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Supersymmetric structures of Dirac oscillators in commutative and noncommutative spaces*

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The supersymmetric properties of a charged planar Dirac oscillator coupling to a uniform perpendicular magnetic field are studied. We find that there is an $N = 2$ supersymmetric structure in both commutative and noncommutative cases. We construct the generators of the supersymmetric algebras explicitly and show that the generators of the supersymmetric algebras can be mapped onto ones which only contain the left or right-handed chiral phonons by unitary transformations.

Keywords: Dirac oscillator, anti-Jaynes–Cummings model, Jaynes–Cummings model, supersymmetry

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1. Introduction

Dirac oscillator was introduced in Ref. [1] and further illustrated in Ref. [2]. The interaction term was introduced by the replacement $p_i \to p_i - \mathrm{i}m\beta\omega x_i$ from the free Dirac Hamiltonian. It could be regarded as a relativistic generalization of the non-relativistic harmonic oscillator since it reduced to the non-relativistic harmonic oscillator together with a strong spin-orbit interaction in the non-relativistic limit.[3] Due to its exact integrabilities and wide applications, Dirac oscillator was studied extensively from various aspects[4–9] since it has been introduced. Interestingly, it has been realized experimentally in recent years.[10,11]

The relations between the planar Dirac oscillator and the quantum optics model were revealed in Ref. [12]. In this paper, authors found an exact map from $2+1$-dimensional Dirac oscillator to the Jaynes–Cummings (JC) model, which described the interaction between a single-mode quantized electromagnetic field and a two-level atom in the rotating-wave approximation.[13] Then, this work was generalized to the model of a charged planar Dirac oscillator coupling to a uniform magnetic field. In Ref. [14] authors showed that this model could be mapped onto a quantum optics model, which contained both the anti-Jaynes–Cummings (AJC) and the Jaynes–Cummings interactions. Interestingly, each interaction only depended on certain chiral phonons, and there was a competition between these two opposite chiral phonons when the magnetic-field intensity changed. Moreover, when the magnetic field intensity passed the critical point, a chirality quantum phase transition[15] would occur. Shortly after, an alternative approach to study this model was suggested.[16]

The symmetry properties of Dirac oscillator were also investigated. In Ref. [17], authors found that the symmetric properties of a $3+1$-dimensional Dirac oscillator were described by $\mathrm{so}(4) \oplus \mathrm{so}(1,3)$. Interestingly, authors of Ref. [18] found a supersymmetric structure in $3+1$-dimensional Dirac oscillator. In the work [19], authors found an $\mathrm{so}(4)$ algebraic structure in $1+1$-dimensional Dirac oscillator.

One the other hand, the noncommutativity becomes one of the foci of the theoretical research nowadays. The concept of the spatial noncommutativity was introduced in physics in the middle of 1940s.[20] At the end of 1990s, it was found that spatial noncommutativity could be derived naturally from the superstring theories.[21–28] Since then, both quantum field theories and quantum mechanics in the noncommutative space have been extensively studied.[29–44] The noncommutative phase space, i.e., both coordinates and momenta are noncommutative, also attracted much interests.[45–49]

In this paper, we will study the supersymmetric structures of the model of a charged Dirac oscillator interacting with a uniform perpendicular magnetic field both on commutative and noncommutative planes. The organization of this paper is as follows. In Section 2, we will study this model on the commutative plane. We will show that this model has an $N = 2$ supersymmetric structure. We construct the generators of the supersymmetric structure explicitly and then map them onto the ones which only contain left or right-handed phonons according to the signs of a dimensionless parameter. Then, in Section 3, we generalize our studies to the noncommutative case. The corrections due to the noncommutativity are studied. Some conclusions and remarks will be given in the last section.

2. The charged Dirac oscillator in the commutative plane

The model we will analyze is a charged planar Dirac oscillator in the background of a uniform perpendicular magnetic

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field. The Hamiltonian of this model is

$$H = c\sum_{i=1}^{2}\sigma_i\left(p_i - \mathrm{i}m\omega\sigma_3 x_i + \frac{e}{c}A_i\right) + mc^2\sigma_3, \tag{1}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices, ω is the frequency of the Dirac oscillator, e is the charges the oscillator carried, c is the speed of light in the vacuum and A_i are the vector potentials. We apply the symmetric gauge $(A_1, A_2) = (B/2)(-y, x)$ with B being the intensity of the magnetic field.

Defining the Larmor frequency

$$\omega_{\mathrm{L}} = \frac{eB}{2mc}, \tag{2}$$

and substituting both Pauli matrices and magnetic potentials $(A_1, A_2) = (B/2)(-y, x)$ into the Hamiltonian (1), we get

$$H = \begin{pmatrix} mc^2 & c(p_1 + \mathrm{i}m\omega\zeta x_1) \\ & -\mathrm{i}c(p_2 + \mathrm{i}m\omega\zeta x_2) \\ c(p_1 - \mathrm{i}m\omega\zeta x_1) & \\ +\mathrm{i}c(p_2 - \mathrm{i}m\omega\zeta x_2) & -mc^2 \end{pmatrix}, \tag{3}$$

in which

$$\zeta = 1 - \frac{\omega_{\mathrm{L}}}{\omega} \tag{4}$$

is a dimensionless parameter. It will be vanished when the intensity of the magnetic field takes the critical value $B_{\mathrm{c}} = (2m\omega c/e)$.

Introducing annihilation–creation operators

$$\begin{aligned} a_i &= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x_i + \mathrm{i}\sqrt{\frac{1}{m\hbar\omega}}p_i\right), \\ a_i^{\dagger} &= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x_i - \mathrm{i}\sqrt{\frac{1}{m\hbar\omega}}p_i\right), \end{aligned} \tag{5}$$

and the chiral annihilation–creation operators with respect to the Dirac oscillator

$$\begin{aligned} a_{\mathrm{l}} &= \sqrt{\frac{1}{2}}(a_1 + \mathrm{i}a_2), \quad a_{\mathrm{l}}^{\dagger} = \sqrt{\frac{1}{2}}(a_1^{\dagger} - \mathrm{i}a_2^{\dagger}), \\ a_{\mathrm{r}} &= \sqrt{\frac{1}{2}}(a_1 - \mathrm{i}a_2), \quad a_{\mathrm{r}}^{\dagger} = \sqrt{\frac{1}{2}}(a_1^{\dagger} + \mathrm{i}a_2^{\dagger}), \end{aligned} \tag{6}$$

one can verify that the non-vanishing commutators among these chiral operators are

$$[a_{\mathrm{l}}, a_{\mathrm{l}}^{\dagger}] = [a_{\mathrm{r}}, a_{\mathrm{r}}^{\dagger}] = 1. \tag{7}$$

Applying the chiral annihilation–creation operators (6) and defining $\sigma_{\pm} = (1/2)(\sigma_1 \pm \mathrm{i}\sigma_2)$, we rewrite the Hamiltonian (3) in the form

$$H = mc^2\sigma_3 + H_{\mathrm{AJC}}(g_{\mathrm{l}}) + H_{\mathrm{JC}}(g_{\mathrm{r}}), \tag{8}$$

in which

$$\begin{aligned} H_{\mathrm{AJC}} &= g_{\mathrm{l}}a_{\mathrm{l}}^{\dagger}\sigma_+ + g_{\mathrm{l}}^{*}a_{\mathrm{l}}\sigma_-, \\ H_{\mathrm{JC}} &= g_{\mathrm{r}}a_{\mathrm{r}}\sigma_+ + g_{\mathrm{r}}^{*}a_{\mathrm{r}}^{\dagger}\sigma_- \end{aligned} \tag{9}$$

are the AJC and JC interactions. The strengths of the AJC and JC interactions are $g_{\mathrm{l}} = \mathrm{i}\mu$ and $g_{\mathrm{r}} = -\mathrm{i}\nu$, respectively, with the parameters μ and ν being two real parameters

$$\mu = c\sqrt{m\hbar\omega}(1+\zeta), \quad \nu = c\sqrt{m\hbar\omega}(1-\zeta). \tag{10}$$

In view of the interaction strengths g_{l} and g_{r}, one finds that the dimensionless parameter ζ separates the model (1) into two distinct regions. They are the $\zeta > 0$ and $\zeta < 0$ regions, respectively. In the region $\zeta > 0$, the AJC interaction is stronger than JC interaction, i.e., $|g_{\mathrm{l}}| > |g_{\mathrm{r}}|$, so the AJC interaction is dominated. Contrarily, in the region $\zeta < 0$, the JC interaction dominates.

It was shown that by applying unitary transformations, one can map the Hamiltonian (8) to the ones which only contain AJC or JC interaction according to the signs of the dimensionless parameter ζ. Furthermore, a quantum phase transition would happen when ζ passed the zero point.[14,16] This quantum phase transition was named chirality quantum phase transition since the chiralities of the phonons would also change abruptly at the critical point.[14]

The eigenvalue equation of the Hamiltonian (1) $H|\psi\rangle = E|\psi\rangle$, where $|\psi\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}$, can be written explicitly as

$$\begin{aligned} (E - mc^2)|\psi_1\rangle &= \mathrm{i}(\mu a_{\mathrm{l}}^{\dagger} - \nu a_{\mathrm{r}})|\psi_2\rangle, \\ (E + mc^2)|\psi_2\rangle &= -\mathrm{i}(\mu a_{\mathrm{l}} - \nu a_{\mathrm{r}}^{\dagger})|\psi_1\rangle. \end{aligned} \tag{11}$$

The diagonal form of the above equations is

$$(E^2 - m^2c^4)|\psi\rangle = H^{\mathrm{S}}|\psi\rangle, \tag{12}$$

where

$$H^{\mathrm{S}} = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix}, \tag{13}$$

with H_{11}, H_{22} being

$$\begin{aligned} H_{11} &= \mu^2 N_{\mathrm{l}} + \nu^2(N_{\mathrm{r}} + 1) - \mu\nu a_{\mathrm{l}}a_{\mathrm{r}} - \mu\nu a_{\mathrm{l}}^{\dagger}a_{\mathrm{r}}^{\dagger}, \\ H_{22} &= \mu^2(N_{\mathrm{l}} + 1) + \nu^2 N_{\mathrm{r}} - \mu\nu a_{\mathrm{l}}a_{\mathrm{r}} - \mu\nu a_{\mathrm{l}}^{\dagger}a_{\mathrm{r}}^{\dagger}. \end{aligned}$$

Here $N_{\mathrm{l}} = a_{\mathrm{l}}^{\dagger}a_{\mathrm{l}}$ and $N_{\mathrm{r}} = a_{\mathrm{r}}^{\dagger}a_{\mathrm{r}}$ are the number operators of left and right-handed phonons, respectively.

Obviously, Hamiltonian (13) shares the common eigenfunctions with Hamiltonian (1). Therefore, it plays the same role as Hamiltonian (1). In Ref. [18], authors found a hidden supersymmetric structure in the $3+1$-dimensional Dirac oscillator and constructed the generators which contained a similar Hamiltonian (13). The supersymmetric structure in model (1)

is described by Hamiltonian (13) and two supercharges. The two non-Hermitian supercharges are

$$\begin{aligned} Q &= -\mathrm{i}(\nu a_{\mathrm{r}} - \mu a_{\mathrm{l}}^{\dagger})\sigma_{+}, \\ Q^{\dagger} &= \mathrm{i}(\nu a_{\mathrm{r}}^{\dagger} - \mu a_{\mathrm{l}})\sigma_{-}, \end{aligned} \tag{14}$$

which obey the commutation relations

$$\begin{aligned} &[H^{\mathrm{S}},\, Q] = [H^{\mathrm{S}},\, Q^{\dagger}] = 0, \\ &\{Q,\, Q^{\dagger}\} = \{Q^{\dagger},\, Q\} = H^{\mathrm{S}}, \\ &Q^2 = (Q^{\dagger})^2 = 0. \end{aligned} \tag{15}$$

where $\{\,,\,\}$ denotes the anti-commutator.

The $N=2$ supersymmetric structure in the model (1) will be transparent if we introduce two Hermitian supercharges Q^1, Q^2 as

$$\begin{aligned} Q^1 &= -\frac{\mathrm{i}}{\sqrt{2}}(Q - Q^{\dagger}), \\ Q^2 &= -\frac{1}{\sqrt{2}}(Q + Q^{\dagger}). \end{aligned} \tag{16}$$

It can be verified that the generators H^{S}, Q^i $(i=1,2)$ satisfy the standard $N=2$ superalgebra

$$\left\{Q^i,\, Q^j\right\} = H^{\mathrm{S}}\delta^{ij}, \quad \left[H^{\mathrm{S}},\, Q^i\right] = 0. \tag{17}$$

Thus, it shows that the model (1) possess an $N=2$ supersymmetric structure.

The generators of the supersymmetric structure, i.e, the Hamiltonian (13) and the supercharges (16), contain both the left and right-handed phonons. Inspired by the work,[14,16] we show that the generators of the supersymmetric algebra can be expressed by the phonons with certain chiralities. This purpose can be realized by a unitary transformation. The unitary operator is

$$U(\theta) = \mathrm{e}^{\theta(a_{\mathrm{l}}a_{\mathrm{r}} - a_{\mathrm{l}}^{\dagger}a_{\mathrm{r}}^{\dagger})}, \tag{18}$$

where θ is a parameter which will be determined according to the regions the parameter ζ characterized. The unitary operator $U(\theta)$ transforms the chiral operators a_{l}, a_{r} as

$$\begin{aligned} U(\theta)a_{\mathrm{l}}U^{\dagger}(\theta) &= \cosh\theta a_{\mathrm{l}} + \sinh\theta a_{\mathrm{r}}^{\dagger}, \\ U(\theta)a_{\mathrm{r}}U^{\dagger}(\theta) &= \cosh\theta a_{\mathrm{r}} + \sinh\theta a_{\mathrm{l}}^{\dagger}. \end{aligned} \tag{19}$$

In the region $\zeta > 0$, the AJC interaction is dominated. We choose the parameter θ

$$\theta = \theta_{\mathrm{l}} = \operatorname{arctanh}\left(\frac{\nu}{\mu}\right). \tag{20}$$

Then, Hamiltonian (13) and supercharges (16) transform as $H^{\mathrm{S}} \to H_{\mathrm{l}}^{\mathrm{S}} = U(\theta_{\mathrm{l}})H^{\mathrm{S}}U^{\dagger}(\theta_{\mathrm{l}})$, $Q^i \to Q_{\mathrm{l}}^i = U(\theta_{\mathrm{l}})Q^iU^{\dagger}(\theta_{\mathrm{l}})$. The results are

$$\begin{aligned} H^{\mathrm{S}} &\to H_{\mathrm{l}}^{\mathrm{S}} = 4m\hbar c^2\omega\zeta \begin{pmatrix} N_{\mathrm{l}} & 0 \\ 0 & N_{\mathrm{l}}+1 \end{pmatrix}, \\ Q^1 &\to Q_{\mathrm{l}}^1 = \sqrt{2m\hbar c^2\omega\zeta}(a_{\mathrm{l}}^{\dagger}\sigma_{+} + a_{\mathrm{l}}\sigma_{-}), \\ Q^2 &\to Q_{\mathrm{l}}^2 = -\mathrm{i}\sqrt{2m\hbar c^2\omega\zeta}(a_{\mathrm{l}}^{\dagger}\sigma_{+} - a_{\mathrm{l}}\sigma_{-}). \end{aligned} \tag{21}$$

Obviously, the generators $H_{\mathrm{l}}^{\mathrm{S}}, Q_{\mathrm{l}}^1, Q_{\mathrm{l}}^2$ only depend on the left-handed phonons and satisfy the same algebraic relations (17).

In the region $\zeta < 0$, the JC interaction is stronger than the AJC interaction, we choose the parameter θ as

$$\theta = \theta_{\mathrm{r}} = \operatorname{arctanh}\left(\frac{\mu}{\nu}\right). \tag{22}$$

In this region, the Hamiltonian (13) and the supercharges (16) transform as $H^{\mathrm{S}} \to H_{\mathrm{r}}^{\mathrm{S}} = U(\theta_{\mathrm{r}})H^{\mathrm{S}}U^{\dagger}(\theta_{\mathrm{r}})$, $Q^i \to Q_{\mathrm{r}}^i = U(\theta_{\mathrm{r}})Q^iU^{\dagger}(\theta_{\mathrm{r}})$. The explicit results are

$$\begin{aligned} H^{\mathrm{S}} &\to H_{\mathrm{r}}^{\mathrm{S}} = -4m\hbar c^2\omega\zeta \begin{pmatrix} N_{\mathrm{r}}+1 & 0 \\ 0 & N_{\mathrm{r}} \end{pmatrix}, \\ Q^1 &\to Q_{\mathrm{r}}^1 = -\sqrt{-2m\hbar c^2\omega\zeta}(a_{\mathrm{r}}\sigma_{+} + a_{\mathrm{r}}^{\dagger}\sigma_{-}), \\ Q^2 &\to Q_{\mathrm{r}}^2 = -\mathrm{i}\sqrt{-2m\hbar c^2\omega\zeta}(a_{\mathrm{r}}^{\dagger}\sigma_{-} - a_{\mathrm{r}}\sigma_{+}). \end{aligned} \tag{23}$$

Therefore, in the region $\zeta < 0$, the generators of the supersymmetric structure only contain the right-handed phonons.

3. The charged Dirac oscillator in the noncommutative plane

In this section, we will generalize the studies in the previous section to the noncommutative plane. The noncommutative plane we considered is characterized by the algebra (in order to avoid the problem of unitary, we only consider the noncommutativities among position-position and momentum-momentum)[45–48]

$$\begin{aligned} &[\hat{x}_i,\, \hat{x}_j] = \theta\varepsilon_{ij}, \\ &[\hat{p}_i,\, \hat{p}_j] = \mathrm{i}\eta\varepsilon_{ij}, \\ &[\hat{x}_i,\, \hat{p}_j] = \left(1 + \frac{\theta\eta}{4\hbar^2}\right)\delta_{ij}, \end{aligned} \tag{24}$$

where θ, η are two parameters which are characteristics of the noncommutativities.

The Hamiltonian in the noncommutative space (24) takes the same form as the commutative one (1). However, the variables x_i, p_i in Hamiltonian (1) are replaced by the noncommutative ones $\hat{x}_i$, $\hat{p}_i$. We map the noncommutative variables $(\hat{x}_i,\, \hat{p}_i)$ to the commutative ones by the Bopp shift

$$\hat{x}_i = x_i - \frac{\theta}{2\hbar}\varepsilon_{ij}p_j, \quad \hat{p}_i = p_i + \frac{\eta}{2\hbar}\varepsilon_{ij}x_j. \tag{25}$$

In Ref. [47], authors proved that in order to keep the gauge invariance of the charged Dirac particle interacting with the magnetic field in noncommutative space, we must drop the noncommutativity between the spatial coordinates. Therefore, we set $\theta = 0$ in the following.

Mapping the noncommutative variables $\hat{x}_i$, $\hat{p}_i$ to the commutative ones via the Bopp shift and defining a new parameter with the frequency dimension

$$\omega_\eta = \frac{\eta}{2m\hbar}, \tag{26}$$

we get the Hamiltonian

$$H_{\rm NC}=\begin{pmatrix} mc^2 & c(p_1+{\rm i}m\omega\zeta' x_1)-{\rm i}c(p_2+{\rm i}m\omega\zeta' x_2) \\ c(p_1-{\rm i}m\omega\zeta' x_1)+{\rm i}c(p_2-im\omega\zeta' x_2) & -mc^2 \end{pmatrix}, \quad (27)$$

where ζ' is also a dimensionless parameter

$$\zeta'=1-\frac{\omega_{\rm L}-\omega_\eta}{\omega}. \quad (28)$$

The relative magnitude of the frequency ω_η and ω had been estimated and the conclusion was $\omega_\eta<\omega$.[50]

Substituting the chiral annihilation–creation operators (6) and the parameter ζ' into the above Hamiltonian, we find that the Hamiltonian (27) can be written as

$$H_{\rm NC}=mc^2\sigma_3+H_{\rm AJC}(g_{\rm l}')+H_{\rm JC}(g_{\rm r}'), \quad (29)$$

where $H_{\rm AJC}(g_{\rm l}')$ and $H_{\rm JC}(g_{\rm r}')$ are the AJC and JC interactions,

$$\begin{aligned} H_{\rm AJC}(g_{\rm l}')&=g_{\rm l}'a_{\rm l}^\dagger\sigma_+ +(g_{\rm l}')^*a_{\rm l}\sigma_-,\\ H_{\rm JC}(g_{\rm r}')&=g_{\rm r}'a_{\rm r}\sigma_+ +(g_{\rm r}')^*a_{\rm r}^\dagger\sigma_-, \end{aligned} \quad (30)$$

in which $g_{\rm l}'={\rm i}\mu'$, $g_{\rm r}'=-{\rm i}\nu'$ with μ' and ν' being

$$\begin{aligned} \mu'&=c\sqrt{m\hbar\omega}\,(1+\zeta'),\\ \nu'&=c\sqrt{m\hbar\omega}\,(1-\zeta'). \end{aligned} \quad (31)$$

It shows that the noncommutative extension of Hamiltonian (1), i.e., Hamiltonian (27) is also equivalent to a quantum optics model which contains both AJC and JC interactions.

Solving the eigenvalue equation of Dirac equation $H_{\rm NC}|\psi\rangle_{\rm NC}=E_{\rm NC}|\psi\rangle_{\rm NC}$ on the noncommutative plane, we get

$$(E_{\rm NC}^2-m^2c^4)|\psi\rangle_{\rm NC}=H_{\rm NC}^{\rm S}|\psi\rangle_{\rm NC}, \quad (32)$$

where

$$H_{\rm NC}^{\rm S}=\begin{pmatrix} (H_{\rm NC}^{\rm S})_{11} & 0 \\ 0 & (H_{\rm NC}^{\rm S})_{22} \end{pmatrix}$$

is a diagonal matrix with $\left(H_{\rm NC}^{\rm S}\right)_{11}$ and $\left(H_{\rm NC}^{\rm S}\right)_{11}$ being

$$\begin{aligned} \left(H_{\rm NC}^{\rm S}\right)_{11}&=(\mu')^2N_{\rm l}+(\nu')^2(N_{\rm r}+1)-\mu'\nu'a_{\rm l}a_{\rm r}-\mu'\nu'a_{\rm l}^\dagger a_{\rm r}^\dagger,\\ \left(H_{\rm NC}^{\rm S}\right)_{22}&=(\mu')^2(N_{\rm l}+1)+(\nu')^2N_{\rm r}-\mu'\nu'a_{\rm l}a_{\rm r}-\mu'\nu'a_{\rm l}^\dagger a_{\rm r}^\dagger. \end{aligned}$$

The noncommutative generalization of the Dirac oscillator (27) also possesses the supersymmetric structure. The generators of the supersymmetric structure consist of $H_{\rm NC}^{\rm S}$ and two non-Hermitian supercharges $Q_{\rm NC}$, $Q_{\rm NC}^\dagger$. The explicit form of these two non-Hermitian supercharges is

$$\begin{aligned} Q_{\rm NC}&=-{\rm i}(\nu'a_{\rm r}-\mu'a_{\rm l}^\dagger)\sigma_+,\\ Q_{\rm NC}^\dagger&={\rm i}(\nu'a_{\rm r}^\dagger-\mu'a_{\rm l})\sigma_-. \end{aligned} \quad (33)$$

It can be checked that the generators $H_{\rm NC}^{\rm S}$ and two non-Hermitian supercharges $Q_{\rm NC}$, $Q_{\rm NC}^\dagger$ satisfy the same commutation relation as (15).

Similar with the commutative case, we can construct two Hermitian supercharges $Q_{\rm NC}^i$ $(i=1,2)$ out of the non-Hermitian ones $Q_{\rm NC}$, $Q_{\rm NC}^\dagger$ in the same manner as (16). The generators $H_{\rm NS}$, $Q_{\rm NS}^i$ obey the same algebraic relation (17). It shows that the $N=2$ supersymmetric structures survive the noncommutative extension.

The generators $H_{\rm NC}^{\rm S}$, $Q_{\rm NC}^i$ contain both the left and right-handed phonons simultaneously. The dimensionless parameter ζ' separates the model (27), (24) into two distinct regions, i.e., $\zeta'>0$ and $\zeta'<0$. In the region $\zeta'>0$, the AJC interaction is dominated. In the other region, the JC interaction dominates. We can map the generators $H_{\rm NC}^{\rm S}$, $Q_{\rm NC}^i$ onto the ones which only contain the left or right-handed phonons by unitary transformations according to the signs of ζ'.

The unitary operator we applied is

$$U(\theta_{\rm NC})={\rm e}^{\theta_{\rm NC}(a_{\rm l}a_{\rm r}-a_{\rm l}^\dagger a_{\rm r}^\dagger)}. \quad (34)$$

It takes the same form as Eq. (18) except the parameter is different from the commutative case. In the region $\zeta'>0$, the parameter $\theta_{\rm NC}$ is

$$\theta_{\rm NC}=\theta_{\rm NC}^{\rm l}={\rm arctanh}\frac{\nu'}{\mu'}. \quad (35)$$

Under this unitary transformation, the generators $H_{\rm NC}^{\rm S}$, $Q_{\rm NC}^i$ transform as $H_{\rm NC}^{\rm S}\to(H_{\rm NC}^{\rm S})^{\rm l}=U\left(\theta_{\rm NC}^{\rm l}\right)H_{\rm NC}^{\rm S}U^\dagger\left(\theta_{\rm NC}^{\rm l}\right)$ and $Q_{\rm NC}^i\to\left(Q_{\rm NC}^i\right)^{\rm l}=U\left(\theta_{\rm NC}^{\rm l}\right)Q_{\rm NC}^iU^\dagger\left(\theta_{\rm NC}^{\rm l}\right)$. The results are

$$\begin{aligned} H_{\rm NC}^{\rm S}&\to(H_{\rm NC}^{\rm S})^{\rm l}=4m\hbar c^2\omega\zeta'\begin{pmatrix} N_{\rm l} & 0\\ 0 & N_{\rm l}+1\end{pmatrix},\\ Q_{\rm NC}^1&\to(Q_{\rm NC}^1)^{\rm l}=\sqrt{2m\hbar c^2\omega\zeta'}(a_{\rm l}^\dagger\sigma_+ +a_{\rm l}\sigma_-),\\ Q_{\rm NC}^2&\to(Q_{\rm NC}^2)^{\rm l}=-{\rm i}\sqrt{2m\hbar c^2\omega\zeta'}(a_{\rm l}^\dagger\sigma_+ -a_{\rm l}\sigma_-). \end{aligned} \quad (36)$$

While in the region $\zeta'<0$, the parameter $\theta_{\rm NC}$ should be chosen as

$$\theta_{\rm NC}=\theta_{\rm NC}^{\rm r}={\rm arctanh}\frac{\mu'}{\nu'}, \quad (37)$$

and the generators $H_{\rm NC}^{\rm S}$, $Q_{\rm NC}^i$ transform as $H_{\rm NC}^{\rm S}\to\left(H_{\rm NC}^{\rm S}\right)^{\rm r}=U\left(\theta_{\rm NC}^{\rm r}\right)H_{\rm NC}^{\rm S}U^\dagger\left(\theta_{\rm NC}^{\rm r}\right)$ and $Q_{\rm NC}^i\to\left(Q_{\rm NC}^i\right)^{\rm r}=U\left(\theta_{\rm NC}^{\rm r}\right)Q_{\rm NC}^iU^\dagger\left(\theta_{\rm NC}^{\rm r}\right)$

$$\begin{aligned} H_{\rm NC}^{\rm S}&\to(H_{\rm NC}^{\rm S})^{\rm r}=-4m\hbar c^2\omega\zeta'\begin{pmatrix} N_{\rm r}+1 & 0\\ 0 & N_{\rm r}\end{pmatrix},\\ Q_{\rm NC}^1&\to(Q_{\rm NC}^1)^{\rm r}=-\sqrt{-2m\hbar c^2\omega\zeta'}(a_{\rm r}\sigma_+ +a_{\rm r}^\dagger\sigma_-),\\ Q_{\rm NC}^2&\to(Q_{\rm NC}^2)^{\rm r}={\rm i}\sqrt{-2m\hbar c^2\omega\zeta'}(a_{\rm r}\sigma_+ -a_{\rm r}^\dagger\sigma_-). \end{aligned} \quad (38)$$

The results (36), (38) not only indicate that the noncommutative generalization of the model (1) possesses an $N=2$ supersymmetric structure, but also imply that similar with the commutative case, the generators of the supersymmetric structures can also be constructed by using phonons with certain chiralities.

4. Conclusion

In this paper, we study the supersymmetric structures of a charge planar Dirac oscillator in the background of a uniform perpendicular magnetic field. We find that this model possesses an $N = 2$ supersymmetric structure in the commutative space.

Previous studies showed that the model (1) would experience a chirality quantum phase transition when the parameter ζ passes the critical point. Our studies show that the chiralities of the phonons which construct the generators of the supersymmetric structures change abruptly when the quantum phase transition occurs. It can certainly be regarded as an evidence of the chirality quantum phase transition.

Then we generalize our studies to the noncommutative space (24). Noticing the fact that the spatial noncommutativity should be dropped for the charged Dirac particle interacting with the magnetic field, we confine ourselves to $\theta = 0$. Our results show that the model (1) also has an $N = 2$ supersymmetric structures in the noncommutative plane (24).

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