressions into the formula gives

$$F(\rho(\lambda^1,\ldots,\lambda^{m+n}),\rho(\lambda^1+\delta\lambda^1,\ldots,\lambda^{m+n}+\delta\lambda^{m+n}))$$
$$= |\langle\widetilde{\phi}(\theta,\lambda^1+\delta\lambda^1,\ldots,\lambda^{m+n}+\delta\lambda^{m+n})|\phi(\theta,\lambda^1,\ldots,\lambda^{m+n})\rangle\langle\widetilde{\phi}(\theta,\lambda^1,\ldots,\lambda^{m+n})|\phi(\theta,\lambda^1+\delta\lambda^1,\ldots,\lambda^{m+n}+\delta\lambda^{m+n})\rangle|^{\frac{1}{2}}. \quad (39)$$

In the spirit of Bures distance,[46] the distance element between $\rho(\lambda^1,\ldots,\lambda^{m+n})$ and $\rho(\lambda^1+\delta\lambda^1,\ldots,\lambda^{m+n}+\delta\lambda^{m+n})$ can be defined as

$$\mathrm{d}s^2 := 2[1-F(\rho(\lambda^1,\ldots,\lambda^{m+n}),\rho(\lambda^1+\delta\lambda^1,\ldots,\lambda^{m+n}+\delta\lambda^{m+n}))]. \quad (40)$$

Substituting Eq. (39) into this defining expression (40) and using Taylor-series expansions of

$$|\phi(\theta,\lambda^1+\delta\lambda^1,\ldots,\lambda^{m+n}+\delta\lambda^{m+n})\rangle$$

and

$$|\widetilde{\phi}(\theta,\lambda^1+\delta\lambda^1,\ldots,\lambda^{m+n}+\delta\lambda^{m+n})\rangle,$$

we obtain, up to second order,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}\lambda^{\mu}\mathrm{d}\lambda^{\nu}, \quad (41)$$

with

$$g_{\mu\nu} = \frac{1}{2}\Re\big(\langle\partial_\mu\widetilde{\phi}|\partial_\nu\phi\rangle - \langle\partial_\mu\widetilde{\phi}|\phi\rangle\langle\widetilde{\phi}|\partial_\nu\phi\rangle + \text{“terms } \widetilde{\phi}\leftrightarrow\phi\text{”}\big). \quad (42)$$

Here, “terms $\widetilde{\phi}\leftrightarrow\phi$” stands for $\langle\partial_\mu\phi|\partial_\nu\widetilde{\phi}\rangle - \langle\partial_\mu\phi|\widetilde{\phi}\rangle\langle\phi|\partial_\nu\widetilde{\phi}\rangle$. The derivation of Eq. (42) is the same as that in Ref. [10]. Hence, we omit it here. Equation (42) is the desired metric tensor, which generalizes the seminal metric tensor[46] and the one proposed in Ref. [10]. Like the seminal one[46] and the one in Ref. [10], the metric tensor (42) is a real symmetric tensor, i.e., $g_{\mu\nu} = g^*_{\mu\nu}$ and $g_{\mu\nu} = g_{\nu\mu}$.

The metric tensor in Eq. (42) reduces to the seminal one[46] when $W(\lambda) = \mathbb{1}$, where $\mathbb{1}$ denotes the identity operator. As an immediate consequence, $\mathrm{d}s^2$ may be Riemannian, as the seminal metric is Riemannian. However, $\mathrm{d}s^2$ may also be pseudo-Riemanniana, a case which is absent in standard quantum mechanics. To comprehend this case, one may recall the Minkowski metric $\mathrm{d}s^2_M$ in special relativity. $\mathrm{d}s^2_M$ is a pseudo-Riemannian metric describing the geometry of space–time, according to which a curve of events in space–time is said to be spacelike, lightlike, or timelike if $\mathrm{d}s^2_M > 0$, $\mathrm{d}s^2_M = 0$, or $\mathrm{d}s^2_M < 0$, respectively. Similar to the Minkowski metric, our metric $\mathrm{d}s^2$, when being pseudo-Riemannian, results in three types of evolutions, classified according to the sign of $\mathrm{d}s^2$, i.e., $\mathrm{d}s^2 > 0$, $\mathrm{d}s^2 = 0$, and $\mathrm{d}s^2 < 0$. Resorting to the language of special relativity, we refer to them as spacelike, lightlike, and timelike evolutions, respectively. An example showing the existence of timelike evolutions will be given later on.

4.5. Quantum geometric tensor

In standard quantum mechanics, the seminal quantum geometric tensor[46] is a gauge-invariant complex Hermitian tensor, with its imaginary part determining the Berry curvature and its real part inducing a metric tensor on the space of rays. Here, we advocate a quantum geometric tensor for time-dependent $\mathcal{PT}$QM.

The quantum geometric tensor advocated here reads

$$Q_{\mu\nu} = \frac{1}{2}\big(\langle\partial_\mu\widetilde{\phi}|\partial_\nu\phi\rangle - \langle\partial_\mu\widetilde{\phi}|\phi\rangle\langle\widetilde{\phi}|\partial_\nu\phi\rangle + \text{“terms } \widetilde{\phi}\leftrightarrow\phi\text{”}\big). \quad (43)$$

The $Q_{\mu\nu}$ shares all the features of the seminal quantum geometric tensor. First, $Q_{\mu\nu}$ is a gauge-invariant complex Hermitian tensor. Indeed, direct calculations show that

$$Q_{\mu\nu} = \frac{1}{2}\mathrm{tr}\big[\partial_\mu(|\phi\rangle\langle\widetilde{\phi}|)(1-|\phi\rangle\langle\widetilde{\phi}|)\partial_\nu(|\phi\rangle\langle\widetilde{\phi}|) + \partial_\mu(|\widetilde{\phi}\rangle\langle\phi|)(1-|\widetilde{\phi}\rangle\langle\phi|)\partial_\nu(|\widetilde{\phi}\rangle\langle\phi|)\big], \quad (44)$$

from which the gauge-invariance of $Q_{\mu\nu}$ follows immediately. Besides, it is easy to see that $Q_{\mu\nu} = Q^*_{\nu\mu}$, implying that $Q_{\mu\nu}$ is a complex Hermitian tensor. Second, the imaginary part of $Q_{\mu\nu}$ determines the curvature in Eq. (35). To see this, note that $\langle\partial_\mu\widetilde{\phi}|\phi\rangle = -\langle\partial_\mu\phi|\widetilde{\phi}\rangle^*$ and $\langle\widetilde{\phi}|\partial_\nu\phi\rangle = -\langle\phi|\partial_\nu\widetilde{\phi}\rangle^*$. It implies that $-\langle\partial_\mu\widetilde{\phi}|\phi\rangle\langle\widetilde{\phi}|\partial_\nu\phi\rangle - \langle\partial_\mu\phi|\widetilde{\phi}\rangle\langle\phi|\partial_\nu\widetilde{\phi}\rangle$, i.e., a

term appearing in Eq. (43), is real, which leads to $\Im Q_{\mu\nu} = \Im(\langle\partial_\mu\widetilde{\phi}|\partial_\nu\phi\rangle + \langle\partial_\mu\phi|\partial_\nu\widetilde{\phi}\rangle)/2$. That is,

$$\Im\left[Q_{\mu\nu}\right] = \Omega_{\mu\nu}. \tag{45}$$

Third, the real part of $Q_{\mu\nu}$ induces the metric tensor $g_{\mu\nu}$ in Eq. (42), i.e.,

$$\Re\left[Q_{\mu\nu}\right] = g_{\mu\nu}. \tag{46}$$

This point can be easily verified by comparing Eq. (42) with Eq. (43).

From Eqs. (45) and (46), it follows immediately that the quantum geometric tensor (43) depicts a unified picture: Its imaginary part gives the Berry curvature (35) and thus further determines the GP (31), whereas its real part induces the metric tensor (42) and thereby further determines the fidelity (39).

5. On the unconventional geometric phase

So far, we have presented our main findings, consisting of a GP and a series of differential geometry concepts, namely, connection, curvature, parallel transport, metric tensor, and quantum geometric tensor. To exemplify the application of our findings, we revisit one well-known example that yields an interesting GP, called the unconventional GP in the literature.[39]

The physical model studied in Ref. [39] is a harmonic oscillator. Its Hamiltonian reads

$$H(t) = \mathrm{i}\Omega_{\mathrm{D}}\left(a^\dagger \mathrm{e}^{-\mathrm{i}\delta t+\mathrm{i}\phi_{\mathrm{L}}} - a\,\mathrm{e}^{\mathrm{i}\delta t-\mathrm{i}\phi_{\mathrm{L}}}\right), \tag{47}$$

where Ω_{D}, δ, ϕ_{L} are real numbers, and $a^\dagger$ and a are the raising and lowering operators, respectively. The evolving state $|\varphi(t)\rangle$ of the system was shown to be $|\varphi(t)\rangle = \mathrm{e}^{\mathrm{i}\gamma(t)}|z(t)\rangle$, provided that the initial state is $|\varphi(0)\rangle = |0\rangle$. Here,

$$\begin{aligned} z(t) &= \mathrm{i}\Omega_{\mathrm{D}}(\mathrm{e}^{-\mathrm{i}\delta t}-1)\mathrm{e}^{\mathrm{i}\phi_{\mathrm{L}}}/\delta, \\ \gamma(t) &= -\frac{i}{2}\int_0^t \mathrm{d}s[z^*(s)\dot{z}(s) - \dot{z}^*(s)z(s)], \end{aligned}$$

and $|z\rangle$ denotes a coherent state. At the time $t = \tau := 2\pi/\delta$, the evolving state $|\varphi(t)\rangle$ returns to its initial physical state, i.e., $|\varphi(\tau)\rangle = \mathrm{e}^{\mathrm{i}\gamma(\tau)}|0\rangle$, and it acquires a total phase $\gamma(\tau)$. A remarkable observation made in Ref. [39] is that $\gamma(\tau)$ has a nonzero DP component but is still of geometric nature, i.e., it is an unconventional GP. In showing this, the DP and GP components of $\gamma(\tau)$, denoted respectively by γ_{d} and γ_{g}, were calculated, and found to satisfy $\gamma_{\mathrm{d}} = \eta\gamma_{\mathrm{g}}$ ($\eta \neq 0, -1$). So, $\gamma(\tau) = (1+\eta)\gamma_{\mathrm{g}}$, indicating that $\gamma(\tau)$ is of geometric nature as it inherits geometric features from γ_{g}. Despite this interesting observation, it remains an open question whether the unconventional GP itself admits a geometric interpretation or not.

To answer this question, we resort to the equivalence of a $\mathcal{PT}$-symmetric system with exact $\mathcal{PT}$-symmetry and a Hermitian system.[2] Consider the $\mathcal{PT}$-symmetric system with $H(z^1) = 0$ and $W(z^1) = \mathrm{e}^{2z^{1*}a}\mathrm{e}^{2z^1a^\dagger}$. Here, the manifold M is the complex plane and its point is designated by z^1. An evolution of the system is induced by a curve $z^1 = z^1(t)$ and determined purely by the gauge field $K(t)$, that is,

$$\mathrm{i}\partial_t|\psi(t)\rangle = \mathrm{i}K(t)|\psi(t)\rangle, \tag{48}$$

with

$$K(t) = -\left[\dot{z}^1(t)a^\dagger + \dot{z}^{1*}(t)a + 2z^1(t)\dot{z}^{1*}(t)\right]. \tag{49}$$

Under the map

$$|\psi(t)\rangle \to |\varphi(t)\rangle := \mathrm{e}^{2z^1(t)a^\dagger}|\psi(t)\rangle, \tag{50}$$

the $\mathcal{PT}$-symmetric system transforms into an equivalent Hermitian system,

$$\mathrm{i}\partial_t|\varphi(t)\rangle = h(t)|\varphi(t)\rangle, \tag{51}$$

with the Hamiltonian

$$h(t) = \mathrm{i}\left[\dot{z}^1(t)a^\dagger - \dot{z}^{1*}(t)a\right]. \tag{52}$$

Physically speaking, the $\mathcal{PT}$-symmetric system and its equivalent Hermitian system may be considered two different interpretations of the dynamics of a same physical system.

Suppose now that $z^1(t) = \mathrm{i}\Omega_{\mathrm{D}}(\mathrm{e}^{-\mathrm{i}\delta t}-1)\mathrm{e}^{\mathrm{i}\phi_{\mathrm{L}}}/\delta$, i.e., $z^1(t) = z(t)$. For this, Eq. (52) reduces to Eq. (47). Hence, the evolution of the Hermitian system is simply the evolution process studied in Ref. [39]. As another interpretation of the dynamics of the same physical system, the $\mathcal{PT}$-symmetric system undergos the corresponding evolution $|\psi(t)\rangle = \mathrm{e}^{-2z^1(t)a^\dagger}|\varphi(t)\rangle$. Since $z^1(\tau) = z^1(0) = 0$ and $|\varphi(\tau)\rangle = \mathrm{e}^{\mathrm{i}\gamma(\tau)}|\varphi(0)\rangle$, this evolution is cyclic, and the evolving state $|\psi(t)\rangle$ of the $\mathcal{PT}$-symmetric system acquires the same total phase $\gamma(\tau)$ as that of the Hermitian system. Note that for the $\mathcal{PT}$-symmetric system, the total phase accumulated in any cyclic evolution is simply the GP γ in Eq. (15), due to the vanishing of its Hamiltonian. Hence, $\gamma(\tau) = \gamma$, i.e., the unconventional GP is precisely the GP expressed by Eq. (15).

To shed more light on the unconventional GP, we calculate the quantum geometric tensor in Eq. (43), with which, we further obtain the curvature Ω and the metric $\mathrm{d}s^2$. To do this, we find the evolution operator of the $\mathcal{PT}$-symmetric system. Using magnus expansion[47] and noting that the commutator of $K(t)$ at different time is a number, we have that the evolution operator reads $\mathrm{e}^{-2z^1(t)a^\dagger}D(z^1(t))$, up to a global phase factor, where $D(z^1) := \mathrm{e}^{z^1a^\dagger - z^{1*}a}$ is the displacement operator. So, starting at an arbitrary coherent state |"some complex number"⟩, the evolving state $|\psi(t)\rangle$ reads $|\psi(t)\rangle = \mathrm{e}^{-2z^1(t)a^\dagger}|z_1(t) + \text{"some complex number"}\rangle$, up to a phase factor. So, the evolving state is of the form $\mathrm{e}^{-2z^1a^\dagger}|z^2\rangle$, where $z^2 = z^1(t) + $ "some complex number". Substituting $\mathrm{e}^{-2z^1a^\dagger}|z^2\rangle$ into Eq. (43), i.e., setting $|\phi\rangle$ and $|\widetilde{\phi}\rangle$ appearing in

Eq. (43) as $|\phi\rangle = \mathrm{e}^{-2z^1 a^\dagger}|z^2\rangle$ and $|\widetilde{\phi}\rangle = \mathrm{e}^{2z^{1*}a}|z^2\rangle$, we obtain, after tedious but straightforward calculations,

$$(Q_{\mu\nu}) = \begin{pmatrix} 0 & 0 & -1 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & -1 \\ -1 & -\mathrm{i} & 1 & \mathrm{i} \\ \mathrm{i} & -1 & -\mathrm{i} & 1 \end{pmatrix}. \tag{53}$$

Here, the real coordinates are $(\lambda^1,\lambda^2,\lambda^3,\lambda^4)$ such that $z^1 = \lambda^1 + \mathrm{i}\lambda^2$ and $z^2 = \lambda^3 + \mathrm{i}\lambda^4$. Now, using Eq. (45), we easily obtain

$$(\Omega_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}. \tag{54}$$

That is,

$$\Omega = -2\mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^4 + 2\mathrm{d}\lambda^2 \wedge \mathrm{d}\lambda^3 + 2\mathrm{d}\lambda^3 \wedge \mathrm{d}\lambda^4. \tag{55}$$

For the evolution process studied in Ref. [39], in which the initial state is $|0\rangle$, we have $z^1(t) = z^2(t)$, leading to the constraints $\lambda^1 = \lambda^3$ and $\lambda^2 = \lambda^4$. Substituting $\lambda^1 = \lambda^3$ and $\lambda^2 = \lambda^4$ into Eq. (55), we have $\Omega = -2\mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2$. From Eq. (45), it follows that

$$\gamma(\tau) = 2\iint \mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2, \tag{56}$$

representing twice the area enclosed by the curve $z^1(t)$. So, the geometric nature of the unconventional GP is confirmed. Moreover, from Eq. (46), we deduce that

$$(g_{\mu\nu}) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \tag{57}$$

That is,

$$\mathrm{d}s^2 = -2\mathrm{d}\lambda^1\mathrm{d}\lambda^3 - 2\mathrm{d}\lambda^2\mathrm{d}\lambda^4 + \mathrm{d}\lambda^3\mathrm{d}\lambda^3 + \mathrm{d}\lambda^4\mathrm{d}\lambda^4. \tag{58}$$

Two of the eigenvalues of $(g_{\mu\nu})$ are positive, i.e., $\frac{1}{2}(1+\sqrt{5})$, whereas the rest are negative, i.e., $\frac{1}{2}(1-\sqrt{5})$. Hence, the metric $\mathrm{d}s^2$ is pseudo-Riemannian. Substituting $\lambda^1 = \lambda^3$ and $\lambda^2 = \lambda^4$ into Eq. (58), we have $\mathrm{d}s^2 = -\mathrm{d}\lambda^1\mathrm{d}\lambda^1 - \mathrm{d}\lambda^2\mathrm{d}\lambda^2 < 0$, indicating that the evolution process studied in Ref. [39] is timelike.

6. A general geometric phase

It may be an interesting topic for future work to generalize the GP found here to a more general setting,[48] like the non-cyclic case considered in Ref. [49]. This is a non-trivial problem, as the Hilbert space $\mathcal{H}(t)$ changes with time t in general.[3] To shed some light on the problem, we provide one possible way to achieve the generalization,

$$\gamma(\tau) := -\frac{\mathrm{i}}{2}\ln\frac{\langle\psi(0)|W(0)|\psi(\tau)\rangle}{\langle\psi(\tau)|W(0)|\psi(0)\rangle} - \Im\int_0^\tau \mathrm{d}t \frac{\langle\psi(t)|W(t)|\dot{\psi}(t)\rangle}{\langle\psi(t)|W(t)|\psi(t)\rangle}, \tag{59}$$

where $|\psi(t)\rangle$ is an (unnormalized) state in $\mathcal{H}(t)$ and $W(t)$ is the (time-dependent) metric operator associated with $\mathcal{H}(t)$.[3] Mathematically, the GP is defined as a functional of a state and the metric operator, $\gamma = \gamma[\psi(t), W(t)]$. It is easy to see that γ defined in Eq. (59) is gauge-invariant and reduces to the GP given by Eq. (15) in cyclic cases. With the choice of Dirac inner-product, i.e., $W(t) = \mathbb{1}$, Eq. (59) reproduces AA phase for the cyclic states in Ref. [48] and non-cyclic states in Ref. [50]. Thus, the GP defined in Eq. (59) unifies real and gauge-invariant geometric phases in non-Hermitian systems.

7. Conclusion

We have presented a series of results on the geometry of time-dependent $\mathcal{PT}$QM. Specifically, they are the GP (15), the connection (28), the curvature (35), the parallel transport condition (37), the metric tensor (42), and the quantum geometric tensor (43). The GP emerges naturally from cyclic evolutions of $\mathcal{PT}$-symmetric systems, and it may be regarded as a counterpart of Aharonov–Anandan's phase. The connection and curvature are responsible for the appearance of the GP, as expressed by Eqs. (30) and (31). The quantum geometric tensor is a unifying concept, of which the imaginary part gives the curvature and the real part induces the metric tensor, as described by Eqs. (45) and (46), respectively.

These results constitute a useful, perhaps indispensible, tool to tackle geometric problems involving $\mathcal{PT}$-symmetric systems with time-varying system's parameters. As an illustration of their usefulness, we have solved the open question whether the unconventional GP admits a geometric interpretation or not. Specifically, we have shown that the unconventional GP, instead of being the sum of a DP and a GP, can be expressed as the single GP in Eq. (15), thus making its geometric nature undoubtedly clear.

Of particular interest is the finding that the metric tensor (42) may be pseudo-Riemannian, which is elusive in standard quantum mechanics. This leads to the intriguing fact that there can exist three types of evolutions for a $\mathcal{PT}$-symmetric system, i.e., spacelike, lightlike, and timelike evolutions, as shown in the example involving the unconventional GP. Further studies on the implications/applications of the pseudo-Riemannian feature are highly desirable.

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