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Explosive synchronization of multi-layer complex networks based on inter-layer star network connection

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Explosive synchronization (ES) is a first-order transition phenomenon that is ubiquitous in various physical and biological systems. In recent years, researchers have focused on explosive synchronization in a single-layer network, but few in multi-layer networks. This paper proposes a frequency-weighted Kuramoto model in multi-layer complex networks with star connection between layers and analyzes the factors affecting the backward critical coupling strength by both theoretical analysis and numerical validation. Our results show that the backward critical coupling strength of each layer network is influenced by the inter-layer interaction strength and the average degree. The number of network layers, the number of nodes, and the network topology can not directly affect the synchronization of the network. Enhancing the inter-layer interaction strength can prevent the emergence of explosive synchronization and increasing the average degree can promote the generation of explosive synchronization.

Keywords: explosive synchronization, Kuramoto model, multi-layer networks

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1. Introduction

The upsurge of complex network research was caused by two famous papers published in 1998 and 1999.[1,2] These two papers have reported the common topological statistical properties of many real networks, namely, small-world and scale-free properties. Therefore, a large number of real networks are abstracted as complex network models, such as protein networks, brain neural networks, social networks, transportation networks, and the Internet. A complex network is a model composed of nodes and connections between them called edges. Complex networks have become important research topics in various disciplines such as statistical physics, computer science, biology, sociology, and nonlinear dynamics.[3–6] Synchronization is a common dynamic phenomenon in complex networks and has become a major focus of research.[7–9] At present, the Kuramoto model is the most classical phase oscillator model for studying phase synchronization in complex networks, and it plays an important role in researching the synchronization of networks.[10,11]

In recent ten years, many important results have been achieved in synchronization of a complex network. So far, most research in the field of complex networks has focused on a singlelayer network.[12–15] In the recent few decades, the research on coupled Kuramoto phase oscillator and coupled chaotic oscillator has shown that, in most cases, the synchronization of the system, that is, the phase transition from a non-coordinate state to coordinative state, is a second-order phase transition.[12] In 2011, Gómez Gardenes *et al.* found a phenomenon in which the phase transition processes of forward and backward were inconsistent in the study of phase transition of a scale-free network, and they named this phenomenon of the first-order phase transition as explosive synchronization (ES).[13] After that, researchers realized the circuit verification in a star network.[14] Very recently, Zhang *et al.* have proposed a Kuramoto model of natural frequency-weighted and have found the ES in a general complex network. They have extended the research on ES from the scale-free networks to the general network and have found the critical coupling strength of ES in a single layer.[15] In addition, the research also includes identifying the topology of the network,[16] the influence of time delay on synchronization[17,18] and network evolution, etc. Lately, researchers have gradually shifted the study on ES from a single-layer network to multi-layer networks.[19–23] Shu *et al.* have discussed the synchronization optimization of coupled Kuramoto oscillators in single-layer structure and multi-layer star structure networks.[19] The network structure of each layer is a star structure and the connection between the layers is a chain structure. Aleta *et al.* have provided an overview of the basic methods used to describe multi-layer systems and some representative dynamical processes.[20] Liu *et al.* have proposed a framework to simulate the co-evolution of interdependent networks.[21] Jalan *et al.* have shown that by tuning the properties of one layer (network) of a multilayer network, one can regulate the dy-

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namical behavior of another layer (network).[22] Yao *et al.* have proposed a frequency-weighted Kuramoto model on a two-layer network and the critical coupling strength of explosive synchronization is obtained by both theoretical analysis and numerical validation.[23] However, the paper only summarizes the two-layer network model, not the multi-layer network model.

The main work of this paper is as follows: (1) We consider a frequency-weighted coupled Kuramoto model of multi-layer networks and each layer of networks interacts one-to-one. Different from other researches on multi-layer complex networks, the connection between layers is a star connection and each layer can be an arbitrary topology in our model. The star connection between layers can extend the inter-layer interaction strength from one-to-one to one-to-many. (2) We discuss the impact of different parameters on ES in multi-layer networks including the inter-layer interaction strength, the average node degree, the number of the network layers, the network topology, and the number of nodes. (3) We conduct numerical simulation experiments and verify the simulation results through theoretical analysis.

The rest of this paper is organized as follows: Section 2 presents the Kuramoto model of multi-layer networks and provides the mean-field analysis of the model. In Section 3, the simulation results of the model are given. Concluding is given in Section 4.

2. Model

2.1. The Kuramoto model of multi-layer networks

For a single-layer network, the evolution of each oscillator on the natural frequency-weighted coupled Kuramoto model[15] can be expressed by

$$\dot{\theta}_i = w_i + \frac{\lambda\,|w_i|}{k_i}\sum_{j=1}^{N} A_{ij}\sin(\theta_j - \theta_i), \tag{1}$$

where $i = 1,\ldots,N$, θ_i and θ_j represent the instant phase of oscillator i and j, λ is the coupling strength between two nodes, ω_i stands for the natural frequency of the ith oscillator, $k_i = \sum_{j=1}^{N} A_{ij}$ is the ith oscillator's degree, and A_{ij} are the elements of adjacency matrix $\boldsymbol{A}$. When the nodes i and j are connected, $A_{ij} = 1$, otherwise, $A_{ij} = 0$. The order parameter R that measures the coherence of the collective motion is defined as[24]

$$R\mathrm{e}^{\mathrm{i}\Psi} = \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}\theta_j}, \tag{2}$$

where ψ denotes the average phase of network, and $R \subseteq [0,1]$. $R = 1$ stands for the fully synchronized state and $R = 0$ represents the incoherent solution.

In this paper, we investigate a frequency-weighted coupled model of Kuramoto oscillators in a multi-layer network. We consider a multi-layer network comprising m layers. The connections between these layers are star network connections, as shown in Fig. 1(d).

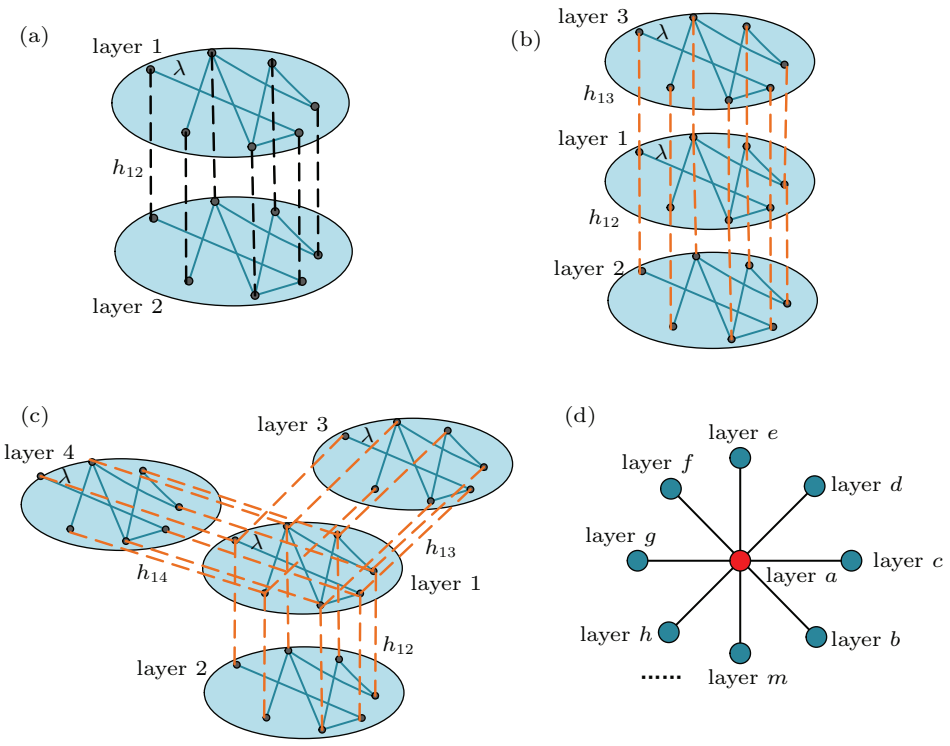


Fig. 1. The topology diagram of a multi-layer network, with λ representing the intra-layer strength expressed by solid lines and h_{12} the inter-layer strength between layer 1 and layer 2: (a) 2-layer network ($m = 2$), (b) 3-layer network ($m = 3$), (c) 4-layer network ($m = 4$), (d) m-layer network. The dotted lines indicate the inter-layer strength.

Figures 1(a)–1(c) show the network for $m = 2,3,4$, respectively. The network in the center is called the center layer, and the other layers are called leaf layers. Layer 1 in Fig. 1 is the center layer. Each subnetwork contains N oscillators, and the dynamic behavior of oscillators can be expressed by

$$\dot{\theta}_i = \omega_i + \frac{\lambda_1 |\omega_i|}{\langle k\rangle + \sum_{a=2}^{m} h_{1a}} \Big[\sum_{j=1}^{N} A_{ij}\sin(\theta_j - \theta_i) + \sum_{a=2}^{m} h_{1a}\sin(\theta_i' - \theta_i)\Big], \tag{3}$$

$$\dot{\theta}_i = \omega_i + \frac{\lambda_a |\omega_i|}{\langle k\rangle + h_{1a}} \Big[\sum_{j=1}^{N} A_{ij}\sin(\theta_j - \theta_i) + h_{1a}\sin(\theta_i' - \theta_i)\Big]. \tag{4}$$

Here Eq. (3) is the model of the center layer, and Eq. (4) is the model of the leaf layers; $i = 1,\ldots,N$ and $a = 2,\ldots,m$; $\langle k\rangle$ represents the average node degree of each layer in the network, h_{1a} stands for the inter-layer strength between the first layer and the ath layer, and θ_i' stands for the instant phase of oscillator i in the ath layer network. A_{ij} is the element of adjacency matrix of $\boldsymbol{M}$ in multi-layer networks denoted as

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A}_1 & h\boldsymbol{I} & h\boldsymbol{I} & h\boldsymbol{I} & \ldots & h\boldsymbol{I} \\ h\boldsymbol{I} & \boldsymbol{A}_2 & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{0} \\ h\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{A}_3 & \boldsymbol{0} & \ldots & \ldots \\ h\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_4 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \boldsymbol{0} \\ h\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{0} & \boldsymbol{A}_m \end{bmatrix}, \tag{5}$$

where $\boldsymbol{A}_1,\ldots,\boldsymbol{A}_m$ are the adjacency matrices of each layer, respectively, and $\boldsymbol{I}$ is the identity matrix that denotes the one-to-one interaction between layers.

2.2. Analytical explanations

To simplify the research, we consider the multi-layer network composed of the non-fully connected networks of the same topology and the same natural frequency distribution. Since the models of the center layer and the leaf layer are different, the dynamic evolutions of the Kuramoto oscillators on the center layer and the leaf layer can be described separately. First, we analyze the center layer.

We ignore the time fluctuations under the thermodynamic limit condition when the average degree is large enough, and Eq. (3) can be rewritten as

$$\dot{\theta}(t) = \omega + \frac{\lambda|\omega|}{\langle k\rangle + \sum_{a=2}^{m} h_{1a}} k \int \mathrm{d}k' \int \mathrm{d}\theta' \times \frac{k'P(k')}{\langle k\rangle}\rho(k';\theta',t)\sin(\theta_i' - \theta_i) + \frac{\lambda|\omega|}{\langle k\rangle + \sum_{a=2}^{m} h_{1a}} \sum_{a=2}^{m} h_{1a}\sin(\theta_i'' - \theta_i). \tag{6}$$

Here, $\langle k\rangle$, $P(k')$, $\rho(k';\theta',t)$ stand for the average degree, degree distribution, and density of the oscillators with phase θ at time t for a k. Moreover, Eq. (2) can be rewritten as

$$R\mathrm{e}^{\mathrm{i}\Psi} = \frac{1}{\langle k\rangle} k \int \mathrm{d}k' \int \mathrm{d}\theta' P(k')\rho(k';\theta',t)\,\mathrm{e}^{\mathrm{i}\theta}. \tag{7}$$

Based on the self-consistent theoretical analysis and mean-field theory,[25,26] substituting Eq. (7) into Eq. (6) yields the following mean-field form:

$$\dot{\theta}(t) = \omega + \frac{\lambda|\omega|\langle k\rangle}{\langle k\rangle + \sum_{a=2}^{m} h_{1a}} R\sin(\Psi - \theta_i) + \frac{\lambda|\omega|\sum_{a=2}^{m} h_{1a}}{\langle k\rangle + \sum_{a=2}^{m} h_{1a}}\sin(\theta_i'' - \theta_i). \tag{8}$$

We define the phase difference between the instant phase and the average phase $\Delta\theta_i = \theta_i - \psi$, then Eq. (8) can be rewritten as

$$\dot{\theta}(t) = \omega_i - \frac{\lambda|\omega|\langle k\rangle}{\langle k\rangle + \sum_{a=2}^{m} h_{1a}} R\sin(\Delta\theta_i) + \frac{\lambda|\omega|\sum_{a=2}^{m} h_{1a}}{\langle k\rangle + \sum_{a=2}^{m} h_{1a}}\sin(\theta_i'' - \theta_i). \tag{9}$$

When the system reaches a complete synchronized state, the phase of all the oscillators in the system and the average phase will maintain a constant difference, that is, the phase lock state is reached, we have the $\dot{\theta}(t) = 0$ and $\theta_i'' - \theta_i \approx 0$, the phase difference can be solved as

$$\Delta\dot{\theta}_i = \begin{cases} \arcsin\left(\dfrac{\langle k\rangle + \sum_{a=2}^{m} h_{1a}}{\lambda R\langle k\rangle}\right), & \omega_i > 0, \\ \arcsin\left(-\dfrac{\langle k\rangle + \sum_{a=2}^{m} h_{1a}}{\lambda R\langle k\rangle}\right), & \omega_i < 0. \end{cases} \tag{10}$$

Through Eq. (10) we can realize that the $\Delta\theta_i$ will gradually approach 0 when the coupling strength λ is increased. In this case, from Eq. (2) we can obtain

$$R = \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\Delta\theta_j} = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\Delta\theta_+} + \mathrm{e}^{\mathrm{i}\Delta\theta_-}), \tag{11}$$

where $\mathrm{e}^{\mathrm{i}\Delta\theta_+}$ and $\mathrm{e}^{\mathrm{i}\Delta\theta_-}$ represent the two synchronized phases when the natural frequency is positive or negative, respectively. Substituting Eq. (10) into Eq. (11), we have

$$R^4 - R^2 + \frac{[\langle k\rangle + \sum_{a=2}^{m} h_{1a}]^2}{\lambda^2\langle k\rangle^2} = 0. \tag{12}$$

Through the stability analysis, the range of the coupling strength is $\lambda \geq 2(1 + \frac{\sum_{a=2}^{m} h_{1a}}{\langle k\rangle})$ when Eq. (12) has a non-zero solution, and we can obtain the backward critical coupling strength

$$\lambda_{b1} = 2\left(1 + \frac{\sum_{a=2}^{m} h_{1a}}{\langle k\rangle}\right). \tag{13}$$

Similarly, the backward critical coupling strength of the leaf layers can be calculated with

$$\lambda_{b2} = 2\left(1 + \frac{h_{1a}}{\langle k\rangle}\right). \tag{14}$$

Observing Eqs. (13) and (14), we can know that the backward critical coupling strength value λ_b of ES is directly related to

the inter-layer strength h and the average degree $\langle k\rangle$ of the network. However, the center layer network model represented by Eq. (13) is different from other layer network models represented by Eq. (14). For the center layer, $h=\sum_{a=2}^{m}h_{1a}$ is the sum of the strength of all other layers in the center layer. For each other layer, $h=h_{1a}$ stands for the inter-layer strength between the center layer and the ath layer. Therefore, λ_b of each layer is different and $\lambda_b=2$ is the ES threshold of the model in a single layer network when $h=0$.

If $h=h_{12}=h_{13}=h_{14}=\cdots=h_{1m}$, according to Eqs. (13) and (14), we can obtain

$$\lambda_{b1}=2\left(1+\frac{(m-1)h}{\langle k\rangle}\right), \tag{15}$$

$$\lambda_{b2}=2\left(1+\frac{h}{\langle k\rangle}\right). \tag{16}$$

If the average degree $\langle k\rangle=N$, the network topology tends to be a fully connected network. Therefore, the fully connected network model can be written as

$$\lambda_{b1}=2\left(1+\frac{(m-1)h}{N}\right), \tag{17}$$

$$\lambda_{b2}=2\left(1+\frac{h}{N}\right). \tag{18}$$

3. Numerical simulations

In this section, we discuss various simulation results in detail. For our numerical experiment, we not only consider the inter-layer interaction strength and the average node degree in the theoretical results but also test the influence of the number of the network layers, the number of nodes, and the network topology on the ES. In the process of numerical simulation, the parameters and parameter values are listed in Table 1.

Table 1. Simulation parameters.

Variable	Parameter	Value
$\sum_{a=2}^{m}h_{1a}$	the inter-layer interaction strength of the center layer	0, 40, 80, 120
h_{1a}	the inter-layer interaction strength of the leaf layers	$h_{12}=5,10,4,15,20,40,60$ $h_{14}=26,25$ $h_{13}=35,30,10,25,20,40,60$
m	the number of network layers	2, 3, 4
$\langle k\rangle$	the average node degree	40, 60, 199, 200
N	the number of nodes	200, 400, 600, 1000
Topology	the network topology	fully, ER, SW, BA

For all the simulations, the natural frequency distribution follows the Lorentzian distribution, and the initial phases of oscillators are followed by a random uniform distribution. Firstly, the coupling strength λ is adiabatically increased starting from 0 with the step of $\delta\lambda=0.02$. Secondly, we decrease the coupling strength λ adiabatically from 5 with the same step. We compute the order parameter for each λ and make a graph of the relationship between the critical coupling strength and the interaction strength.

3.1. The effects of inter-layer interaction strength on ES

First of all, we simulate the influence of the inter-layer interaction strength h on ES. We discuss two situations.

3.1.1. Numerical simulations of ES in the center layer

In this part, we consider an Erdös–Rényi random network. The average degree of each layer is 60. To simplify the calculation, the number of nodes in each layer of the network is $N=200$, and research on the influence of the number of nodes on synchronization will be discussed in the following part.

Figure 2 shows the dependence of R and λ. Figures 2(a) and 2(b) are three-layer networks, and Figs. 2(c) and 2(d) are four-layer networks. We mark the center layer as R1, and the other layers as R2, R3, and R4, respectively. $R1_{\rm F}$ and $R1_{\rm B}$ represent the synchronous order parameter of forward and backward processes on R1; h_{12}, h_{13}, and h_{14} represent the interaction strength between R1 and R2, R3, R4, respectively. According to Fig. 2, we can find something as follows:

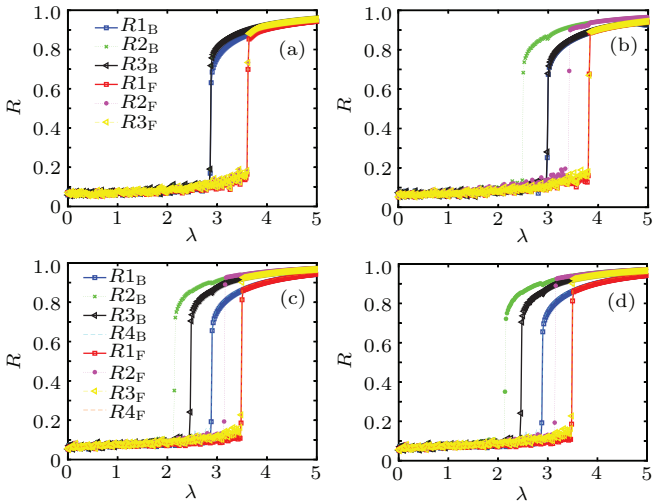


Fig. 2. Synchronous phase transition diagrams of order parameter variation with coupling strength for the same $\sum_{a=2}^{m}h_{1a}=40$ but different h_{1a}: (a) $h_{12}=20$, $h_{13}=20$, (b) $h_{12}=10$, $h_{13}=30$, (c) $h_{12}=4$, $h_{13}=10$, $h_{14}=26$, (d) $h_{12}=5$, $h_{13}=10$, $h_{14}=25$.

(1) It can be seen that the sum of the inter-layer interaction strength, $\sum_{a=2}^{m} h_{1a}$, in the four graphs of Fig. 2 is the same, and the $R1_{\mathrm{B}}$ is basically the same. Therefore, when $\sum_{a=2}^{m} h_{1a}$ remains unchanged, although h_{1a} of each layer changes, the backward critical coupling value of R1 does not change.

(2) The experimental results show that the number of the network layers has no effect on the backward critical coupling value of R1 when the sum of the inter-layer interaction strength, $\sum_{a=2}^{m} h_{1a}$, remains the same.

If the sum of the inter-layer interaction strength is increased, the backward critical value will gradually become larger, which demonstrates that increasing h will weaken the emerge of ES on a multi-layer network, as shown in Fig. 3. The average degree of each layer is 60 and the number of the layers is 2. The number of the network is 200.

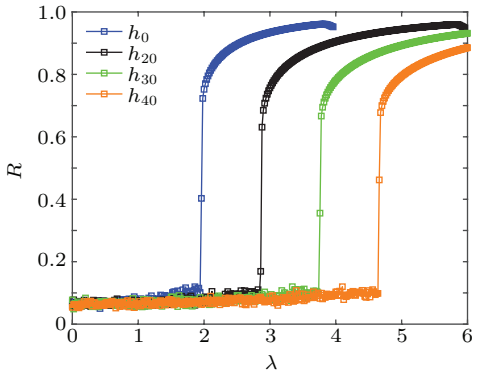


Fig. 3. Synchronous phase transition diagrams of order parameter variation with coupling strength for different inter-layer interaction strengths ($\sum_{a=2}^{m} h_{1a}$): (a) $\sum_{a=2}^{m} h_{1a} = 0$, (b) $\sum_{a=2}^{m} h_{1a} = 40$, (c) $\sum_{a=2}^{m} h_{1a} = 80$, (d) $\sum_{a=2}^{m} h_{1a} = 120$.

3.1.2. Numerical simulations of ES in the other layers

Next, we discuss the changes in the inter-layer interaction strength between the other layers. Figure 4 is the simulation result.

The results in Fig. 4 show that when the values of h_{12} and h_{13} are quite different, the backward critical coupling strength values of the two layers $R2_{\mathrm{B}}$, $R3_{\mathrm{B}}$ are significantly different. When the value of h_{12} increases, $R2_{\mathrm{B}}$ also increases.

Therefore, the backward critical coupling strength of each layer network is influenced by the inter-layer interaction strength. Enhancing the inter-layer interaction strength can prevent the emergence of explosive synchronization. The result of influence of the inter-layer interaction strength is consistent with the theoretical result.

3.2. The effects of average degree on ES

For the average degree, we think about a three-layer Erdös–Rényi random network with the number of nodes $N = 200$ in each layer and inter-layer interaction strength in each layer is $h_{12} = h_{13} = h = 20$. According to Eqs. (13) and (14), the influence of the average node degree in each layer network is the same. Thus, in this section the average node degree of each layer of the network $\langle k \rangle$ is the same.

Correspondingly, we notice that the phase transition of ES appears in each panel of Fig. 5. The critical point λ on ES decreases when increasing $\langle k \rangle$. It shows that increasing the average degree of the network can promote the generation of ES.

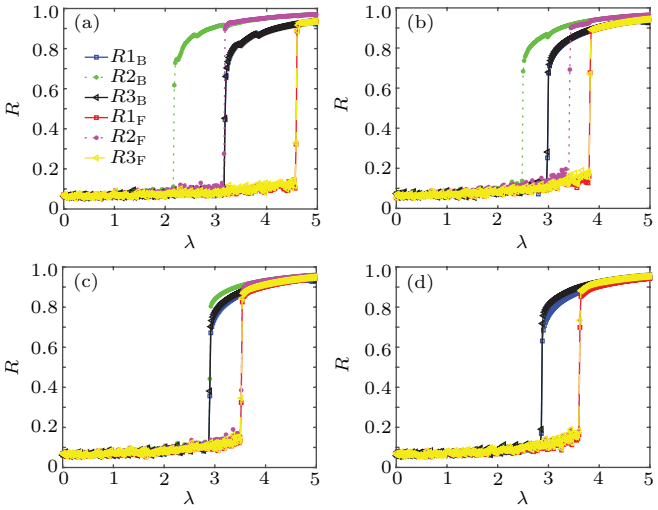


Fig. 4. Synchronous phase transition diagrams of order parameter variation with coupling strength for different inter-layer interaction strength (h): (a) $h_{12} = 5$, $h_{13} = 35$, (b) $h_{12} = 10$, $h_{13} = 30$, (c) $h_{12} = 15$, $h_{13} = 25$, (d) $h_{12} = 20$, $h_{13} = 20$.

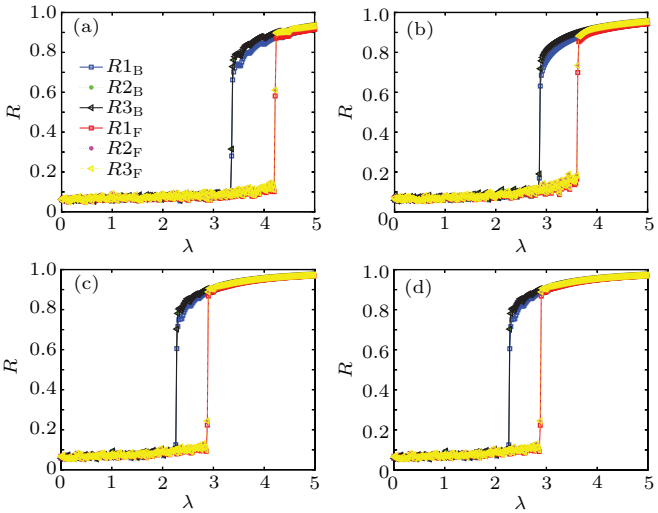


Fig. 5. Synchronous phase transition diagrams of order parameter variation with coupling strength for different average degrees: (a) $\langle k \rangle = 40$, (b) $\langle k \rangle = 60$, (c) $\langle k \rangle = 199$, and (d) $\langle k \rangle = N = 200$.

Figures 5(c) and 5(d) show that when the average node degree is close to N, the network topology tends to be fully connected. For a fully connected network, changing the number of nodes in the network is to change the node degree of the network. However, for a non-fully connected network, changing the number of nodes for fixed average degree of nodes does not affect the explosive synchronization in the network.

To verify this finding, we conduct the following simulation experiment. Change the number of nodes, but the average node degree remains the same. For this research, a two-layer

Erdös–Rényi random network is considered with the average degree $\langle k_1 \rangle = \langle k_2 \rangle = 40$. The inter-layer interaction strength h in each layer is 20.

It can be seen from Fig. 6 when the number of nodes in the network is increased, the critical coupling strength remains the same. Therefore, the number of nodes in the network does not affect explosive synchronization. The numerical simulation results are consistent with the theoretical derivation.

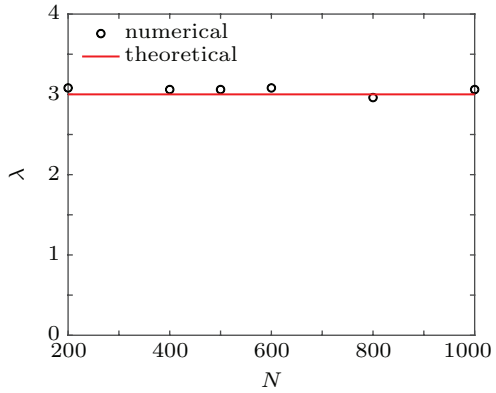


Fig. 6. The critical backward coupling strength value with the number of nodes (N): $N = 200, 400, 600, 800, 1000$.

3.3. The effects of different network topology on ES

The network topology used in the previous experiments in Subsections 3.1 and 3.2 is the Erdös–Rényi random network. To verify the universality of this model to other kinds of network topologies, we use other topologies for testing. For this study, we use a three-layer network with the number of nodes in each layer is $N = 200$.

In Figs. 7(a) and 7(b) $\langle k \rangle = 10$, $h = 5$ and in Figs. 7(c) and 7(d) $\langle k \rangle = 100$, $h = 20$ in each layer.

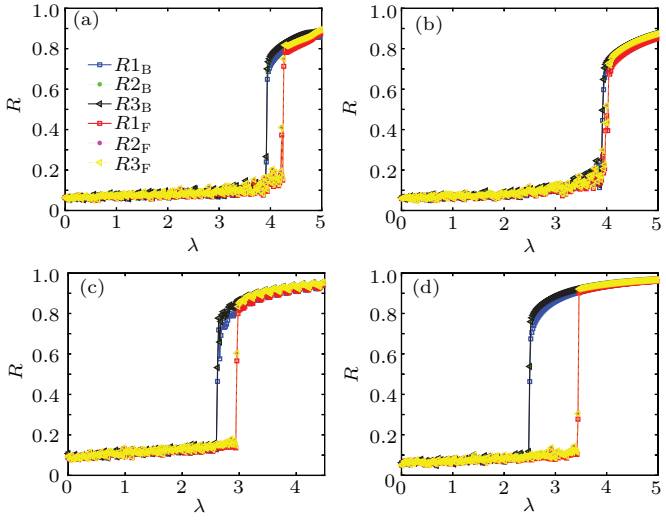


Fig. 7. Synchronous phase transition diagrams of order parameter variation with coupling strength for different network topologies. (a) Erdös–Rényi random network (ER), (b) scale-free network (BA) (c) Erdös–Rényi random network (ER), (d) small-world network (SW).

From Fig. 7 we can find that when only the topological structure is different, the backward critical coupling strength value of the network does not change. Therefore, it is shown that the network topology is not a factor that affects explosive synchronization.

4. Conclusion

In summary, we have investigated the frequency-weighted coupled Kuramoto model of multi-layer networks with a star network connection between layers. ES in the multi-layer network can be influenced by the inter-layer interaction strength and the average degree. We can find that increasing the inter-layer interaction strength will prevent the generation of ES and increasing the average degree of the network will promote the emergence of ES in multi-layer networks. Then, we discuss the impact of different parameters on ES in multi-layer networks and find that the number of network layers, the number of nodes, and the network topology can not directly affect the synchronization of the network. Through mean-field analysis, we obtain the theoretical value of critical coupling strength in the process direction of backward and prove the correctness of the experiment results.

It is significant to research generation and control of multi-layer complex systems in real life and we will further study the synchronization behavior in multi-layer complex systems in the future. The multi-layer Kuramoto model proposed in this paper is a one-to-one interaction between the network layers. In real life, there is a situation where one node affects multiple nodes in another layer of the network at the same time. Therefore, it is worth analyzing the synchronization phenomenon of one-to-many and many-to-many connected multi-layer networks.

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