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Connes distance of 2D harmonic oscillators in quantum phase space*

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We study the Connes distance of quantum states of two-dimensional (2D) harmonic oscillators in phase space. Using the Hilbert–Schmidt operatorial formulation, we construct a boson Fock space and a quantum Hilbert space, and obtain the Dirac operator and a spectral triple corresponding to a four-dimensional (4D) quantum phase space. Based on the ball condition, we obtain some constraint relations about the optimal elements. We construct the corresponding optimal elements and then derive the Connes distance between two arbitrary Fock states of 2D quantum harmonic oscillators. We prove that these two-dimensional distances satisfy the Pythagoras theorem. These results are significant for the study of geometric structures of noncommutative spaces, and it can also help us to study the physical properties of quantum systems in some kinds of noncommutative spaces.

Keywords: Connes distance, noncommutative geometry, harmonic oscillator

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1. Introduction

In quantum physics, the physical quantities are represented by some operators in a Hilbert space. Some of the operators are noncommutative with each other. So it is natural to study physical systems by noncommutative geometry. In the 1980's, Connes formulated the mathematically rigorous framework of noncommutative geometry.[1] Many kinds of noncommutative spaces have been studied by physicists and mathematicians, such as the Moyal plane and fuzzy space.[2–11] The phase space in quantum mechanics is also a Moyal-type noncommutative space, because the position and momentum operators satisfy the noncommutative Heisenberg algebras.

Due to the noncommutativity, there is no traditional point in a noncommutative space. This is different from normal commutative spaces. We only have states in noncommutative spaces. In a noncommutative space, a pure state is the analog of a traditional point in a normal commutative space, and the spectral distance between pure states corresponds to the geodesic distance between points. So there is no normal distance between two points in a noncommutative space, but we can calculate some kinds of distance measures between the states, such as the Connes distance.[12] Many researchers have already studied the Connes distance in some kinds of noncommutative spaces. Bimonte *et al.* calculated the distances between the points of a lattice on which the usual discretized Dirac operator has been defined.[13] Cagnache *et al.* have studied the Connes spectral distance in the Moyal plane.[14] They explicitly computed Connes spectral distance between the pure states which corresponding to eigenfunctions of the quantum harmonic oscillators. Martinetti *et al.* have studied the metric aspect of the Moyal plane from Connes' noncommutative geometry point of view.[15] They obtained the spectral distance between coherent states of the quantum harmonic oscillator as the Euclidean distance on the plane. They also computed the spectral distance in the so-called double Moyal plane. D'Andrea *et al.* have studied the Pythagoras' theorem in noncommutative geometry.[16] They showed that for non-pure states it is replaced by some Pythagoras inequalities. Franco and Wallet also studied metrics and causality on Moyal planes.[17] Scholtz and his collaborators have done many works on the studies of Connes spectral distances in Moyal plane and fuzzy sphere.[18–20] They developed the Hilbert–Schmidt operatorial formulation, and obtained the distances of harmonic oscillator states and also coherent states. Kumar *et al.* used Dirac eigen-spinor method to compute spectral distances in doubled Moyal plane.[21] Chakraborty *et al.* also studied the spectral distance on Lorentzian Moyal plane.[22]

In the present work, we consider the Connes distance of the quantum states (namely the Fock states) of 2D harmonic oscillators. It is known that in noncommutative spaces, for example, the quantum phase space, due to the noncommutativity, there are no simultaneous eigenstates such as $|\hat{x}_i, \hat{p}_i\rangle$.

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Therefore there is no traditional coordinate representation in the whole quantum phase space. So in order to describe the states of physical systems in noncommutative spaces, one need some other representations. For example, Fock state representation and coherent state representation. These representations are powerful tools for the studies of physical systems in noncommutative spaces.[5] Fock state representation is one of the simplest and most useful representations in quantum mechanics. The Fock states can be regarded as the discrete points of the abstract background space. So one can obtain some information of geometric structures of the noncommutative spaces and also the physical properties of quantum systems by studying the mathematical structure of Fock states in noncommutative spaces, such as the Connes spectral distance. Actually, some researchers have already studied the Connes distance between the states of one-dimensional (1D) harmonic oscillators.[14,18,20] But we find that the calculations and results of the Connes distance of 2D harmonic oscillators are much more complicated than those of 1D oscillator system. So it is significant to study the explicit formulae of the Connes distance of 2D harmonic oscillator systems. The method used in the present work is also different from those used in the literatures.

This paper is organized as follows. In Section 2, we consider the 4D quantum phase space and construct a corresponding spectral triple. Using the Hilbert–Schmidt operatorial formulation, we construct a boson Fock space and a quantum Hilbert space, and obtain the Dirac operator. In Section 3, we review the definition of Connes spectral distance, and derive some inequalities corresponding to the ball condition. Based on these inequalities, we derive some constraint relations about the optimal elements in Section 4. The Connes distance between two Fock states of 2D harmonic oscillators is calculated in Section 5. Some discussions and conclusions are given in Section 6. Some calculation details of the Dirac operator and the Connes spectral distances are presented in Appendix A and Appendix B.

2. The 4D quantum phase space and spectral triple

Let us consider the 4D quantum phase space, and the coordinate operators $\hat{x}_i$, $\hat{p}_i$ satisfy the following commutation relation $[\hat{x}_1,\hat{p}_1]=[\hat{x}_2,\hat{p}_2]=\mathrm{i}\hbar$, and others vanish. Consider the 2D harmonic oscillator system, the corresponding Hamiltonian is

$$\hat{H}=\frac{1}{2m}\hat{p}_1^2+\frac{1}{2m}\hat{p}_2^2+\frac{m\omega^2}{2}\hat{x}_1^2+\frac{m\omega^2}{2}\hat{x}_2^2, \tag{1}$$

and we can define the following creation and annihilation operators:

$$\hat{a}_i=\frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\hat{x}_i+\mathrm{i}\frac{\hat{p}_i}{\sqrt{m\omega}}\right),$$

$$\hat{a}_i^\dagger=\frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\hat{x}_i-\mathrm{i}\frac{\hat{p}_i}{\sqrt{m\omega}}\right). \tag{2}$$

For simplicity, we will denote

$$\tilde{x}_i=\sqrt{m\omega}\hat{x}_i,\quad \tilde{p}_i=\frac{\hat{p}_i}{\sqrt{m\omega}}, \tag{3}$$

and we have

$$[\tilde{x}_i,\tilde{p}_j]=\mathrm{i}\delta_{ij}\hbar,\quad i,j=1,2. \tag{4}$$

There are

$$\hat{a}_i=\frac{1}{\sqrt{2\hbar}}(\tilde{x}_i+\mathrm{i}\tilde{p}_i),\quad \hat{a}_i^\dagger=\frac{1}{\sqrt{2\hbar}}(\tilde{x}_i-\mathrm{i}\tilde{p}_i). \tag{5}$$

These creation and annihilation operators (2) or (5) satisfy the commutation relations $[\hat{a}_i,\hat{a}_j^\dagger]=\delta_{ij}$, and $[\hat{a}_i,\hat{a}_j]=[\hat{a}_i^\dagger,\hat{a}_j^\dagger]=0$.

In the present work, we will use the Hilbert–Schmidt operatorial formulation developed in Ref. [23]. We can construct the following boson Fock space:

$$\begin{aligned}\mathcal{F}=\mathrm{span}\Bigg\{&|n_1,n_2\rangle=\frac{1}{\sqrt{n_1!n_2!}}(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}|00\rangle,\\ &n_1,n_2=0,1,2,\dots\Bigg\},\end{aligned} \tag{6}$$

where $|00\rangle$ is the vacuum state, $\hat{a}_i|00\rangle=0$. The Fock states $|n_1,n_2\rangle$ are eigenstates of the number operators $\hat{N}_i=\hat{a}_i^\dagger\hat{a}_i$, $\hat{N}_i|n_1,n_2\rangle=n_i|n_1,n_2\rangle$, and

$$\begin{aligned}&\hat{a}_1|n_1,n_2\rangle=\sqrt{n_1}|n_1-1,n_2\rangle,\\ &\hat{a}_1^\dagger|n_1,n_2\rangle=\sqrt{n_1+1}|n_1+1,n_2\rangle,\\ &\hat{a}_2|n_1,n_2\rangle=\sqrt{n_2}|n_1,n_2-1\rangle,\\ &\hat{a}_2^\dagger|n_1,n_2\rangle=\sqrt{n_2+1}|n_1,n_2+1\rangle.\end{aligned} \tag{7}$$

Furthermore, one can construct the following quantum Hilbert space:

$$\mathcal{Q}=\mathrm{span}\{|m_1,m_2\rangle\langle n_1,n_2|\}. \tag{8}$$

In the followings, we will denote the elements $\psi(\hat{x}_i,\hat{p}_i)$ of the quantum Hilbert space by $|\psi)$ (The elements ϕ of the boson Fock space $\mathcal{F}$ are denoted by $|\phi\rangle$),

$$\psi(\hat{x}_i,\hat{p}_i)\equiv|\psi)=\sum_{i,j,k,l}\psi_{k,l}^{i,j}|i,j\rangle\langle k,l|. \tag{9}$$

The corresponding inner product is

$$\begin{aligned}(\phi|\psi)&=(\phi(\hat{x}_i,\hat{p}_i),\psi(\hat{x}_i,\hat{p}_i))=\mathrm{tr}_{\mathcal{F}}\left(\phi(\hat{x}_i,\hat{p}_i)^\dagger\psi(\hat{x}_i,\hat{p}_i)\right)\\ &=\sum_{m,n=0}^{\infty}\langle m,n|\phi(\hat{x}_i,\hat{p}_i)^\dagger\psi(\hat{x}_i,\hat{p}_i)|m,n\rangle,\end{aligned} \tag{10}$$

where $\mathrm{tr}_{\mathcal{F}}(\cdot)$ denotes the trace over $\mathcal{F}$.

The quantum phase space (4) is also a Moyal-type noncommutative space. In general, a noncommutative space corresponds to a spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ with $\mathcal{A}$ an involutive

algebra acting on a Hilbert space $\mathcal{H}$, and $\mathcal{D}$ is the Dirac operator on $\mathcal{H}$.[1] Moyal spaces are non-compact spectral triples.[4] One can construct a spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ for the 4D quantum phase space (4) as follows:

$$\mathcal{A}=\mathcal{Q},\quad \mathcal{H}=\mathcal{F}\otimes\mathbb{C}^4, \tag{11}$$

and an element $e\in\mathcal{A}$ acts on $\Psi=\begin{pmatrix}|\psi_1\rangle\\|\psi_2\rangle\\|\psi_3\rangle\\|\psi_4\rangle\end{pmatrix}\in\mathcal{H}$ through the diagonal representation π as

$$\begin{aligned}\pi(e)\Psi &= \pi(e)\begin{pmatrix}|\psi_1\rangle\\|\psi_2\rangle\\|\psi_3\rangle\\|\psi_4\rangle\end{pmatrix}=\begin{pmatrix}e&0&0&0\\0&e&0&0\\0&0&e&0\\0&0&0&e\end{pmatrix}\begin{pmatrix}|\psi_1\rangle\\|\psi_2\rangle\\|\psi_3\rangle\\|\psi_4\rangle\end{pmatrix}\\ &=\begin{pmatrix}e|\psi_1\rangle\\e|\psi_2\rangle\\e|\psi_3\rangle\\e|\psi_4\rangle\end{pmatrix}.\end{aligned} \tag{12}$$

The Dirac operator $\mathcal{D}$ for the 4D quantum phase space (4) can be defined as (see Appendix A for more details)

$$\mathcal{D}=\frac{1}{\hbar}\begin{pmatrix}0&0&-\tilde{x}_2+\mathrm{i}\tilde{p}_2&-\tilde{x}_1+\mathrm{i}\tilde{p}_1\\0&0&\tilde{x}_1+\mathrm{i}\tilde{p}_1&-\tilde{x}_2-\mathrm{i}\tilde{p}_2\\-\tilde{x}_2-\mathrm{i}\tilde{p}_2&\tilde{x}_1-\mathrm{i}\tilde{p}_1&0&0\\-\tilde{x}_1-\mathrm{i}\tilde{p}_1&-\tilde{x}_2+\mathrm{i}\tilde{p}_2&0&0\end{pmatrix}, \tag{13}$$

or

$$\mathcal{D}=\sqrt{\frac{2}{\hbar}}\begin{pmatrix}0&0&-\hat{a}_2^\dagger&-\hat{a}_1^\dagger\\0&0&\hat{a}_1&-\hat{a}_2\\-\hat{a}_2&\hat{a}_1^\dagger&0&0\\-\hat{a}_1&-\hat{a}_2^\dagger&0&0\end{pmatrix}. \tag{14}$$

3. Connes spectral distance and ball condition

The Connes spectral distance between two states ω and ω' is defined as[12]

$$d(\omega,\omega')\equiv\sup_{e\in B}|\omega(e)-\omega'(e)|, \tag{15}$$

where

$$B=\left\{e\in\mathcal{A}:\left\|[\mathcal{D},\pi(e)]\right\|_{\mathrm{op}}\leqslant 1\right\}, \tag{16}$$

$$\|A\|_{\mathrm{op}}\equiv\sup_{\psi\in\mathcal{H}}\frac{\|A\psi\|}{\|\psi\|},\quad \|A\|^2\equiv(A,A)=\mathrm{tr}_{\mathcal{F}}(A^\dagger A). \tag{17}$$

The inequality in formula (16) is the so-called ball condition.

Similar to Ref. [18], here we only consider the case where the quantum states ω are normal and bounded so that they are representable by density matrices ρ. The action of the state ω on an element $e\in\mathcal{A}$ can be written as

$$\omega(e)=\mathrm{tr}_{\mathcal{F}}(\rho e). \tag{18}$$

Suppose the quantum states ω and ω' correspond to the density matrices ρ and ρ', respectively. We have

$$\begin{aligned}d(\omega,\omega') &= \sup_{e\in B}|\mathrm{tr}_{\mathcal{F}}(\rho e)-\mathrm{tr}_{\mathcal{F}}(\rho' e)|\\ &= \sup_{e\in B}|\mathrm{tr}_{\mathcal{F}}(\Delta\rho\, e)|=\sup_{e\in B}|(\Delta\rho,e)|,\end{aligned} \tag{19}$$

where $\Delta\rho=\rho-\rho'$, and $(\Delta\rho)^\dagger=\Delta\rho$.

Now let us calculate the commutator $[\mathcal{D},\pi(e)]$ for a Hermitian element $e\in\mathcal{A}$. Using the Dirac operator $\mathcal{D}$ [Eq. (14)], we have

$$\begin{aligned}[\mathcal{D},\pi(e)] &= \sqrt{\frac{2}{\hbar}}\begin{pmatrix}0&0&[-\hat{a}_2^\dagger,e]&[-\hat{a}_1^\dagger,e]\\0&0&[\hat{a}_1,e]&[-\hat{a}_2,e]\\[-\hat{a}_2,e]&[\hat{a}_1^\dagger,e]&0&0\\[-\hat{a}_1,e]&[-\hat{a}_2^\dagger,e]&0&0\end{pmatrix}\\ &=\sqrt{\frac{2}{\hbar}}\begin{pmatrix}0&0&[\hat{a}_2,e]^\dagger&[\hat{a}_1,e]^\dagger\\0&0&[\hat{a}_1,e]&-[\hat{a}_2,e]\\-[\hat{a}_2,e]&-[\hat{a}_1,e]^\dagger&0&0\\-[\hat{a}_1,e]&[\hat{a}_2,e]^\dagger&0&0\end{pmatrix}\\ &=\sqrt{\frac{2}{\hbar}}\begin{pmatrix}0&D_1\\-D_1^\dagger&0\end{pmatrix},\end{aligned} \tag{20}$$

where

$$\begin{aligned}D_1&=\begin{pmatrix}[\hat{a}_2,e]^\dagger&[\hat{a}_1,e]^\dagger\\[\hat{a}_1,e]&-[\hat{a}_2,e]\end{pmatrix},\\ D_1^\dagger&=\begin{pmatrix}[\hat{a}_2,e]&[\hat{a}_1,e]^\dagger\\[\hat{a}_1,e]&-[\hat{a}_2,e]^\dagger\end{pmatrix}.\end{aligned} \tag{21}$$

It is easy to verify that

$$\begin{aligned}&\left\|[\mathcal{D},\pi(e)]^\dagger[\mathcal{D},\pi(e)]\right\|_{\mathrm{op}}\\ &=\frac{2}{\hbar}\max\left\{\|D_1^\dagger D_1\|_{\mathrm{op}},\|D_1D_1^\dagger\|_{\mathrm{op}}\right\}.\end{aligned} \tag{22}$$

It is known that for any bounded operator A, there is $\|A\|_{\mathrm{op}}^2=\|A^\dagger\|_{\mathrm{op}}^2=\|A^\dagger A\|_{\mathrm{op}}$. Therefore we have

$$\begin{aligned}\left\|[\mathcal{D},\pi(e)]\right\|_{\mathrm{op}}^2 &= \left\|[\mathcal{D},\pi(e)]^\dagger[\mathcal{D},\pi(e)]\right\|_{\mathrm{op}}\\ &=\frac{2}{\hbar}\|D_1^\dagger D_1\|_{\mathrm{op}}=\frac{2}{\hbar}\|D_1D_1^\dagger\|_{\mathrm{op}}\\ &=\frac{2}{\hbar}\|D_1\|_{\mathrm{op}}^2=\frac{2}{\hbar}\|D_1^\dagger\|_{\mathrm{op}}^2.\end{aligned} \tag{23}$$

So for any Hermitian element $e\in B$, using the ball condition in formula (16), one can obtain the following inequality:

$$\|D_1^\dagger D_1\|_{\mathrm{op}}=\|D_1D_1^\dagger\|_{\mathrm{op}}=\frac{\hbar}{2}\left\|[\mathcal{D},\pi(e)]\right\|_{\mathrm{op}}^2\leqslant\frac{\hbar}{2}. \tag{24}$$

From Eq. (21), one can obtain

$$D_1^\dagger D_1=\begin{pmatrix}[\hat{a}_2,e]&[\hat{a}_1,e]^\dagger\\[\hat{a}_1,e]&-[\hat{a}_2,e]^\dagger\end{pmatrix}\begin{pmatrix}[\hat{a}_2,e]^\dagger&[\hat{a}_1,e]^\dagger\\[\hat{a}_1,e]&-[\hat{a}_2,e]\end{pmatrix}=\begin{pmatrix}[\hat{a}_2,e][\hat{a}_2,e]^\dagger+[\hat{a}_1,e]^\dagger[\hat{a}_1,e]&[\hat{a}_2,e][\hat{a}_1,e]^\dagger-[\hat{a}_1,e]^\dagger[\hat{a}_2,e]\\[\hat{a}_1,e][\hat{a}_2,e]^\dagger-[\hat{a}_2,e]^\dagger[\hat{a}_1,e]&[\hat{a}_1,e][\hat{a}_1,e]^\dagger+[\hat{a}_2,e]^\dagger[\hat{a}_2,e]\end{pmatrix}. \tag{25}$$

So we have

$$\begin{aligned}\|D_1^\dagger D_1\|_{\rm op} &= \|D_1\|_{\rm op}^2 = \sup_{\substack{\psi\in\mathcal{F}\otimes\mathbb{C}^2,\\ \langle\psi|\psi\rangle=1}} \|D_1\psi\|^2 = \sup_{\substack{\psi\in\mathcal{F}\otimes\mathbb{C}^2,\\ \langle\psi|\psi\rangle=1}} {\rm tr}(\psi^\dagger D_1^\dagger D_1\psi) = \sup_{\substack{\psi\in\mathcal{F}\otimes\mathbb{C}^2,\\ \langle\psi|\psi\rangle=1}} \langle\psi|D_1^\dagger D_1|\psi\rangle \\ &\geqslant \sup_{\substack{\psi\in\mathcal{F}\otimes\mathbb{C}^2,\\ \langle\psi|\psi\rangle=1}} \langle\phi|[\hat{a}_2,e][\hat{a}_2,e]^\dagger+[\hat{a}_1,e]^\dagger[\hat{a}_1,e]|\phi\rangle \qquad \left(\text{here } \psi=\begin{pmatrix}|\phi\rangle\\0\end{pmatrix}\right) \\ &= \sup_{\phi\in\mathcal{F},\langle\phi|\phi\rangle=1} \langle\phi|[\hat{a}_1,e]^\dagger[\hat{a}_1,e]+[\hat{a}_2,e][\hat{a}_2,e]^\dagger|\phi\rangle. \end{aligned} \tag{26}$$

From expressions (24) and (26), for any Hermitian element $e\in B$, there is the following inequality:

$$\sup_{\phi\in\mathcal{F},\langle\phi|\phi\rangle=1} \langle\phi|[\hat{a}_1,e]^\dagger[\hat{a}_1,e]+[\hat{a}_2,e][\hat{a}_2,e]^\dagger|\phi\rangle \leqslant \frac{\hbar}{2}. \tag{27}$$

Similarly, we have

$$\sup_{\phi\in\mathcal{F},\langle\phi|\phi\rangle=1} \langle\phi|[\hat{a}_1,e][\hat{a}_1,e]^\dagger+[\hat{a}_2,e]^\dagger[\hat{a}_2,e]|\phi\rangle \leqslant \frac{\hbar}{2}. \tag{28}$$

From Eq. (21), we also have

$$D_1D_1^\dagger = \begin{pmatrix} [\hat{a}_2,e]^\dagger[\hat{a}_2,e]+[\hat{a}_1,e]^\dagger[\hat{a}_1,e] & [\hat{a}_2,e]^\dagger[\hat{a}_1,e]^\dagger-[\hat{a}_1,e]^\dagger[\hat{a}_2,e]^\dagger \\ [\hat{a}_1,e][\hat{a}_2,e]-[\hat{a}_2,e][\hat{a}_1,e] & [\hat{a}_1,e][\hat{a}_1,e]^\dagger+[\hat{a}_2,e][\hat{a}_2,e]^\dagger \end{pmatrix}, \tag{29}$$

and there are also the following inequalities:

$$\begin{aligned} &\sup_{\phi\in\mathcal{F},\langle\phi|\phi\rangle=1} \langle\phi|[\hat{a}_1,e]^\dagger[\hat{a}_1,e]+[\hat{a}_2,e]^\dagger[\hat{a}_2,e]|\phi\rangle \leqslant \frac{\hbar}{2}, \\ &\sup_{\phi\in\mathcal{F},\langle\phi|\phi\rangle=1} \langle\phi|[\hat{a}_1,e][\hat{a}_1,e]^\dagger+[\hat{a}_2,e][\hat{a}_2,e]^\dagger|\phi\rangle \leqslant \frac{\hbar}{2}. \end{aligned} \tag{30}$$

4. Optimal element and constraint relations

Consider the 2D harmonic oscillator states $|p,q\rangle$ and $|m,n\rangle$, and the corresponding density matrices are $\rho_{p,q}=|p,q\rangle\langle p,q|$ and $\rho_{m,n}=|m,n\rangle\langle m,n|$. The Connes distance between these two states is

$$\begin{aligned} d(\omega_{p,q},\omega_{m,n}) &= \sup_{e\in B}|{\rm tr}_{\mathcal{F}}(\rho_{p,q}e)-{\rm tr}_{\mathcal{F}}(\rho_{m,n}e)| \\ &= \sup_{e\in B}|{\rm tr}_{\mathcal{F}}(\Delta\rho\, e)| \\ &= \sup_{e\in B}|\langle p,q|e|p,q\rangle-\langle m,n|e|m,n\rangle|, \end{aligned} \tag{31}$$

where $\Delta\rho=\rho_{p,q}-\rho_{m,n}=|p,q\rangle\langle p,q|-|m,n\rangle\langle m,n|$. The element $e\in B$ can be expressed as

$$e=\sum_{i,j,k,l=0}^{\infty} C_{k,l}^{i,j}|i,j\rangle\langle k,l|. \tag{32}$$

Consider the element e attaining the supremum in Eq. (31), which is the so-called optimal element. Since Hermitian elements can give the supremum in the Connes distance function,[26] one can assume $C_{k,l}^{i,j}=(C_{i,j}^{k,l})^*$, and $C_{i,j}^{i,j}$ are real numbers. After some straightforward calculations, one can obtain

$$d(\omega_{p,q},\omega_{m,n})=\sup_{e\in B}|C_{p,q}^{p,q}-C_{m,n}^{m,n}|. \tag{33}$$

For the element e [Eq. (32)], using the relations (7), we have

$$\begin{aligned}[\hat{a}_1,e] &= \hat{a}_1e-e\hat{a}_1=\sum_{i,j,k,l=0}^{\infty}C_{k,l}^{i,j}\sqrt{i}|i-1,j\rangle\langle k,l|-\sum_{i,j,k,l=0}^{\infty}C_{k,l}^{i,j}|i,j\rangle\langle k+1,l|\sqrt{k+1} \\ &= \sum_{i,j,k,l=0}^{\infty}\left(C_{k,l}^{i+1,j}\sqrt{i+1}-C_{k-1,l}^{i,j}\sqrt{k}\right)|i,j\rangle\langle k,l|, \end{aligned} \tag{34}$$

$$\begin{aligned}[\hat{a}_1,e][\hat{a}_1,e]^\dagger &= \left(\sum_{i,j,k,l=0}^{\infty}\left(C_{k,l}^{i+1,j}\sqrt{i+1}-C_{k-1,l}^{i,j}\sqrt{k}\right)|i,j\rangle\langle k,l|\right)\left(\sum_{i,j,k,l=0}^{\infty}\left(C_{k,l}^{i+1,j}\sqrt{i+1}-C_{k-1,l}^{i,j}\sqrt{k}\right)^*|k,l\rangle\langle i,j|\right) \\ &= \sum_{i,j=0}^{\infty}\left(\sum_{k,l=0}^{\infty}\left|C_{k,l}^{i+1,j}\sqrt{i+1}-C_{k-1,l}^{i,j}\sqrt{k}\right|^2\right)|i,j\rangle\langle i,j|+n.d., \end{aligned} \tag{35}$$

where "$n.d.$" denotes the sum of terms $|k,l\rangle\langle i,j|$ with $k\neq i$ and/or $l\neq j$, and these terms can be ignored in the calculations in the following content.

We also have

$$[\hat{a}_2,e]=\sum_{i,j,k,l=0}^{\infty}\left(C_{k,l}^{i,j+1}\sqrt{j+1}-C_{k,l-1}^{i,j}\sqrt{l}\right)|i,j\rangle\langle k,l|, \tag{36}$$

$$[\hat{a}_2,e]^{\dagger}[\hat{a}_2,e]=\sum_{i,j=0}^{\infty}\left(\sum_{k,l=0}^{\infty}\left|C_{k,l+1}^{i,j}\sqrt{l+1}-C_{k,l}^{i,j-1}\sqrt{j}\right|^2\right)|i,j\rangle\langle i,j|+n.d. \tag{37}$$

So there is

$$[\hat{a}_1,e][\hat{a}_1,e]^{\dagger}+[\hat{a}_2,e]^{\dagger}[\hat{a}_2,e]=\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\left(\left|C_{k,l}^{i+1,j}\sqrt{i+1}-C_{k-1,l}^{i,j}\sqrt{k}\right|^2+\left|C_{k,l+1}^{i,j}\sqrt{l+1}-C_{k,l}^{i,j-1}\sqrt{j}\right|^2\right)|i,j\rangle\langle i,j|+n.d. \tag{38}$$

Similarly, we also have

$$[\hat{a}_1,e]^{\dagger}[\hat{a}_1,e]+[\hat{a}_2,e][\hat{a}_2,e]^{\dagger}=\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\left(\left|C_{k+1,l}^{i,j}\sqrt{k+1}-C_{k,l}^{i-1,j}\sqrt{i}\right|^2+\left|C_{k,l}^{i,j+1}\sqrt{j+1}-C_{k,l-1}^{i,j}\sqrt{l}\right|^2\right)|i,j\rangle\langle i,j|+n.d. \tag{39}$$

From expressions (27), (28), (38), and (39), one can obtain the following inequalities:

$$\begin{aligned}&\sum_{k,l=0}^{\infty}\left(\left|C_{k+1,l}^{i,j}\sqrt{k+1}-C_{k,l}^{i-1,j}\sqrt{i}\right|^2+\left|C_{k,l}^{i,j+1}\sqrt{j+1}-C_{k,l-1}^{i,j}\sqrt{l}\right|^2\right)\leqslant\frac{\hbar}{2},\\&\sum_{k,l=0}^{\infty}\left(\left|C_{k,l}^{i+1,j}\sqrt{i+1}-C_{k-1,l}^{i,j}\sqrt{k}\right|^2+\left|C_{k,l+1}^{i,j}\sqrt{l+1}-C_{k,l}^{i,j-1}\sqrt{j}\right|^2\right)\leqslant\frac{\hbar}{2}.\end{aligned} \tag{40}$$

Similar to Refs. [14,20], one can just consider the optimal elements being diagonal elements. So we can set $C_{k,l}^{i,j}=0$ if $k\neq i$ and/or $l\neq j$, and denote $C_{i,j}^{i,j}=c_{i,j}$. For $i,j=0,1,2,\ldots$, one can obtain the following inequalities:

$$\begin{aligned}&\left|c_{i,j}\sqrt{i}-c_{i-1,j}\sqrt{i}\right|^2+\left|c_{i,j+1}\sqrt{j+1}-c_{i,j}\sqrt{j+1}\right|^2\leqslant\frac{\hbar}{2},\\&\left|c_{i+1,j}\sqrt{i+1}-c_{i,j}\sqrt{i+1}\right|^2+\left|c_{i,j}\sqrt{j}-c_{i,j-1}\sqrt{j}\right|^2\leqslant\frac{\hbar}{2}.\end{aligned} \tag{41}$$

Since $c_{i,j}$ are real, we have

$$\begin{aligned}&i\left(c_{i,j}-c_{i-1,j}\right)^2+(j+1)\left(c_{i,j+1}-c_{i,j}\right)^2\leqslant\frac{\hbar}{2},\\&(i+1)\left(c_{i+1,j}-c_{i,j}\right)^2+j\left(c_{i,j}-c_{i,j-1}\right)^2\leqslant\frac{\hbar}{2}.\end{aligned} \tag{42}$$

Denote

$$c_{i,j}-c_{i-1,j}=E_{i,j},\quad c_{i,j}-c_{i,j-1}=F_{i,j}, \tag{43}$$

then the inequalities (42) can be rewritten as

$$\begin{aligned}&iE_{i,j}^2+(j+1)F_{i,j+1}^2\leqslant\frac{\hbar}{2},\\&(i+1)E_{i+1,j}^2+jF_{i,j}^2\leqslant\frac{\hbar}{2}.\end{aligned} \tag{44}$$

Similarly, from inequality (30), one can obtain the following inequalities:

$$\begin{aligned}&i\left(c_{i,j}-c_{i-1,j}\right)^2+j\left(c_{i,j}-c_{i,j-1}\right)^2\leqslant\frac{\hbar}{2},\\&(i+1)\left(c_{i+1,j}-c_{i,j}\right)^2+(j+1)\left(c_{i,j+1}-c_{i,j}\right)^2\leqslant\frac{\hbar}{2},\end{aligned} \tag{45}$$

or

$$\begin{aligned}&iE_{i,j}^2+jF_{i,j}^2\leqslant\frac{\hbar}{2},\\&(i+1)E_{i+1,j}^2+(j+1)F_{i,j+1}^2\leqslant\frac{\hbar}{2}.\end{aligned} \tag{46}$$

So in order to obtain the supremum in Eq. (31), in general, one only need to consider the so-called optimal elements $e_{\rm o}\in B$ which can be expressed as

$$e_{\rm o}=\sum_{i,j=0}^{\infty}c_{i,j}|i,j\rangle\langle i,j|, \tag{47}$$

where the coefficients $c_{i,j}$ satisfy the constraint relations (42), (45) or (44), (46), and the Connes distance (31) is obtained by

$$\begin{aligned}d(\omega_{p,q},\omega_{m,n})&=|{\rm tr}_{\mathcal{F}}(\rho_{p,q}e_{\rm o})-{\rm tr}_{\mathcal{F}}(\rho_{m,n}e_{\rm o})|\\&=|\langle p,q|e_{\rm o}|p,q\rangle-\langle m,n|e_{\rm o}|m,n\rangle|.\end{aligned} \tag{48}$$

5. Connes distance between harmonic oscillator states

Now let us explicitly calculate the Connes distances between the quantum states of 2D harmonic oscillators. First, let us consider the adjacent states $|m+1,n\rangle$ and $|m,n\rangle$, and the corresponding Connes distance is

$$\begin{aligned}d(\omega_{m+1,n},\omega_{m,n})&=\sup_{e\in B}|{\rm tr}_{\mathcal{F}}(\rho_{m+1,n}e)-{\rm tr}_{\mathcal{F}}(\rho_{m,n}e)|\\&=\sup_{e\in B}|c_{m+1,n}-c_{m,n}|=\sup_{e\in B}|E_{m+1,n}|.\end{aligned} \tag{49}$$

From expressions (44) and (46), we have

$$(m+1)E_{m+1,n}^2+(n+1)F_{m+1,n+1}^2\leqslant\frac{\hbar}{2},$$

$$
\begin{aligned}
&(m+1)E_{m+1,n}^2+nF_{m,n}^2\leqslant\frac{\hbar}{2},\\
&(m+1)E_{m+1,n}^2+(n+1)F_{m,n+1}^2\leqslant\frac{\hbar}{2},\\
&(m+1)E_{m+1,n}^2+nF_{m+1,n}^2\leqslant\frac{\hbar}{2}.
\end{aligned}\tag{50}
$$

So in order to attain the supremum of $|c_{m+1,n}-c_{m,n}|=|E_{m+1,n}|$, there should be

$$
(m+1)E_{m+1,n}^2=\frac{\hbar}{2},\tag{51}
$$

or

$$
|c_{m+1,n}-c_{m,n}|=|E_{m+1,n}|=\sqrt{\frac{\hbar}{2(m+1)}},\tag{52}
$$

and then there should be

$$
F_{m,n+1}=F_{m+1,n+1}=0.\tag{53}
$$

Without loss of generality, we can assume $E_{m+1,n}\geqslant 0$, and

$$
\begin{aligned}
E_{m+1,n}&=F_{m+1,n+1}+E_{m+1,n}=c_{m+1,n+1}-c_{m,n}\\
&=E_{m+1,n+1}+F_{m,n+1}=E_{m+1,n+1}.
\end{aligned}\tag{54}
$$

So it is easy to see that, if $E_{m+1,n}$ satisfy Eq. (51), there must be

$$
E_{m+1,i}=\sqrt{\frac{\hbar}{2(m+1)}},\quad F_{m,i+1}=F_{m+1,i+1}=0,\tag{55}
$$

where $i=0,1,2,\ldots$.

Furthermore, let us consider the states $|m+k,n\rangle$ and $|m,n\rangle$, $k>1$, and the corresponding Connes distance is

$$
\begin{aligned}
&d(\omega_{m+k,n},\omega_{m,n})\\
&=\sup_{e\in B}|\mathrm{tr}_{\mathcal{F}}(\rho_{m+k,n}e)-\mathrm{tr}_{\mathcal{F}}(\rho_{m,n}e)|\\
&=\sup_{e\in B}|c_{m+k,n}-c_{m,n}|\\
&=\sup_{e\in B}|c_{m+k,n}-c_{m+k-1,n}+c_{m+k-1,n}\\
&\quad-c_{m+k-2,n}+\ldots+c_{m+1,n}-c_{m,n}|\\
&\leqslant\sup_{e\in B}\big(|c_{m+k,n}-c_{m+k-1,n}|+|c_{m+k-1,n}\\
&\quad-c_{m+k-2,n}|+\cdots+|c_{m+1,n}-c_{m,n}|\big)\\
&\leqslant\sqrt{\frac{\hbar}{2}}\left(\frac{1}{\sqrt{m+k}}+\frac{1}{\sqrt{m+k-1}}+\cdots+\frac{1}{\sqrt{m+1}}\right)\\
&=\sqrt{\frac{\hbar}{2}}\sum_{i=1}^{k}\frac{1}{\sqrt{m+i}}=\zeta_{m;m+k}\sqrt{\frac{\hbar}{2}},
\end{aligned}\tag{56}
$$

where

$$
\zeta_{p;q}=\sum_{i=p+1}^{q}\frac{1}{\sqrt{i}}=\zeta\left(\frac{1}{2},p+1\right)-\zeta\left(\frac{1}{2},q+1\right),\tag{57}
$$

and $\zeta(s,q)$ is the Hurwitz zeta function

$$
\zeta(s,q)=\sum_{i=0}^{\infty}\frac{1}{(q+i)^s}.\tag{58}
$$

Obviously, there is $\zeta_{i;j}=-\zeta_{j;i}$ and $\zeta_{i;i}=0$.

In order to attain the supremum of $|c_{m+k,n}-c_{m,n}|$, for example, one can choose

$$
\begin{aligned}
&c_{0,i}=0,\quad c_{1,i}=\sqrt{\frac{\hbar}{2}},\quad c_{2,i}=\sqrt{\frac{\hbar}{2}}\left(1+\frac{1}{\sqrt{2}}\right),\quad\cdots\\
&c_{p,i}=\sqrt{\frac{\hbar}{2}}\sum_{j=1}^{p}\frac{1}{\sqrt{j}}=\zeta_{0;p}\sqrt{\frac{\hbar}{2}},\quad\cdots
\end{aligned}\tag{59}
$$

where $p,i=0,1,2\ldots$, and there is

$$
\begin{aligned}
&c_{m+i,n}-c_{m+i-1,n}=E_{m+i,n}=\sqrt{\frac{\hbar}{2(m+i)}},\\
&i=1,2,\ldots,k.
\end{aligned}\tag{60}
$$

So there is

$$
e_{\mathrm{o}}'=\sum_{p,i=0}^{\infty}c_{p,i}|p,i\rangle\langle p,i|=\sqrt{\frac{\hbar}{2}}\sum_{p,i=0}^{\infty}\zeta_{0;p}|p,i\rangle\langle p,i|.\tag{61}
$$

In fact, it is easy to see that the element e_{o}' [Eq. (61)] will diverge in the Hilbert–Schmidt norm of $\mathcal{Q}$. Therefore one can not directly use the above element e_{o}' [Eq. (61)] as the optimal element in the calculation of the Connes distance (56). One can just obtain the result by taking the limit. So the Connes distance between the states $|m+k,n\rangle$ and $|m,n\rangle$ is

$$
\begin{aligned}
&d(\omega_{m+k,n},\omega_{m,n})\\
&=\lim_{e\to e_{\mathrm{o}}'}|\mathrm{tr}_{\mathcal{F}}(\rho_{m+k,n}e)-\mathrm{tr}_{\mathcal{F}}(\rho_{m,n}e)|=|c_{m+k,n}-c_{m,n}|\\
&=\sqrt{\frac{\hbar}{2}}\sum_{i=1}^{k}\frac{1}{\sqrt{m+i}}=\zeta_{m;m+k}\sqrt{\frac{\hbar}{2}}\\
&=\sqrt{\frac{\hbar}{2}}\left(\zeta\left(\frac{1}{2},m+1\right)-\zeta\left(\frac{1}{2},m+k+1\right)\right).
\end{aligned}\tag{62}
$$

Of course, there is $d(\omega_{m+k,n},\omega_{m,n})=d(\omega_{m,n},\omega_{m+k,n})$. These distances also satisfy

$$
\begin{aligned}
&d(\omega_{m+k+l,n},\omega_{m,n})\\
&=d(\omega_{m+k,n},\omega_{m,n})+d(\omega_{m+k+l,n},\omega_{m+k,n}),\quad k,l\geqslant 0.
\end{aligned}\tag{63}
$$

These results coincide with the results of the Connes distance between the quantum states of 1D harmonic oscillators in Refs. [14,18,20]. This means that the Connes spectral distance between the states $|m,n\rangle$ and $|m+k,n\rangle$ of 2D harmonic oscillators is the same as that between the states $|m\rangle$ and $|m+k\rangle$ of 1D harmonic oscillators.

Similarly, there is

$$
\begin{aligned}
&d(\omega_{m,n+k},\omega_{m,n})\\
&=\sqrt{\frac{\hbar}{2}}\sum_{i=1}^{k}\frac{1}{\sqrt{n+i}}=\zeta_{n;n+k}\sqrt{\frac{\hbar}{2}}\\
&=\sqrt{\frac{\hbar}{2}}\left(\zeta\left(\frac{1}{2},n+1\right)-\zeta\left(\frac{1}{2},n+k+1\right)\right).
\end{aligned}\tag{64}
$$

Furthermore, using the above method, one can find that for any integers $m,n,k,l \geqslant 0$, the Connes distance between the states $|m,n\rangle$ and $|k,l\rangle$ is (see Appendix B for more details)

$$d(\omega_{k,l},\omega_{m,n}) = c_0\zeta_{m;k} + d_0\zeta_{n;l} = \sqrt{\frac{\hbar}{2}}\sqrt{\zeta_{m;k}^2+\zeta_{n;l}^2}, \quad (65)$$

where

$$c_0 = \sqrt{\frac{\hbar}{2}}\frac{\zeta_{m;k}}{\sqrt{\zeta_{m;k}^2+\zeta_{n;l}^2}}, \quad d_0 = \sqrt{\frac{\hbar}{2}}\frac{\zeta_{n;l}}{\sqrt{\zeta_{m;k}^2+\zeta_{n;l}^2}}. \quad (66)$$

Obviously, there are

$$\begin{aligned} d(\omega_{k,l},\omega_{m,n}) &= d(\omega_{m,l},\omega_{k,n}) = d(\omega_{k,n},\omega_{m,l}) \\ &= d(\omega_{m,n},\omega_{k,l}). \end{aligned} \quad (67)$$

From Eqs. (62), (64), and (65), it is easy to verify that there is

$$\begin{aligned} d(\omega_{k,l},\omega_{m,n}) &= \sqrt{d(\omega_{k,l},\omega_{k,n})^2 + d(\omega_{k,n},\omega_{m,n})^2} \\ &= \sqrt{d(\omega_{k,l},\omega_{m,l})^2 + d(\omega_{m,l},\omega_{m,n})^2}. \end{aligned} \quad (68)$$

So these spectral distances also satisfy the Pythagoras theorem. This also coincides with the result in Ref. [16].

Furthermore, one can also analyse the Connes distance between mixed states. But usually, the calculations and results will be much more cumbersome.

6. Discussion and conclusions

In this paper, we study the Connes distance of quantum states of 2D harmonic oscillators in phase space. By virtue of the Hilbert–Schmidt operatorial formulation, we construct a boson Fock space and a quantum Hilbert space, and obtain the Dirac operator and a spectral triple corresponding to a 4D quantum phase space. From the ball condition, we obtain some constraint relations about the optimal elements. Using these constraint relations, we derive the explicit expressions of the Connes distance between the states of 2D quantum harmonic oscillators.

We find that the Connes distance between two arbitrary Fock states is not just a simple sum of distances of adjacent Fock states. Furthermore, these spectral distances also satisfy the Pythagoras theorem. The calculations and results of the Connes distance of quantum states of 2D harmonic oscillators are much more complicated than those of the states of 1D harmonic oscillators. These results are not just trivial generalizations of the result of 1D case. Our method used in the present work is also different from those used in the literatures. So these results are significant for the study of the Connes distances of physical systems in noncommutative spaces.

Here we only analyse the Connes distance between two pure states. One can also study the distance between two mixed states. Furthermore, one can use our method to study the Connes distance between the quantum states of higher-dimensional harmonic oscillators. But usually the calculations and results will be much more complicated.

We are still exploring the intuitive physical meaning of these distances of physical systems. Our methods can be used to study other physical systems in other kinds of noncommutative spaces. We hope that our results can help to study the mathematical structures and physical properties of noncommutative spaces.

Appendix A: Dirac operator for 4D quantum phase space

Similar to Ref. [20], in order to construct the Dirac operator for the 4D quantum phase space, one can consider the following extended noncommutative phase space in which the coordinate operators $\hat{X}_i$, $\hat{Y}_i$, and $\hat{P}_i$, $\hat{Q}_i$ satisfy the following commutation relations:[24,25]

$$\begin{aligned} &[\hat{X}_i,\hat{P}_j] = [\hat{Y}_i,\hat{Q}_j] = \mathrm{i}\delta_{ij}\hbar, \\ &[\hat{X}_i,\hat{Y}_j] = \mathrm{i}\delta_{ij}\mu, \quad [\hat{P}_i,\hat{Q}_j] = \mathrm{i}\delta_{ij}\nu, \end{aligned} \quad (A1)$$

and others vanish. Here $i,j = 1,2$, and μ, ν are some parameters. It is easy to verify that, a unitary representation of the noncommutative Heisenberg algebra (A1) can be obtained by the following actions on the quantum Hilbert space $\mathcal{Q}$:

$$\begin{aligned} &\hat{X}_i|\phi) = |\tilde{x}_i\phi), \quad \hat{Y}_i|\phi) = \frac{\mu}{\hbar}|\tilde{p}_i\phi) - \frac{\sqrt{\hbar^2-\mu\nu}}{\hbar}|\phi\tilde{x}_i), \\ &\hat{P}_i|\phi) = |\tilde{p}_i\phi), \quad \hat{Q}_i|\phi) = -\frac{\nu}{\hbar}|\tilde{x}_i\phi) + \frac{\sqrt{\hbar^2-\mu\nu}}{\hbar}|\phi\tilde{p}_i). \end{aligned} \quad (A2)$$

By virtue of the result in Ref. [4], the Dirac operator $\mathcal{D}$ can be written as

$$\mathcal{D} = \frac{1}{\mu}\gamma^1\hat{Y}_1 + \frac{1}{\nu}\gamma^2\hat{Q}_1 + \frac{1}{\mu}\gamma^3\hat{Y}_2 + \frac{1}{\nu}\gamma^4\hat{Q}_2, \quad (A3)$$

where γ^k's are the Euclidean–Dirac matrices satisfying

$$\gamma^k\gamma^l + \gamma^l\gamma^k = 2\delta_{kl}\boldsymbol{I}_4, \quad k,l = 1,2,3,4, \quad (A4)$$

and $\boldsymbol{I}_4$ is the 4×4 identity matrix. For example, one can choose

$$\begin{aligned} &\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ &\gamma^3 = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (A5)$$

and the Dirac operator (A3) can be expressed as

$$\mathcal{D}=\begin{pmatrix} 0 & 0 & \mathrm{i}\frac{1}{\mu}\hat{Y}_2+\frac{1}{\nu}\hat{Q}_2 & \mathrm{i}\frac{1}{\mu}\hat{Y}_1+\frac{1}{\nu}\hat{Q}_1 \\ 0 & 0 & \mathrm{i}\frac{1}{\mu}\hat{Y}_1-\frac{1}{\nu}\hat{Q}_1 & -\mathrm{i}\frac{1}{\mu}\hat{Y}_2+\frac{1}{\nu}\hat{Q}_2 \\ -\mathrm{i}\frac{1}{\mu}\hat{Y}_2+\frac{1}{\nu}\hat{Q}_2 & -\mathrm{i}\frac{1}{\mu}\hat{Y}_1-\frac{1}{\nu}\hat{Q}_1 & 0 & 0 \\ -\mathrm{i}\frac{1}{\mu}\hat{Y}_1+\frac{1}{\nu}\hat{Q}_1 & \mathrm{i}\frac{1}{\mu}\hat{Y}_2+\frac{1}{\nu}\hat{Q}_2 & 0 & 0 \end{pmatrix}. \tag{A6}$$

After some straightforward calculations, one can obtain the commutator $[\mathcal{D},\pi(a)]$ acting on an element $\Phi\in\mathcal{Q}\otimes\mathbb{C}^4$ as

$$[\mathcal{D},\pi(a)]\Phi=[\mathcal{D},\pi(a)]\begin{pmatrix}|\phi_1\rangle\\|\phi_2\rangle\\|\phi_3\rangle\\|\phi_4\rangle\end{pmatrix}=\frac{1}{\hbar}\begin{pmatrix} 0 & 0 & [-\tilde{x}_2+\mathrm{i}\tilde{p}_2,a] & [-\tilde{x}_1+\mathrm{i}\tilde{p}_1,a] \\ 0 & 0 & [\tilde{x}_1+\mathrm{i}\tilde{p}_1,a] & [-\tilde{x}_2-\mathrm{i}\tilde{p}_2,a] \\ [-\tilde{x}_2-\mathrm{i}\tilde{p}_2,a] & [\tilde{x}_1-\mathrm{i}\tilde{p}_1,a] & 0 & 0 \\ [-\tilde{x}_1-\mathrm{i}\tilde{p}_1,a] & [-\tilde{x}_2+\mathrm{i}\tilde{p}_2,a] & 0 & 0 \end{pmatrix}\begin{pmatrix}|\phi_1\rangle\\|\phi_2\rangle\\|\phi_3\rangle\\|\phi_4\rangle\end{pmatrix}. \tag{A7}$$

Regarding Φ as a test function, one can identify the Dirac operator $\mathcal{D}$ as

$$\mathcal{D}=\frac{1}{\hbar}\begin{pmatrix} 0 & 0 & -\tilde{x}_2+\mathrm{i}\tilde{p}_2 & -\tilde{x}_1+\mathrm{i}\tilde{p}_1 \\ 0 & 0 & \tilde{x}_1+\mathrm{i}\tilde{p}_1 & -\tilde{x}_2-\mathrm{i}\tilde{p}_2 \\ -\tilde{x}_2-\mathrm{i}\tilde{p}_2 & \tilde{x}_1-\mathrm{i}\tilde{p}_1 & 0 & 0 \\ -\tilde{x}_1-\mathrm{i}\tilde{p}_1 & -\tilde{x}_2+\mathrm{i}\tilde{p}_2 & 0 & 0 \end{pmatrix}. \tag{A}$$

Appendix B: Calculations of the distance between $|m,n\rangle$ and $|m+k,n+l\rangle$

First, let us consider the states $|m+1,n+1\rangle$ and $|m,n\rangle$, and the corresponding Connes distance is

$$\begin{aligned} &d(\omega_{m+1,n+1},\omega_{m,n})\\ &=\sup_{e\in B}|\mathrm{tr}_{\mathcal{F}}(\rho_{m+1,n+1}e)-\mathrm{tr}_{\mathcal{F}}(\rho_{m,n}e)|\\ &=\sup_{e\in B}|c_{m+1,n+1}-c_{m,n}|=\sup_{e\in B}|E_{m+1,n+1}+F_{m,n+1}|\\ &=\sup_{e\in B}\left|\frac{1}{\sqrt{m+1}}E_{m+1,n+1}\sqrt{m+1}+\frac{1}{\sqrt{n+1}}F_{m,n+1}\sqrt{n+1}\right|\\ &\leqslant\sup_{e\in B}\sqrt{\frac{1}{m+1}+\frac{1}{n+1}}\sqrt{(m+1)E^2_{m+1,n+1}+(n+1)F^2_{m,n+1}}\\ &\leqslant\sqrt{\frac{1}{m+1}+\frac{1}{n+1}}\sqrt{\frac{\hbar}{2}}. \end{aligned} \tag{B1}$$

In the first inequality, we have used the Cauchy–Schwartz inequality

$$(x_1y_1+x_2y_2)^2\leqslant(x_1^2+x_2^2)(y_1^2+y_2^2), \tag{B2}$$

where equality holds if $x_1y_2=x_2y_1$.

So in order to attain the supremum of $|E_{m+1,n+1}+F_{m,n+1}|$, from expression (44), there should be

$$(m+1)E^2_{m+1,n+1}+(n+1)F^2_{m,n+1}=\frac{\hbar}{2}, \tag{B3}$$

$$(m+1)E_{m+1,n+1}=(n+1)F_{m,n+1}. \tag{B4}$$

Without loss of generality, we can assume that $E_{m+1,n+1},F_{m,n+1}\geqslant 0$. From expression (46), we have

$$\begin{aligned} &(m+1)E^2_{m+1,n+1}+(n+1)F^2_{m+1,n+1}\leqslant\frac{\hbar}{2},\\ &(m+1)E^2_{m+1,n}+(n+1)F^2_{m,n+1}\leqslant\frac{\hbar}{2}. \end{aligned} \tag{B5}$$

Compare the above inequalities with Eq. (B3), one can obtain

$$E_{m+1,n}\leqslant E_{m+1,n+1},\quad F_{m+1,n+1}\leqslant F_{m,n+1}. \tag{B6}$$

But

$$\begin{aligned} E_{m+1,n}+F_{m+1,n+1}&=E_{m+1,n+1}+F_{m,n+1}\\ &=c_{m+1,n+1}-c_{m,n}, \end{aligned} \tag{B7}$$

so there must be

$$E_{m+1,n}=E_{m+1,n+1},\quad F_{m+1,n+1}=F_{m,n+1}. \tag{B8}$$

Next, let us consider the Fock states $|m+2,n+1\rangle$ and $|m,n\rangle$, and the corresponding Connes distance is

$$\begin{aligned} &d(\omega_{m+2,n+1},\omega_{m,n})\\ &=\sup_{e\in B}|\mathrm{tr}_{\mathcal{F}}(\rho_{m+2,n+1}e)-\mathrm{tr}_{\mathcal{F}}(\rho_{m,n}e)|\\ &=\sup_{e\in B}|c_{m+2,n+1}-c_{m,n}|\\ &=\sup_{e\in B}|E_{m+2,n+1}+F_{m+1,n+1}+E_{m+1,n}|. \end{aligned} \tag{B9}$$

In order to attain the supremum of $|E_{m+2,n+1}+F_{m+1,n+1}+E_{m+1,n}|$, there must be

$$\begin{aligned} &(m+1)E^2_{m+1,n}+(n+1)F^2_{m+1,n+1}=\frac{\hbar}{2},\\ &(m+2)E^2_{m+2,n+1}+(n+1)F^2_{m+1,n+1}=\frac{\hbar}{2}, \end{aligned} \tag{B10}$$

and then

$$(m+1)E^2_{m+1,n}=(m+2)E^2_{m+2,n+1}. \tag{B11}$$

Without loss of generality, we can assume $E_{m+2,n+1}$, $E_{m+1,n}$, $F_{m+1,n+1}\geqslant 0$. It is easy to see that, there must be

$$\begin{aligned} E_{m+1,n+1}=E_{m+1,n}&=\sqrt{\frac{m+2}{m+1}}E_{m+2,n+1}\\ &=\sqrt{\frac{m+2}{m+1}}E_{m+2,n}, \end{aligned} \tag{B12}$$

$$F_{m+2,n+1} = F_{m+1,n+1} = F_{m,n+1}. \tag{B13}$$

So we have

$$\begin{aligned}
&d(\omega_{m+2,n+1}, \omega_{m,n}) \\
&= \sup_{e\in B} |E_{m+2,n+1} + F_{m+1,n+1} + E_{m+1,n}| \\
&= \sup_{e\in B} \left| \left(\frac{1}{\sqrt{m+1}} + \frac{1}{\sqrt{m+2}} \right) E_{m+1,n}\sqrt{m+1} \right. \\
&\quad \left. + \frac{1}{\sqrt{n+1}} F_{m+1,n+1}\sqrt{n+1} \right| \\
&\leqslant \sup_{e\in B} \sqrt{(m+1)E_{m+1,n}^2 + (n+1)F_{m+1,n+1}^2} \\
&\quad \times \sqrt{\left(\frac{1}{\sqrt{m+1}} + \frac{1}{\sqrt{m+2}} \right)^2 + \frac{1}{n+1}} \\
&\leqslant \sqrt{\frac{\hbar}{2}} \sqrt{\left(\frac{1}{\sqrt{m+1}} + \frac{1}{\sqrt{m+2}} \right)^2 + \frac{1}{n+1}}.
\end{aligned} \tag{B14}$$

Here in the first inequality, we have used the Cauchy–Schwartz inequality (B2).

Finally, let us consider the Connes distance between two arbitrary Fock states $|m,n\rangle$ and $|m+k,n+l\rangle$. For simplicity, we assume $k,l \geqslant 0$. The corresponding Connes distance is

$$\begin{aligned}
&d(\omega_{m+k,n+l}, \omega_{m,n}) \\
&= \sup_{e\in B} |\mathrm{tr}_{\mathcal{F}}(\rho_{m+k,n+l}e) - \mathrm{tr}_{\mathcal{F}}(\rho_{m,n}e)| \\
&= \sup_{e\in B} |c_{m+k,n+l} - c_{m,n}| \\
&= \sup_{e\in B} |c_{m+k,n+l} - c_{m+k,n} + c_{m+k,n} - c_{m,n}| \\
&= \sup_{e\in B} \left| \sum_{i=1}^{l} (c_{m+k,n+i} - c_{m+k,n+i-1}) \right. \\
&\quad \left. + \sum_{j=1}^{k} (c_{m+j,n} - c_{m+j-1,n}) \right| \\
&= \sup_{e\in B} \left| \sum_{i=1}^{l} F_{m+k,n+i} + \sum_{j=1}^{k} E_{m+j,n} \right|.
\end{aligned} \tag{B15}$$

Similarly, in order to attain the supremum of $|c_{m+k,n+l} - c_{m,n}|$, there must be

$$(m+1)E_{m+1,n}^2 + (n+1)F_{m,n+1}^2 = \frac{\hbar}{2}, \tag{B16}$$

$$E_{m+i,n+j} = \sqrt{\frac{m+1}{m+i}} E_{m+1,n}, \quad 1 \leqslant i \leqslant k, 0 \leqslant j \leqslant l, \tag{B17}$$

$$F_{m+i,n+j} = \sqrt{\frac{n+1}{n+j}} F_{m,n+1}, \quad 0 \leqslant i \leqslant k, 1 \leqslant j \leqslant l. \tag{B18}$$

So we have

$$\begin{aligned}
&d(\omega_{m+k,n+l}, \omega_{m,n}) \\
&= \sup_{e\in B} \left| \sum_{i=1}^{k} E_{m+i,n} + \sum_{j=1}^{l} F_{m+k,n+j} \right| \\
&= \sup_{e\in B} \left| \sqrt{m+1} E_{m+1,n} \sum_{i=1}^{k} \frac{1}{\sqrt{m+i}} + \sqrt{n+1} F_{m,n+1} \sum_{j=1}^{l} \frac{1}{\sqrt{n+j}} \right| \\
&= \sup_{e\in B} \left| \sqrt{m+1} E_{m+1,n} \zeta_{m;m+k} + \sqrt{n+1} F_{m,n+1} \zeta_{n;n+l} \right| \\
&\leqslant \sup_{e\in B} \sqrt{(m+1)E_{m+1,n}^2 + (n+1)F_{m,n+1}^2} \sqrt{\zeta_{m;m+k}^2 + \zeta_{n;n+l}^2} \\
&= \sqrt{\frac{\hbar}{2}} \sqrt{\zeta_{m;m+k}^2 + \zeta_{n;n+l}^2}.
\end{aligned} \tag{B19}$$

Here we have used the Cauchy–Schwartz inequality (B2) in the above inequality, and the equality holds if

$$\sqrt{m+1} E_{m+1,n} \zeta_{n;n+l} = \sqrt{n+1} F_{m,n+1} \zeta_{m;m+k}. \tag{B20}$$

From Eqs. (B16) and (B20), one can derive

$$E_{m+1,n} = \sqrt{\frac{\hbar}{2(m+1)}} \frac{\zeta_{m;m+k}}{\sqrt{\zeta_{m;m+k}^2 + \zeta_{n;n+l}^2}} = c_0 \frac{1}{\sqrt{m+1}}, \tag{B21}$$

$$F_{m,n+1} = \sqrt{\frac{\hbar}{2(n+1)}} \frac{\zeta_{n;n+l}}{\sqrt{\zeta_{m;m+k}^2 + \zeta_{n;n+l}^2}} = d_0 \frac{1}{\sqrt{n+1}}, \tag{B22}$$

where

$$c_0 = \sqrt{\frac{\hbar}{2}} \frac{\zeta_{m;m+k}}{\sqrt{\zeta_{m;m+k}^2 + \zeta_{n;n+l}^2}}, \quad d_0 = \sqrt{\frac{\hbar}{2}} \frac{\zeta_{n;n+l}}{\sqrt{\zeta_{m;m+k}^2 + \zeta_{n;n+l}^2}}. \tag{B23}$$

In order to attain the supremum of $|c_{m+k,n+l} - c_{m,n}|$, for example, one can choose

$$\begin{aligned}
&c_{0,0} = 0, \quad c_{1,0} = c_0, \quad c_{0,1} = d_0, \quad c_{1,1} = c_0 + d_0, \ldots \\
&c_{p,q} = c_0 \sum_{i=1}^{p} \frac{1}{\sqrt{i}} + d_0 \sum_{j=1}^{q} \frac{1}{\sqrt{j}} = c_0 \zeta_{0;p} + d_0 \zeta_{0;q} \\
&\quad = \sqrt{\frac{\hbar}{2}} \frac{\zeta_{0;p}\zeta_{m;m+k} + \zeta_{0;q}\zeta_{n;n+l}}{\sqrt{\zeta_{m;m+k}^2 + \zeta_{n;n+l}^2}}, \ldots
\end{aligned} \tag{B24}$$

where $p,q = 0,1,2,\ldots$. Obviously, there is

$$\begin{aligned}
E_{i+1,j} &= c_{i+1,j} - c_{i,j} = \frac{1}{\sqrt{i+1}} c_0, \\
F_{i,j+1} &= c_{i,j+1} - c_{i,j} = \frac{1}{\sqrt{j+1}} d_0.
\end{aligned} \tag{B25}$$

So we have

$$\begin{aligned}
e_{\mathrm{o}}' &= \sum_{p,q=0}^{\infty} c_{p,q} |p,q\rangle\langle p,q| \\
&= \sqrt{\frac{\hbar}{2}} \sum_{p,q=0}^{\infty} \frac{\zeta_{0;p}\zeta_{m;m+k} + \zeta_{0;q}\zeta_{n;n+l}}{\sqrt{\zeta_{m;m+k}^2 + \zeta_{n;n+l}^2}} |p,q\rangle\langle p,q|,
\end{aligned} \tag{B26}$$

and the Connes distance between the states $|m+k,n+l\rangle$ and $|m,n\rangle$ is

$$d(\omega_{m+k,n+l}, \omega_{m,n})$$

$$
\begin{aligned}
&= \lim_{e\to e'_{\rm o}} |{\rm tr}_{\mathcal{F}}(\rho_{m+k,n+l}e) - {\rm tr}_{\mathcal{F}}(\rho_{m,n}e)| \\
&= |c_{m+k,n+l} - c_{m,n}| \\
&= c_0\zeta_{m;m+k} + d_0\zeta_{n;n+l} \\
&= \sqrt{\frac{\hbar}{2}}\sqrt{\zeta^2_{m;m+k} + \zeta^2_{n;n+l}}. \qquad \text{(B27)}
\end{aligned}
$$

It is easy to verify that, these results (B26) and (B27) are also true for $k < 0$ and/or $l < 0$.

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