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Analysis of the rogue waves in the blood based on the high-order NLS equations with variable coefficients*

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The research of rogue waves is an advanced field which has important practical and theoretical significances in mathematics, physics, biological fluid mechanics, oceanography, etc. Using the reductive perturbation theory and long wave approximation, the equations governing the movement of blood vessel walls and the flow of blood are transformed into high-order nonlinear Schrödinger (NLS) equations with variable coefficients. The third-order nonlinear Schrödinger equation is degenerated into a completely integrable Sasa–Satsuma equation (SSE) whose solutions can be used to approximately simulate the real rogue waves in the vessels. For the first time, we discuss the conditions for generating rogue waves in the blood vessels and effects of some physiological parameters on the rogue waves. Based on the traveling wave solutions of the fourth-order nonlinear Schrödinger equation, we analyze the effects of the higher order terms and the initial deformations of the blood vessel on the wave propagation and the displacement of the tube wall. Our results reveal that the amplitude of the rogue waves are proportional to the initial stretching ratio of the tube. The high-order nonlinear and dispersion terms lead to the distortion of the wave, while the initial deformation of the tube wall will influence the wave amplitude and wave steepness.

Keywords: rogue wave, variable-coefficients high-order nonlinear Schrödinger equation, deformable blood vessels, Sasa–Satsuma equation

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1. Introduction

The flow and circulation of blood are the most basic and important part in the process of life. In order to understand the mechanism of blood circulation, many famous hydromechanics experts have made important contributions to it. Womersley[1,2] and McDonald[3,4] systematically studied the pulsating nature of arterial blood flow, which laid the mathematical foundation for blood research. They treated the arterial tubes as elastic thin-walled circular tubes and discussed the rules of arterial blood flow characteristics in the tubes on the basis of considering the restraint effect of surrounding tissues on the blood vessels. Yomosa, Belardinelli, *et al*.[5–8] proposed a simplified and idealized soliton theory of the blood movement model system in great vessels and applied it to study the propagation characteristics of pressure waves in blood vessels. Using the long wave approximation, Choy[9] employed the reduced perturbation technique to study the propagation of weak nonlinear pressure waves in an elastic tube filled with non-viscous fluid, and the general governing equations were reduced to the well-known standard nonlinear Schrödinger equation (NLSE) by proper scaling. However, in recent years, researchers found that the standard nonlinear Schrödinger equation is far from enough to properly describe the nonlinear phenomena including various physical factors, making the studies of high-order nonlinear Schrödinger equation (HNLSE) and the effect of high-order terms increasingly important.[10–13]

The concept of rogue waves originates from the extremely violent isolated waves generated in the ocean. Due to the increasing number of accidents caused by rogue waves, the physical mechanism of the phenomenon has aroused wide attraction in several fields of nonlinear systems representing ubiquitous physical phenomena. With the limitations of the complexity and uncertainty of the actual observation environment, people have to seek other ways to understand the phenomenon of rogue waves. In particular, the NLSE has played a pivotal role as the simplest universal nonlinear model to describe the rogue wave approximately. Due to the limitations of NLSE, it is necessary to go beyond the standard NLSE based model to explain the rogue wave phenomenon.[14] Even in the one-dimensional case, high-order terms such as third-order dispersion, self-steepening, and other nonlinear effects must be taken into account in the equation. In current research, the behavior of rogue waves and their solutions for integrable HNLSE are mainly focused.[15–20]

Several methods for solving the explicit traveling wave

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solutions of nonlinear evolution equations have been presented, such as the extended modified direct algebraic method,[21] the homotopy perturbation method,[22] the Jocobi elliptic function expansion method,[23] the homogeneous balance method,[24] etc.[25–30] In order to obtain solutions of the extended high-order nonlinear Schrodinger equation with variable coefficients, we use a method which is called the (G'/G)-expansion method[31,32] and has been proposed to construct new bell and kink profile solitary wave solutions, triangular periodic wave solutions, and singular solutions to many nonlinear evolution equations. The core idea of this method is to construct the traveling wave solutions of the complex nonlinear evolution equations by using the solutions of the second-order linear ordinary differential equations.

The structure of this paper is as follows: In Section 2, the momentum flux conservation equation and the Naiver–Stokes equation, the equations describing blood flow and tube motion are established. Using the reductive perturbation theory[33,34] and long wave approximation, the equations are transformed to the standard NLSE with variable coefficients. In Section 3, by expanding the small parameter ε into higher order, a third-order NLSE with variable coefficients is obtained. According to the existing research on the solution of rogue waves, we discuss the conditions under which the rogue waves are generated in the blood vessels and the effect of some parameter changes on the rogue waves. In Section 4, we further derive the HNLSE, and the G'/G-expansion method is used to find the traveling wave solutions of the fourth-order NLSE with variable coefficients. And we analyze the effects of higher order terms and initial vessel deformation on the pulse waveform and radial displacement of the tube wall.

2. Equation for the standard NLSE

Assume that the blood vessels are uniform, incompressible, cylindrical thin elastic tubes filled with inviscid, incompressible blood. In this section, based on the stress–strain relationship of the tube wall, the momentum flux conservation equation and the Naiver–Stokes equation, the equations describing blood flow and tube motion are established. In the long wave approximation, the equations are transformed to the standard NLSE with variable coefficients.

2.1. Establishment of basic equations

Setting z and R as axial coordinate and radial coordinate respectively within the cylindrical coordinates (R,θ,z). Assuming that under the effects of uniform internal pressure P_0 and axial stretch λ_z, the internal radius r_0 of artery in equilibrium state will be deformed. Considering the external constraints, the axial displacement can be ignored. A finite radial displacement u^* to be a function with respect to z^* and t^* is introduced based on the static deformation. z^* represents the axial coordinate after static deformation and t^* is the time parameter. The position vector after deformation of the general point of the tube can be expressed by the unit base vectors in the cylindrical polar coordinates e_r, e_θ, e_z as

$$R^* = (r+u^*)\,e_r + z^* e_z, \quad z^* = \lambda_z Z^*, \tag{1}$$

where Z^* is the axial coordinate of a point in the cylindrical coordinate system before deformation occurs, r and λ_θ represent the internal radius of the artery after finite static deformation, and the circumferential stretching ratio.

The stretches along axial and circumferential directions are given as the follows:

$$\begin{aligned}\lambda_1 &= \lambda_z\left[1+\left(\frac{\partial u^*}{\partial z^*}\right)^2\right]^{\frac{1}{2}},\\ \lambda_2 &= \lambda_\theta\left(1+\frac{u^*}{r}\right), \quad \lambda_\theta = \frac{r}{r_0}.\end{aligned} \tag{2}$$

In the analysis of the propagation of pressure waves in arterial vessels, the dynamic deformation of the vessel wall caused by pressure deviation can be considered as a small disturbance superimposed on the initial static deformation. Then, we establish the equation describing the radial motion of the tube

$$\begin{aligned}&\mu r_0 \frac{\partial}{\partial z^*}\left\{\frac{\dfrac{\partial u^*}{\partial z^*}}{\left[1+\left(\dfrac{\partial u^*}{\partial z^*}\right)^2\right]^{\frac{1}{2}}}\frac{\partial \Sigma}{\partial \lambda_1}\right\}\\ &-\frac{\mu}{\lambda_z}\frac{\partial \Sigma}{\partial \lambda_2}+(r+u^*)\frac{P^*}{H} = \frac{\rho_0 r_0}{\lambda_z}\frac{\partial^2 u^*}{\partial t^{*2}},\end{aligned} \tag{3}$$

where $\mu\Sigma$ indicates the strain energy density function, μ is the shear modulus of the tube, P^* and H represent the fluid pressure received by the artery and the thickness of the tube before deformation, respectively, and the mass density of the artery is written as ρ_0. Refer to the study of Demiray,[35] we will use the strain energy density function of soft biological tissue in two-dimensional case as the energy density function of the soft biological tissue to carry out numerical analysis later,

$$\Sigma = \frac{\exp\left[\alpha\left({\lambda_z}^2+{\lambda_\theta}^2+\dfrac{1}{{\lambda_z}^2{\lambda_\theta}^2}-3\right)\right]-1}{2\alpha}, \tag{4}$$

where α is a material constant. In order to simplify our model, we suppose the artery to be an infinite, straight, circular, uniform elastic tube filled with incompressible, inviscid blood, and the axial velocity v^*, the fluid pressure P^* are uniformly distributed. Then, the model including the one that describes the fluid flow in the longitudinal direction based on the Naiver–Stokes equation and the other which expresses the incompressibility of blood is given by

$$\frac{\partial v^*}{\partial t^*}+v^*\frac{\partial v^*}{\partial z^*}=-\frac{1}{\rho_0}\frac{\partial P^*}{\partial z^*},$$
$$2\frac{\partial u^*}{\partial t^*}+2\frac{\partial u^*}{\partial z^*}v^*+(\lambda_\theta+u^*)\frac{\partial v^*}{\partial z^*}=0. \tag{5}$$

At this point, it may be convenient to introduce the dimensionless quantities as follows:

$$t^*=\left(\frac{r_0}{c_0}\right)t,\quad z^*=r_0z,\quad u^*=r_0u,\quad v^*=c_0v,$$
$$P^*=\rho_f{c_0}^2p,\quad {c_0}^2=\frac{\mu H}{\rho_f r_0},\quad m=\frac{\rho_0 H}{\rho_f r_0}. \tag{6}$$

Substituting Eq. (6) into Eqs. (3) and (6), the equations describing the motion of tube and fluid are finally transformed into

$$\begin{cases} 2\dfrac{\partial u}{\partial t}+2\dfrac{\partial u}{\partial z}v+(\lambda_\theta+u)\dfrac{\partial v}{\partial z}=0,\\ \dfrac{\partial v}{\partial t}+v\dfrac{\partial v}{\partial z}+\dfrac{\partial p}{\partial z}=0,\\ p=\dfrac{m}{\lambda_z(\lambda_\theta+u)}\dfrac{\partial^2 u}{\partial t^2}+\dfrac{1}{\lambda_z(\lambda_\theta+u)}\dfrac{\partial \Sigma}{\partial \lambda_2}\\ \qquad -\dfrac{1}{(\lambda_\theta+u)}\dfrac{\partial}{\partial z}\left\{\dfrac{\partial u/\partial z}{[1+(\partial u/\partial z)^2]^{\frac{1}{2}}}\dfrac{\partial \Sigma}{\partial \lambda_1}\right\}. \end{cases} \tag{7}$$

2.2. Derivation of the standard NLSE with variable coefficients

Considering the dispersion relation of the linearized field equations, we introduce the following slow stretched coordinates:[36]

$$\xi=\varepsilon(z-gt),\quad \tau_1=\varepsilon^2t,\quad \tau_2=\varepsilon^3t,\quad \tau_3=\varepsilon^4t,\quad \cdots, \tag{8}$$

where ε is a small parameter to describe the dispersion and nonlinearity, which should be taken as a positive value to balance the nonlinearity and dissipation, g is the scale parameter to be determined from the solution. Noting the following differential relations:

$$\frac{\partial}{\partial z}\to\frac{\partial}{\partial z}+\varepsilon\frac{\partial}{\partial \xi},$$
$$\frac{\partial}{\partial t}\to\frac{\partial}{\partial t}-\varepsilon g\frac{\partial}{\partial \xi}+\varepsilon^2\frac{\partial}{\partial \tau_1}+\varepsilon^3\frac{\partial}{\partial \tau_2}+\varepsilon^4\frac{\partial}{\partial \tau_3}+\cdots. \tag{9}$$

Using Taylor expansion of the strain energy function Σ around $u=0$ and $\partial u/\partial z=0$, the pressure p can be approximately written as

$$\begin{aligned} p=&\sum_{i=1}^{5}(-1)^i\frac{m}{{\lambda_\theta}^i\lambda_z}u^{n-1}\frac{\partial^2 u}{\partial t^2}+\sum_{i=1}^{5}\beta_iu^i\\ &-\alpha_0\frac{\partial^2 u}{\partial z^2}-\alpha_1\left(\frac{\partial u}{\partial z}\right)^2-\left(2\alpha_1-\frac{\alpha_0}{\lambda_\theta}\right)u\frac{\partial^2 u}{\partial z^2}\\ &-\left(\alpha_2-\frac{2\alpha_1}{\lambda_\theta}+\frac{\alpha_0}{{\lambda_\theta}^2}\right)u^2\frac{\partial^2 u}{\partial z^2}-3\left(\gamma_1-\frac{\alpha_0}{2}\right)\\ &\times\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}-\left(\alpha_2-\frac{\alpha_1}{\lambda_\theta}\right)u\left(\frac{\partial u}{\partial z}\right)^2\\ &-\left(\alpha_3-\frac{\alpha_2}{\lambda_\theta}+\frac{2\alpha_1}{{\lambda_\theta}^2}-\frac{\alpha_0}{{\lambda_\theta}^3}\right)u^3\frac{\partial^2 u}{\partial z^2}\\ &+\left(3\gamma_1+3\alpha_1-\frac{3\alpha_0}{2\lambda_\theta}\right)u\frac{\partial^2 u}{\partial z^2}\left(\frac{\partial u}{\partial z}\right)^2+\alpha_1\left(\frac{\partial u}{\partial z}\right)^4\\ &-\left(3\alpha_4-\frac{2\alpha_3}{\lambda_\theta}+\frac{\alpha_2}{{\lambda_\theta}^2}-\frac{\alpha_1}{{\lambda_\theta}^3}\right)u^3\left(\frac{\partial u}{\partial z}\right)^2\\ &-\left(\alpha_4-\frac{\alpha_3}{\lambda_\theta}+\frac{\alpha_2}{{\lambda_\theta}^2}-\frac{2\alpha_1}{{\lambda_\theta}^3}+\frac{\alpha_0}{{\lambda_\theta}^4}\right)u^4\frac{\partial^2 u}{\partial z^2}\\ &-\left(12\alpha_3-\frac{\alpha_2}{\lambda_\theta}+\frac{\alpha_1}{{\lambda_\theta}^2}\right)u^2\left(\frac{\partial u}{\partial z}\right)^2\\ &+\left(\frac{\gamma_1}{2}-5\gamma_2-\frac{15\alpha_0}{8}\right)\left(\frac{\partial u}{\partial z}\right)^4\frac{\partial^2 u}{\partial z^2}\\ &+\left(\alpha_2-\frac{\alpha_1}{\lambda_\theta}\right)u\left(\frac{\partial u}{\partial z}\right)^4\\ &-\left(\frac{2\gamma_1}{{\lambda_\theta}^2}-\frac{\alpha_2}{2}+\frac{3\alpha_1}{\lambda_\theta}-\frac{3\alpha_0}{2{\lambda_\theta}^2}\right)u^2\\ &\times\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}+\cdots, \end{aligned} \tag{10}$$

where the coefficients α_i, β_i $(i=0,1,2,3,4,5,\ldots)$ and γ_1, γ_2 are defined by

$$\alpha_0=\frac{1}{\lambda_\theta}\frac{\partial \Sigma}{\partial \lambda_z},\quad \alpha_1=\frac{1}{2\lambda_\theta}\frac{\partial^2 \Sigma}{\partial \lambda_z\partial \lambda_\theta},$$
$$\alpha_2=\frac{1}{2\lambda_\theta}\frac{\partial^3 \Sigma}{\partial {\lambda_\theta}^2\partial \lambda_z},\quad \alpha_3=\frac{1}{8\lambda_\theta}\frac{\partial^4 \Sigma}{\partial \lambda_z\partial {\lambda_\theta}^3},$$
$$\alpha_4=\frac{1}{8\lambda_\theta}\frac{\partial^5 \Sigma}{\partial \lambda_z\partial {\lambda_\theta}^4},\quad \beta_0=\frac{1}{\lambda_z\lambda_\theta}\frac{\partial \Sigma}{\partial \lambda_\theta},$$
$$\beta_1=\frac{1}{\lambda_z\lambda_\theta}\frac{\partial^2 \Sigma}{\partial {\lambda_\theta}^2}-\frac{\beta_0}{\lambda_\theta},\quad \beta_2=\frac{1}{2\lambda_z\lambda_\theta}\frac{\partial^3 \Sigma}{\partial {\lambda_\theta}^3}-\frac{\beta_1}{\lambda_\theta},$$
$$\beta_3=\frac{1}{6\lambda_z\lambda_\theta}\frac{\partial^4 \Sigma}{\partial {\lambda_\theta}^4}-\frac{\beta_2}{\lambda_\theta},\quad \beta_4=\frac{1}{24\lambda_\theta\lambda_z}\frac{\partial^5 \Sigma}{\partial \lambda_\theta^5}-\frac{\beta_3}{\lambda_\theta},$$
$$\beta_5=\frac{1}{120\lambda_\theta\lambda_z}\frac{\partial^6 \Sigma}{\partial {\lambda_\theta}^6}-\frac{\beta_4}{\lambda_\theta},$$
$$\gamma_1=\frac{\lambda_z}{2\lambda_\theta}\frac{\partial^2 \Sigma}{\partial {\lambda_\theta}^2},\quad \gamma_2=\frac{\lambda_z}{6\lambda_\theta}\frac{\partial^3 \Sigma}{\partial {\lambda_z}^3}. \tag{11}$$

In general, it is impractical to deal with nonlinear equations by analytic method. To deal with the governing equations, we will use the long wave approximation method and reduced perturbation which should first introduce the following perturbation expansions of field variables u, v, and p with respect to ε:

$$u=\sum_{i=1}^{n}\varepsilon^nu_n,\quad v=\sum_{i=1}^{n}\varepsilon^nv_n,\quad p=\sum_{i=1}^{n}\varepsilon^np_n. \tag{12}$$

Substituting Eq. (12) into Eqs. (7) and (10), and equating the coefficients of like powers of ε. From the first-order equations in ε, we obtain

$$2\frac{\partial u_1}{\partial t}+\lambda_\theta\frac{\partial v_1}{\partial z}=0,\quad \frac{\partial v_1}{\partial t}+\frac{\partial p_1}{\partial z}=0,$$
$$p_1=\frac{m}{\lambda_\theta\lambda_z}u_1\frac{\partial^2 u_1}{\partial t^2}-\alpha_0\frac{\partial^2 u_1}{\partial z^2}+\beta_1 u_1. \tag{13}$$

Searching for the solution in the following form according to the differential Eq. (13):

$$\begin{aligned}&u_1(\xi,\tau_1,\tau_2,\ldots)=U_1\,\mathrm{e}^{\mathrm{i}\theta}+\mathrm{c.c},\\&v_1(\xi,\tau_1,\tau_2,\ldots)=V_1\,\mathrm{e}^{\mathrm{i}\theta}+\mathrm{c.c},\\&p_1(\xi,\tau_1,\tau_2,\ldots)=\left(\frac{-m\omega^2}{\lambda_\theta\lambda_z}+\alpha_0k^2+\beta_1\right)U_1\,\mathrm{e}^{\mathrm{i}\theta}+\mathrm{c.c}.\end{aligned} \tag{14}$$

We set $U_1=U_1(\xi,\tau_1,\tau_2,\ldots)$ and $V_1=V_1(\xi,\tau_1,\tau_2,\ldots)$, where U_1 and V_1 are unknown functions and their governing equation will be obtained later. k, ω, and c.c represent the wave number, the angular frequency, and the complex conjugate of the corresponding expressions, respectively, and the phase can be defined as $\theta=\omega t-kz$. Then, combining Eq. (13) with Eq. (14), we obtain

$$\begin{aligned}&U_1=U(\xi,\tau_1,\tau_2,\ldots),\quad V_1=\frac{2\omega}{\lambda_\theta k}U,\\&g=\frac{2\omega^2\lambda_z-m\omega^2k^2+3\alpha_0\lambda_\theta\lambda_zk^4+\beta_1\lambda_\theta\lambda_zk^2}{4\lambda_\theta\omega k+2m\omega k^3},\end{aligned} \tag{15}$$

besides, the differential relation satisfies

$$\omega^2=\frac{\lambda_\theta\lambda_zk^2\left(\alpha_0k^2+\beta_1\right)}{2\lambda_z+mk^2}. \tag{16}$$

If we consider the coefficients of $o(\varepsilon^2)$ and $o(\varepsilon^3)$, the following equations are available.

The $o(\varepsilon^2)$ order equations are

$$\begin{cases}2\dfrac{\partial u_2}{\partial t}-2g\dfrac{\partial u_1}{\partial\xi}+2v_1\dfrac{\partial u_1}{\partial z}\\\quad+\lambda_\theta\left(\dfrac{\partial v_2}{\partial z}+\dfrac{\partial v_1}{\partial\xi}\right)+u_1\dfrac{\partial v_1}{\partial\xi}=0,\\\dfrac{\partial v_2}{\partial t}-g\dfrac{\partial v_1}{\partial\xi}+v_1\dfrac{\partial v_1}{\partial z}+\dfrac{\partial v_1}{\partial\xi}+\dfrac{\partial p_2}{\partial z}+\dfrac{\partial p_1}{\partial\xi}=0,\\p_2=\dfrac{m}{\lambda_\theta\lambda_z}\left(\dfrac{\partial^2u_2}{\partial t^2}-2g\dfrac{\partial^2u_1}{\partial\xi\partial t}\right)\\\quad-\alpha_0\left(\dfrac{\partial^2u_2}{\partial z^2}+2\dfrac{\partial^2u_1}{\partial\xi\partial z}\right)\\\quad+\beta_1u_2-\dfrac{m}{{\lambda_\theta}^2\lambda_z}u_1\dfrac{\partial^2u_1}{\partial t^2}-\alpha_1\left(\dfrac{\partial u_1}{\partial z}\right)^2\\\quad-\left(2\alpha_1-\dfrac{\alpha_0}{\lambda_\theta}\right)u_1\dfrac{\partial^2u_1}{\partial z^2}+\beta_2{u_1}^2.\end{cases} \tag{17}$$

Introducing the assumptions of the solutions in the following form:

$$\begin{aligned}&u_n=u_n^{(0)}+\left(\sum_{j=1}^{n}u_n^{(j)}\,\mathrm{e}^{j(\mathrm{i}\theta)}+\mathrm{c.c}\right),\\&v_n=v_n^{(0)}+\left(\sum_{j=1}^{n}v_n^{(j)}\,\mathrm{e}^{j(\mathrm{i}\theta)}+\mathrm{c.c}\right),\\&p_n=p_n^{(0)}+\left(\sum_{j=1}^{n}p_n^{(j)}\,\mathrm{e}^{j(\mathrm{i}\theta)}+\mathrm{c.c}\right),\end{aligned} \tag{18}$$

then, we get

$$\begin{cases}2\mathrm{i}\omega u_2^{(1)}+\left(\dfrac{2\omega}{k}-2g\right)\dfrac{\partial U}{\partial\xi}-\mathrm{i}\lambda_\theta kv_2^{(1)}=0,\\\mathrm{i}\omega v_2^{(1)}-\left(\dfrac{m\omega^2}{\lambda_\theta\lambda_z}+\dfrac{2g\omega}{\lambda_\theta k}+\dfrac{2mg\omega k}{\lambda_\theta\lambda_z}-3\alpha_0k^2-\beta_1\right)\dfrac{\partial U}{\partial\xi}\\\quad+\mathrm{i}\left(\dfrac{m\omega^2k}{\lambda_\theta\lambda_z}-\alpha_0k^3-\beta_1k\right)u_2^{(1)}=0,\\2\mathrm{i}\omega u_2^{(2)}-\mathrm{i}\lambda_\theta kv_2^{(2)}-\dfrac{3\mathrm{i}\omega}{\lambda_\theta}U^2=0,\\\mathrm{i}\omega v_2^{(2)}-\mathrm{i}\left(\dfrac{m\omega^2k}{{\lambda_\theta}^2\lambda_z}+\dfrac{2\omega^2}{{\lambda_\theta}^2k}+3\alpha_1k^3-\dfrac{\alpha_0}{\lambda_\theta}k^3+\beta_2k\right)U^2\\\quad+\mathrm{i}\left(\dfrac{4m\omega^2k}{\lambda_\theta\lambda_z}-4\alpha_0k^2-\beta_1\right)u_2^{(2)}=0,\\p_2^{(0)}=\beta_1u_2^{(0)}+2\left(\dfrac{m\omega^2}{{\lambda_\theta}^2\lambda_z}+\alpha_1k^2-\dfrac{\alpha_0k^2}{\lambda_\theta}+\beta_2\right)|U|^2,\\p_2^{(1)}=\left(\dfrac{-m\omega^2}{\lambda_\theta\lambda_z}+\alpha_0k^2+\beta_1\right)u_2^{(1)}\\\quad+2\mathrm{i}\left(\alpha_0k-\dfrac{m\omega g}{\lambda_\theta\lambda_z}\right)\dfrac{\partial U}{\partial\xi},\\p_2^{(2)}=\left(\dfrac{-4m\omega^2}{\lambda_\theta\lambda_z}+\alpha_0k^2+\beta_1\right)u_2^{(2)}\\\quad+\left(\dfrac{m\omega^2}{{\lambda_\theta}^2\lambda_z}+3\alpha_1k^2-\dfrac{\alpha_0k^2}{\lambda_\theta}+\beta_2\right)U^2.\end{cases} \tag{19}$$

Here, we set $u_i^{(1)}=0$, $(i=1,2,3,\ldots)$. According to Eq. (19), the solutions of u_2 and v_2 can be obtained as

$$\begin{aligned}&u_2^{(1)}=0,\quad v_2^{(1)}=\mathrm{i}\left(\frac{2g}{\lambda_\theta k}-\frac{2\omega}{\lambda_\theta k^2}\right)\frac{\partial U}{\partial\xi},\quad u_2^{(2)}=\Phi_0U^2,\\&v_2^{(2)}=\left(\frac{2\omega}{\lambda_\theta k}\Phi_0-\frac{3\omega}{{\lambda_\theta}^2k}\right)U^2,\\&g=\frac{2\omega^2\lambda_z-m\omega^2k^2+3\alpha_0\lambda_\theta\lambda_zk^4+\beta_1\lambda_\theta\lambda_zk^2}{4\lambda_\theta\omega k+2m\omega k^3},\end{aligned} \tag{20}$$

where

$$\begin{aligned}\Phi_0=&\frac{1}{3\left(\beta_1\lambda_\theta k^2-2\omega^2\right)}\\&\times\left(\frac{3\omega^2}{\lambda_\theta}+\beta_1k^2+3\alpha_1\lambda_\theta k^4+\beta_2\lambda_\theta k^2\right).\end{aligned} \tag{21}$$

The $o(\varepsilon^3)$ order equations are given by

$$
\begin{cases}
2\dfrac{\partial u_3}{\partial t}-2g\dfrac{\partial u_2}{\partial \xi}+2\dfrac{\partial u_1}{\partial \tau_1}+2v_1\left(\dfrac{\partial u_2}{\partial z}+\dfrac{\partial u_1}{\partial \xi}\right)+2v_2\dfrac{\partial u_1}{\partial z}+\lambda_\theta\left(\dfrac{\partial v_3}{\partial z}+\dfrac{\partial v_2}{\partial \xi}\right)+u_1\left(\dfrac{\partial v_2}{\partial z}+\dfrac{\partial v_1}{\partial \xi}\right)+u_2\dfrac{\partial v_1}{\partial z}=0,\\
\dfrac{\partial v_3}{\partial t}-g\dfrac{\partial v_2}{\partial \xi}+\dfrac{\partial v_1}{\partial \tau_1}+v_1\dfrac{\partial v_2}{\partial z}+v_2\dfrac{\partial v_1}{\partial z}+v_1\dfrac{\partial v_1}{\partial \xi}+\dfrac{\partial p_3}{\partial z}+\dfrac{\partial p_2}{\partial \xi}=0,\\
p_3=\dfrac{m}{\lambda_\theta\lambda_z}\left(\dfrac{\partial^2 u_3}{\partial t^2}-2g\dfrac{\partial^2 u_2}{\partial \xi\partial t}+2\dfrac{\partial^2 u_1}{\partial \tau_1\partial t}+g^2\dfrac{\partial^2 u_1}{\partial \xi^2}\right)+\beta_2{u_1}^2+2\beta_2u_1u_2+\beta_1u_3\\
\quad-\dfrac{m}{{\lambda_\theta}^2\lambda_z}\left[u_1\left(\dfrac{\partial^2 u_2}{\partial t^2}-2\dfrac{\partial^2 u_1}{\partial \xi\partial t}\right)+u_2\dfrac{\partial^2 u_1}{\partial t^2}\right]-\alpha_1\left[2\left(\dfrac{\partial u_1}{\partial z}\dfrac{\partial u_2}{\partial z}+\dfrac{\partial u_1}{\partial z}\dfrac{\partial u_1}{\partial \xi}\right)\right]\\
\quad-\left(2\alpha_1-\dfrac{\alpha_0}{\lambda_\theta}\right)\left[u_1\left(\dfrac{\partial^2 u_2}{\partial z^2}+2\dfrac{\partial^2 u_1}{\partial \xi\partial z}\right)+u_2\dfrac{\partial^2 u_1}{\partial z^2}\right]-\alpha_0\left(\dfrac{\partial^2 u_3}{\partial z^2}+2\dfrac{\partial^2 u_2}{\partial \xi\partial z}+\dfrac{\partial^2 u_1}{\partial \xi^2}\right)\\
\quad+\dfrac{m}{{\lambda_\theta}^3\lambda_z}{u_1}^2\dfrac{\partial^2 u_1}{\partial t^2}-\left(\alpha_2-\dfrac{\alpha_1}{\lambda_\theta}\right)u_1\left(\dfrac{\partial u_1}{\partial z}\right)^2-\left(\alpha_2-\dfrac{2\alpha_1}{\lambda_\theta}+\dfrac{2{\alpha_0}^2}{\lambda_\theta}\right){u_1}^2\dfrac{\partial^2 u_1}{\partial z^2}\\
\quad-3\left(\gamma_1-\dfrac{\alpha_0}{2}\right)\left[\left(\dfrac{\partial u_1}{\partial z}\right)^2+2\dfrac{\partial u_1}{\partial z}\dfrac{\partial u_1}{\partial \xi}\right]\dfrac{\partial^2 u_1}{\partial z^2}+\beta_3{u_1}^3.
\end{cases}
\tag{22}
$$

Introducing Eq. (18) into Eq. (22), we obtain

$$
\begin{cases}
-2g\dfrac{\partial u_2^{(0)}}{\partial \xi}+\left(\dfrac{8\omega}{\lambda_\theta k}-\dfrac{2g}{\lambda_\theta}\right)\dfrac{\partial |U|^2}{\partial \xi}+\lambda_\theta\dfrac{\partial v_2^{(0)}}{\partial \xi}=0,\\
-g\dfrac{\partial v_2^{(0)}}{\partial \xi}+\dfrac{4\omega^2}{{\lambda_\theta}^2k^2}\dfrac{\partial |U|^2}{\partial \xi}+\dfrac{\partial p_2^{(0)}}{\partial \xi}=0,\\
2\mathrm{i}\omega u_3^{(1)}+2\dfrac{\partial U}{\partial \tau_1}-2\mathrm{i}kUv_2^{(0)}-\dfrac{2\mathrm{i}\omega}{\lambda_\theta}\left(Uu_2^{(0)}-3U^*u_2^{(2)}\right)+\lambda_\theta\left(\dfrac{\partial v_2^{(1)}}{\partial \xi}-\mathrm{i}kv_3^{(1)}\right)=0,\\
\mathrm{i}\omega v_3^{(1)}-g\dfrac{\partial v_2^{(1)}}{\partial \xi}+\dfrac{2\omega}{\lambda_\theta k}\dfrac{\partial U}{\partial \tau_1}-\dfrac{2\mathrm{i}\omega}{\lambda_\theta}\left(Uv_2^{(0)}+U^*v_2^{(2)}\right)-\mathrm{i}kp_3^{(1)}+\dfrac{\partial p_2^{(1)}}{\partial \xi}=0,\\
p_3^{(1)}=\left(\alpha_0k^2-\dfrac{m\omega^2}{\lambda_\theta\lambda_z}+\beta_1\right)U_3^{(1)}+\left(\dfrac{mg^2}{\lambda_\theta\lambda_z}-\alpha_0\right)\dfrac{\partial^2 U}{\partial \xi^2}+\dfrac{2m\omega\mathrm{i}}{\lambda_\theta\lambda_z}\dfrac{\partial U}{\partial \tau_1}\\
\quad+\left(\dfrac{5m\omega^2}{{\lambda_\theta}^2\lambda_z}+6\alpha_1k^2-\dfrac{5\alpha_0k^2}{\lambda_\theta}+2\beta_2\right)U^*U_2^{(2)}+\left(\dfrac{m\omega^2}{{\lambda_\theta}^2\lambda_z}+2\alpha_1k^2-\dfrac{\alpha_0k^2}{\lambda_\theta}+2\beta_2\right)UU_2^{(0)}\\
\quad+\left[-\dfrac{3m\omega^2}{{\lambda_\theta}^3\lambda_z}+2\alpha_2k^2-\dfrac{5\alpha_1k^2}{\lambda_\theta}+\dfrac{3\alpha_0k^2}{{\lambda_\theta}^2}+3\left(\gamma_1-\dfrac{\alpha_0}{2}\right)k^4+3\beta_3\right]|U|^2U.
\end{cases}
\tag{23}
$$

The following results are easily available:

$$
\begin{aligned}
u_2^{(0)} &= \Phi_1U|U|^2,\quad v_2^{(0)}=\left(\frac{2g}{\lambda_\theta}\Phi_1-\frac{6\omega}{{\lambda_\theta}^2k}\right),\\
\Phi_1 &= \frac{\lambda_\theta}{2g^2-\lambda_\theta\beta_1}\left[2\left(\frac{m\omega^2}{{\lambda_\theta}^2\lambda_z}+\alpha_1k^2-\frac{\alpha_0k^2}{\lambda_\theta}+\beta_2\right)\right.\\
&\quad\left.+\frac{8\omega g}{{\lambda_\theta}^2k}-\frac{2g^2}{{\lambda_\theta}^2}\right].
\end{aligned}
\tag{24}
$$

Substituting Eq. (24) into Eq. (23) and eliminating the third-order perturbed quantities v_3, u_3, and p_3, we finally obtain

$$
\mathrm{i}\frac{\partial U}{\partial \tau_1}+\mu_1\frac{\partial^2 U}{\partial \xi^2}+\mu_2|U|^2U=0,
\tag{25}
$$

where the coefficients μ_1 and μ_2 are defined by

$$
\begin{aligned}
\mu_1 &= \frac{1}{\dfrac{4\omega}{\lambda_\theta k}+\dfrac{2m\omega gk}{\lambda_\theta\lambda_z}}\left(\frac{2mg\omega k}{\lambda_z}+\frac{mg^2k^2}{\lambda_z}-\frac{4\omega g}{k}+\frac{2\omega^2}{k^2}+2g^2-3\lambda_\theta\alpha_0k^2\right),\\
\mu_2 &= \frac{1}{\dfrac{4\omega}{\lambda_\theta k}+\dfrac{2m\omega gk}{\lambda_\theta\lambda_z}}\left\{\left[\frac{10\omega^2}{\lambda_\theta}+\lambda_\theta k^2\left(\frac{5m\omega^2}{{\lambda_\theta}^2\lambda_z}-\frac{5\alpha_0k^2}{\lambda_\theta}+6\alpha_1k^2+2\beta_2\right)\right]\Phi_0\right.\\
&\quad+\left[\frac{8\omega gk}{\lambda_\theta}+\frac{2\omega^2}{\lambda_\theta}+\lambda_\theta k^2\left(\frac{m\omega^2}{{\lambda_\theta}^2\lambda_z}-\frac{\alpha_0k^2}{\lambda_\theta}+2\alpha_1k^2+2\beta_2\right)\right]\Phi_1\\
&\quad\left.-\frac{30\omega^2}{\lambda_\theta}+\lambda_\theta k^2\left[\frac{3m\omega^2}{{\lambda_\theta}^3\lambda_z}-\frac{5\alpha_1k^2}{\lambda_\theta}+\frac{3\alpha_0k^2}{{\lambda_\theta}^2}+3\left(\gamma_1-\frac{\alpha_0}{2}\right)+3\beta_3\right]\right\}.
\end{aligned}
\tag{26}
$$

Equation (25) above is the standard nonlinear Schrödinger equation (NLSE) which has been a traditional basic model for nonlinear wave researches. As we all known, the NLSE includes only basic effects on waves such as the lowest-order dispersion and the lowest-order nonlinearity. With the development of nonlinear model, it is found that the nonlinear Schrodinger model can be used to describe many nonlinear phenomena in almost all branches of physics. However, in the process of practical research, researchers found that the standard nonlinear Schrödinger equation is not enough to properly describe the nonlinear phenomena including various physical factors, so high-order effects are growing in importance when the characteristics of the solution extend beyond the simple approximations which are used in deriving the NLSE. The nonlinear Schrödinger equation was extended to more complex forms such as variable coefficients, higher dimensions, and higher orders for practical research. The high-order NLSE model plays an vital role in describing wave processes in weakly dispersive and weakly nonlinear media. Thus, the models to be studied in the rest sections of our paper are mainly focusing on the high-order nonlinear Schrodinger equation (HNLSE) with variable coefficient.

3. Research of rogue waves in deformable vessels filled with blood

3.1. Derivation of the third-order NLS equation

Similarly, collecting the terms with coefficient $e^{2i\theta}$ in Eq. (22), the equations in the following form are obvious:

$$\begin{cases} 4i\omega {u_3}^{(2)} - 2i\lambda_\theta k {v_3}^{(2)} + \left[\left(4g + \dfrac{4\omega}{k}\right)\Phi_0 + \dfrac{6g}{\lambda_\theta} - \dfrac{6\omega}{\lambda_\theta k}\right] U\dfrac{\partial U}{\partial \xi} = 0, \\ 2i\omega {v_3}^{(2)} - 2ik {p_3}^{(2)} + \dfrac{\partial p_2^{(2)}}{\partial \xi} + \left(-\dfrac{4\omega g}{\lambda_\theta k}\Phi_0 + \dfrac{10\omega g}{{\lambda_\theta}^2 k} + \dfrac{4g}{{\lambda_\theta}^2} - \dfrac{4\omega}{{\lambda_\theta}^2 k}\right) U\dfrac{\partial U}{\partial \xi} = 0, \\ 6\omega {u_3}^{(3)} - \dfrac{10\omega}{\lambda_\theta} U {u_2}^{(2)} - 4kU{v_2}^{(2)} - 3\lambda_\theta k {v_3}^{(3)} = 0, \\ \omega {v_3}^{(3)} - \dfrac{2\omega}{\lambda_\theta} U {v_2}^{(2)} - k {p_3}^{(3)} = 0, \\ {p_3}^{(2)} = \left(4\alpha_0 k^2 - \dfrac{4m\omega^2}{\lambda_\theta \lambda_z} + \beta_1\right) {u_3}^{(2)} + i\left[\left(8\alpha_0 k - \dfrac{8m\omega^2 g}{\lambda_\theta \lambda_z}\right)\Phi_0 + \dfrac{2m\omega g}{{\lambda_\theta}^2 \lambda_z} + 6\alpha_1 k - \dfrac{2\alpha_0 k}{\lambda_\theta}\right] U\dfrac{\partial U}{\partial \xi}, \\ {p_3}^{(3)} = \left(-\dfrac{9m\omega^2}{\lambda_\theta \lambda_z} + 9\alpha_0 k^2 + \beta_1\right) {u_3}^{(3)} + \left[\left(\dfrac{5m\omega^2}{{\lambda_\theta}^2 \lambda_z} - \dfrac{5\alpha_0 k^2}{\lambda_\theta} + 14\alpha_1 k^2 + 2\beta_2\right)\Phi_0 \right. \\ \qquad \left. -\dfrac{m\omega^2}{{\lambda_\theta}^3 \lambda_z} + 2\alpha_2 k^2 - \dfrac{3\alpha_1 k^2}{\lambda_\theta} + \dfrac{\alpha_0 k^2}{{\lambda_\theta}^2} + \dfrac{3\alpha_0 k^4}{2} - 3\gamma_1 k^4 + \beta_3\right] U^3. \end{cases} \tag{27}$$

Solving Eq. (27) results in

$$ {u_3}^{(2)} = iP_2 U\frac{\partial U}{\partial \xi}, \quad {v_3}^{(2)} = iQ_2 U\frac{\partial U}{\partial \xi}, \quad {u_3}^{(3)} = P_3 U^3, \quad {v_3}^{(3)} = Q_3 U^3, \tag{28}$$

where

$$\begin{aligned} P_2 &= \frac{1}{\dfrac{4\omega^2}{\lambda_\theta k} - 8\alpha_0 k^3 + \dfrac{8m\omega^2 k}{\lambda_\theta \lambda_z} - 2\beta_1 k}\left[\left(\frac{4\omega^2}{\lambda_\theta k^2} + 18\alpha_0 k^2 - \frac{16m\omega^2 gk}{\lambda_\theta \lambda_z} - \frac{8m\omega^2}{\lambda_\theta \lambda_z} + 2\beta_1\right)\right. \\ &\quad \left.\Phi_0 + \frac{16\omega g}{{\lambda_\theta}^2 k} - \frac{6\omega^2}{{\lambda_\theta}^2 k^2} + \frac{4g^2}{{\lambda_\theta}^2} - \frac{4\omega}{{\lambda_\theta}^2 k} + \frac{4m\omega gk}{{\lambda_\theta}^2 \lambda_z} + 18\alpha_1 k^2 - \frac{6\alpha_0 k^2}{\lambda_\theta} + \frac{2m\omega^2}{{\lambda_\theta}^2 \lambda_z} + 2\beta_2\right], \\ P_3 &= \frac{1}{\dfrac{2\omega^2}{\lambda_\theta k} + \dfrac{9m\omega^2 k}{\lambda_\theta \lambda_z} - 9\alpha_0 k^3 - \beta_1 k}\left[\left(\frac{10\omega^2}{{\lambda_\theta}^2 k} + \frac{5m\omega^2 k}{{\lambda_\theta}^2 \lambda_z} - \frac{5\alpha_0 k^3}{\lambda_\theta} + 14\alpha_1 k^3 + 2\beta_2 k\right)\right. \\ &\quad \left.\Phi_0 - \frac{10\omega^2}{{\lambda_\theta}^3 k} - \frac{m\omega^2 k}{{\lambda_\theta}^3 \lambda_z} + 2\alpha_2 k^3 - \frac{3\alpha_1 k^3}{\lambda_\theta} + \frac{\alpha_0 k^3}{{\lambda_\theta}^2} + \frac{3\alpha_0 k^5}{2} - 3\gamma_1 k^5 + \beta_3 k\right], \\ Q_2 &= \frac{2\omega}{\lambda_\theta k} P_2 - \left[\left(\frac{2g}{\lambda_\theta k} + \frac{2\omega}{\lambda_\theta k^2}\right)\Phi_0 + \frac{3g}{{\lambda_\theta}^2 k} - \frac{3\omega}{{\lambda_\theta}^2 k^2}\right], \quad Q_3 = \frac{2\omega}{\lambda_\theta k} P_3 - \frac{6\omega}{{\lambda_\theta}^2 k}\Phi_0 + \frac{4\omega}{{\lambda_\theta}^3 k}. \end{aligned} \tag{29}$$

The governing Eq. (7) at $o(\varepsilon^4)$ can be expressed as

$$\begin{cases}
2\dfrac{\partial u_4}{\partial t}-2g\dfrac{\partial u_3}{\partial \xi}+2\dfrac{\partial u_2}{\partial \tau_1}+2\dfrac{\partial u_1}{\partial \tau_2}+2v_1\left(\dfrac{\partial u_3}{\partial z}+\dfrac{\partial u_2}{\partial \xi}\right)+2v_2\left(\dfrac{\partial u_2}{\partial z}+\dfrac{\partial u_1}{\partial \xi}\right)+2v_3\dfrac{\partial u_1}{\partial z}\\
\quad+\lambda_\theta\left(\dfrac{\partial v_4}{\partial z}+\dfrac{\partial v_3}{\partial \xi}\right)+u_1\left(\dfrac{\partial v_3}{\partial z}+\dfrac{\partial v_2}{\partial \xi}\right)+u_2\left(\dfrac{\partial v_2}{\partial z}+\dfrac{\partial v_1}{\partial \xi}\right)+u_3\left(\dfrac{\partial v_1}{\partial z}\right)=0,\\
\dfrac{\partial v_4}{\partial t}-g\dfrac{\partial v_3}{\partial \xi}+\dfrac{\partial v_2}{\partial \tau_1}+\dfrac{\partial v_1}{\partial \tau_2}+v_1\left(\dfrac{\partial v_3}{\partial z}+\dfrac{\partial v_2}{\partial \xi}\right)+v_2\left(\dfrac{\partial v_2}{\partial z}+\dfrac{\partial v_1}{\partial \xi}\right)+v_3\dfrac{\partial v_1}{\partial z}+\dfrac{\partial p_4}{\partial z}+\dfrac{\partial p_3}{\partial \xi}=0,\\
p_4=\dfrac{m}{\lambda_\theta\lambda_z}\left(\dfrac{\partial^2 u_4}{\partial t^2}-2g\dfrac{\partial^2 u_3}{\partial \xi\partial t}+2\dfrac{\partial^2 u_2}{\partial \tau_1\partial t}+2\dfrac{\partial^2 u_1}{\partial \tau_2\partial t}+g^2\dfrac{\partial^2 u_2}{\partial \xi^2}-2g\dfrac{\partial^2 u_1}{\partial \xi\partial \tau_1}\right)+\beta_1 u_4\\
\quad-\dfrac{m}{{\lambda_\theta}^2\lambda_z}\left[u_1\left(\dfrac{\partial^2 u_3}{\partial t^2}-2g\dfrac{\partial^2 u_2}{\partial \xi\partial t}+2\dfrac{\partial^2 u_1}{\partial \tau_1\partial t}+g^2\dfrac{\partial^2 u_1}{\partial \xi^2}\right)+u_2\left(\dfrac{\partial^2 u_2}{\partial t^2}-2g\dfrac{\partial^2 u_1}{\partial \xi\partial t}\right)+u_3\dfrac{\partial^2 u_1}{\partial t^2}\right]\\
\quad-\alpha_1\left[\left(\dfrac{\partial u_1}{\partial \xi}\right)^2+2\left(\dfrac{\partial u_1}{\partial z}\dfrac{\partial u_3}{\partial z}+\dfrac{\partial u_1}{\partial z}\dfrac{\partial u_2}{\partial \xi}+\dfrac{\partial u_1}{\partial \xi}\dfrac{\partial u_2}{\partial z}\right)-\alpha_0\left(\dfrac{\partial^2 u_4}{\partial z^2}+2\dfrac{\partial^2 u_3}{\partial \xi\partial z}+\dfrac{\partial^2 u_2}{\partial \xi^2}\right)\right.\\
\quad\left.+\left(\dfrac{\partial u_2}{\partial z}\right)^2\right]-\left(2\alpha_1-\dfrac{\alpha_0}{\lambda_\theta}\right)\left[u_1\left(\dfrac{\partial^2 u_3}{\partial z^2}+2\dfrac{\partial^2 u_2}{\partial \xi\partial z}+\dfrac{\partial^2 u_1}{\partial \xi^2}\right)+u_2\left(\dfrac{\partial^2 u_2}{\partial z^2}+2\dfrac{\partial^2 u_1}{\partial \xi\partial z}\right)+u_3\dfrac{\partial^2 u_1}{\partial z^2}\right]\\
\quad+\beta_2{u_2}^2+2\beta_2u_1u_3+\dfrac{m}{{\lambda_\theta}^3\lambda_z}\left[{u_1}^2\left(\dfrac{\partial^2 u_2}{\partial t^2}-2g\dfrac{\partial^2 u_1}{\partial \xi\partial t}\right)+2u_1u_2\dfrac{\partial^2 u_1}{\partial t^2}\right]\\
\quad-\left(\alpha_2-\dfrac{\alpha_1}{\lambda_\theta}\right)\left[2u_1\left(\dfrac{\partial u_1}{\partial z}\dfrac{\partial u_2}{\partial z}+\dfrac{\partial u_1}{\partial z}\dfrac{\partial u_1}{\partial \xi}\right)+u_2\left(\dfrac{\partial u_1}{\partial z}\right)^2\right]+3\beta_3{u_1}^2u_2-\dfrac{m}{{\lambda_\theta}^4\lambda_z}{u_1}^3\dfrac{\partial^2 u_1}{\partial t^2}\\
\quad-\left(\alpha_2-\dfrac{2\alpha_1}{\lambda_\theta}+\dfrac{\alpha_0}{{\lambda_\theta}^2}\right)\left[{u_1}^2\left(\dfrac{\partial^2 u_2}{\partial z^2}+2\dfrac{\partial^2 u_1}{\partial \xi\partial z}\right)+2u_1u_2\dfrac{\partial^2 u_1}{\partial z^2}\right]-\left(3\gamma_1-\dfrac{3\alpha_0}{2}\right)\left[\left(\dfrac{\partial u_1}{\partial z}\right)^2\right.\\
\quad\left.\times\left(\dfrac{\partial^2 u_2}{\partial z^2}+2\dfrac{\partial^2 u_1}{\partial \xi\partial z}\right)+2\dfrac{\partial u_1}{\partial z}\dfrac{\partial u_1}{\partial \xi}\dfrac{\partial^2 u_1}{\partial z^2}\right]-\left(12\alpha_3-\dfrac{\alpha_2}{\lambda_\theta}+\dfrac{\alpha_1}{{\lambda_\theta}^2}\right){u_1}^2\left(\dfrac{\partial u_1}{\partial z}\right)^2+\alpha_1\left(\dfrac{\partial u_1}{\partial z}\right)^4\\
\quad-\left(\alpha_3-\dfrac{\alpha_2}{\lambda_\theta}+\dfrac{2\alpha_1}{{\lambda_\theta}^2}-\dfrac{\alpha_0}{{\lambda_\theta}^3}\right){u_1}^3\dfrac{\partial^2 u_1}{\partial z^2}+\left(3\gamma_1+3\alpha_1-\dfrac{3\alpha_0}{2\lambda_\theta}\right)u_1\dfrac{\partial^2 u_1}{\partial z^2}\left(\dfrac{\partial u_1}{\partial z}\right)^2+\beta_4{u_1}^4,
\end{cases}\tag{30}$$

setting $u_3^{(1)}=0$, and taking it into Eq. (23), we have

$$v_3^{(1)}=\left(\frac{2g}{\lambda_\theta k^2}-\frac{2\omega}{\lambda_\theta k^3}\right)\frac{\partial^2 U}{\partial \xi^2}-\left[\frac{6\omega}{{\lambda_\theta}^2 k}\Phi_0+\left(\frac{4g}{{\lambda_\theta}^2}+\frac{2\omega}{{\lambda_\theta}^2 k}\right)\Phi_1-\frac{12\omega}{{\lambda_\theta}^3 k}\right]U|U|^2-\frac{2\mathrm{i}}{\lambda_\theta k}\frac{\partial U}{\partial \tau_1}.\tag{31}$$

In this case, the complex conjugate is denoted by '*'. From Eq. (30), we obtain the zeroth-order equation system as follows:

$$\begin{cases}
-2g\dfrac{\partial {u_3}^{(0)}}{\partial \xi}+\lambda_\theta\dfrac{\partial {v_3}^{(0)}}{\partial \xi}+\left(\dfrac{4\mathrm{i}g}{\lambda_\theta k}-\dfrac{4\mathrm{i}w}{\lambda_\theta k^2}\right)\left(U^*\dfrac{\partial^2 U}{\partial \xi^2}-U\dfrac{\partial^2 U^*}{\partial \xi^2}\right)+\left(2\Phi_1+\dfrac{4}{\lambda_\theta}\right)\dfrac{\partial |U|^2}{\partial \tau_1}=0,\\
-g\dfrac{\partial {v_3}^{(0)}}{\partial \xi}+\dfrac{\partial {p_3}^{(0)}}{\partial \xi}+\dfrac{\partial {p_2}^{(0)}}{\partial \tau_1}+\left(\dfrac{4\mathrm{i}\omega g}{{\lambda_\theta}^2k^2}-\dfrac{4\mathrm{i}\omega^2}{{\lambda_\theta}^2k^3}\right)\left(U^*\dfrac{\partial^2 U}{\partial \xi^2}-U\dfrac{\partial^2 U^*}{\partial \xi^2}\right)+\dfrac{2g}{\lambda_\theta}\Phi_1\dfrac{\partial |U|^2}{\partial \tau_1}=0,\\
{p_3}^{(0)}=\beta_1{u_3}^{(0)}+\left(\dfrac{2\mathrm{i}m\omega}{{\lambda_\theta}^2\lambda_z}+2\mathrm{i}\alpha_1k-\dfrac{2\mathrm{i}\alpha_0k}{\lambda_\theta}\right)\left(U^*\dfrac{\partial^2 U}{\partial \xi^2}-U\dfrac{\partial^2 U^*}{\partial \xi^2}\right),
\end{cases}\tag{32}$$

substituting Eq. (25) into Eq. (32), we can obtain the following solutions:

$${u_3}^{(0)}=\mathrm{i}P_0\left(U^*\frac{\partial U}{\partial \xi}-U\frac{\partial U^*}{\partial \xi}\right),\quad {v_3}^{(0)}=\mathrm{i}Q_0\left(U^*\frac{\partial U}{\partial \xi}-U\frac{\partial U^*}{\partial \xi}\right),\tag{33}$$

where

$$\begin{aligned}
P_0&=\frac{\lambda_\theta}{2g^2-\lambda_\theta\beta_1}\left[2\mu_1\left(\frac{m\omega^2}{{\lambda_\theta}^2\lambda_z}+\alpha_1k^2-\frac{\alpha_0k^2}{\lambda_\theta}+\beta_2\right)+\left(\frac{2m\omega}{{\lambda_\theta}^2\lambda_z}+2\alpha_1k-\frac{2\alpha_0k}{\lambda_\theta}\right)+\left(\frac{4\omega g^2}{{\lambda_\theta}^2k^2}-\frac{4\omega^2}{{\lambda_\theta}^2k^3}\right)\right],\\
Q_0&=\frac{2g}{\lambda_\theta}P_0-\mu_1\left(\frac{2\Phi_1}{\lambda_\theta}+\frac{4}{{\lambda_\theta}^2}\right)+\frac{4\omega}{{\lambda_\theta}^2k^2}-\frac{4g}{{\lambda_\theta}^2k}.
\end{aligned}\tag{34}$$

In the same way, collecting the terms with coefficient $e^{i\theta}$ of Eq. (30), we obtain the equations below:

$$\begin{cases} 2i\omega u_4^{(1)} - i\lambda_\theta k v_4^{(1)} + 2\dfrac{\partial U}{\partial \tau_2} - \dfrac{2i}{k}\dfrac{\partial^2 U}{\partial \tau_1 \partial \xi} + \left(\dfrac{2g}{k^2} - \dfrac{2\omega}{k^3}\right)\dfrac{\partial^3 U}{\partial \xi^3} \\ \quad + \left[\left(\dfrac{2\omega}{\lambda_\theta k} - \dfrac{2g}{\lambda_\theta}\right)\Phi_1 + \left(\dfrac{2g}{\lambda_\theta} - \dfrac{2\omega}{\lambda_\theta k}\right)\Phi_0 + \dfrac{6\omega}{{\lambda_\theta}^2 k} - \dfrac{2\omega}{\lambda_\theta}P_0 - 2kQ_0\right] U\dfrac{\partial |U|^2}{\partial \xi} \\ \quad + \left[\dfrac{6\omega}{\lambda_\theta}P_2 + \dfrac{4\omega}{\lambda_\theta}P_0 + 4kQ_0 + \left(\dfrac{2\omega}{\lambda_\theta k} - \dfrac{2g}{\lambda_\theta}\right)\Phi_0 + \left(\dfrac{2g}{\lambda_\theta} - \dfrac{2\omega}{\lambda_\theta k}\right)\Phi_1\right]|U|^2\dfrac{\partial U}{\partial \xi} = 0, \\ i\omega v_4^{(1)} + \dfrac{2\omega}{\lambda_\theta k}\dfrac{\partial U}{\partial \tau_2} + \dfrac{\partial p_3^{(1)}}{\partial \xi} + i\left(\dfrac{4g}{\lambda_\theta k} - \dfrac{2\omega}{\lambda_\theta k^2}\right)\dfrac{\partial^2 U}{\partial \tau_1 \partial \xi} + \left(\dfrac{2\omega g}{\lambda_\theta k^3} - \dfrac{2g^2}{\lambda_\theta k^2}\right)\dfrac{\partial^3 U}{\partial \xi^3} \\ \quad + \left[\left(-\dfrac{8\omega^2}{{\lambda_\theta}^2 k^2} + \dfrac{10\omega g}{{\lambda_\theta}^2 k}\right)\Phi_0 + \left(\dfrac{2\omega g}{{\lambda_\theta}^2 k} + \dfrac{8g^2}{{\lambda_\theta}^2} + \dfrac{8\omega^2}{{\lambda_\theta}^2 k^2}\right)\Phi_1 + \dfrac{4\omega}{\lambda_\theta}Q_0 + \dfrac{2\omega}{\lambda_\theta}Q_2 - \dfrac{24\omega g}{{\lambda_\theta}^3 k}\right]|U|^2\dfrac{\partial U}{\partial \xi} \\ \quad - ikp_4^{(1)} + \left[\left(\dfrac{8\omega^2}{{\lambda_\theta}^2 k^2} + \dfrac{2\omega g}{{\lambda_\theta}^2 k}\right)\Phi_0 + \left(\dfrac{6\omega g}{{\lambda_\theta}^2 k} + \dfrac{4g^2}{{\lambda_\theta}^2}\right)\Phi_1 - \dfrac{2\omega}{\lambda_\theta}Q_0 - \dfrac{12\omega^2}{{\lambda_\theta}^3 k^2} - \dfrac{12\omega g}{{\lambda_\theta}^3 k}\right]U\dfrac{\partial |U|^2}{\partial \xi} = 0, \end{cases} \tag{35}$$

Setting $u_4^{(1)} = 0$ in Eq. (35), we have

$$\begin{aligned} v_4^{(1)} = i\Bigg(& -\frac{2}{\lambda_\theta k}\frac{\partial U}{\partial \tau_2} + N_1\frac{\partial^3 U}{\partial \xi^3} \\ & + N_1' U\frac{\partial |U|^2}{\partial \xi} + N_1''|U|^2\frac{\partial U}{\partial \xi}\Bigg), \end{aligned} \tag{36}$$

where

$$\begin{aligned} N_1 &= \frac{2\omega}{\lambda_\theta k^4} - \frac{2g}{\lambda_\theta k^3} + \frac{2}{\lambda_\theta k^2}\mu_1, \\ N_1' &= -\frac{1}{\lambda_\theta k}\left[\left(\frac{2\omega}{\lambda_\theta k} - \frac{2g}{\lambda_\theta}\right)\Phi_1 + \left(\frac{2g}{\lambda_\theta} - \frac{2\omega}{\lambda_\theta k}\right)\Phi_0 \right. \\ &\quad \left. + \frac{6\omega}{{\lambda_\theta}^2 k} - \frac{2\omega}{\lambda_\theta}P_0 - 2kQ_0\right] + \frac{2}{\lambda_\theta k^2}\mu_2, \\ N_1'' &= -\frac{1}{\lambda_\theta k}\left[\left(\frac{2g}{\lambda_\theta} - \frac{2\omega}{\lambda_\theta k}\right)\Phi_1 + \left(\frac{2\omega}{\lambda_\theta k} - \frac{2g}{\lambda_\theta}\right)\Phi_0 \right. \\ &\quad \left. + \frac{6\omega}{\lambda_\theta}P_2 + \frac{4\omega}{\lambda_\theta}P_0 + 4kQ_0\right] + \frac{2}{\lambda_\theta k^2}\mu_2. \end{aligned} \tag{37}$$

According to the dispersion relation Eq. (16), ${u_4}^{(1)}, {v_4}^{(1)}$, and ${p_4}^{(1)}$ in Eq. (35) are easily eliminated, and we obtain

$$\frac{\partial U}{\partial \tau_2} + \mu_3\frac{\partial^3 U}{\partial \xi^3} + \mu_4 U\frac{\partial |U|^2}{\partial \xi} + \mu_5|U|^2\frac{\partial U}{\partial \xi} = 0. \tag{38}$$

Introducing the following coordinate transformation:

$$\begin{aligned} & Z = z - \lambda g t, \quad T = \lambda t, \quad V = \lambda_1 U, \\ & \lambda_1 = \sqrt{\frac{2\mu_1}{\varepsilon^2\mu_2}}, \quad \lambda = \frac{1}{2\mu_1}, \end{aligned} \tag{39}$$

combining Eqs. (8), (9), and (39), Eq. (38) can be approximately equivalent to the third-order NLS equation with variable coefficient expressed as

$$\begin{aligned} & iV_T + \frac{V_{ZZ}}{2} + V|V|^2 \\ & + i\lambda\left\{\mu_3 V_{ZZZ} + \varepsilon^2{\lambda_1}^2\left[\mu_4\left(|V|^2\right)_Z V + \mu_5 V_Z|V|^2\right]\right\} = 0, \end{aligned} \tag{40}$$

where, $i\lambda\mu_3$, $i\lambda\varepsilon^2{\lambda_1}^2\mu_4$, and $i\lambda\varepsilon^2{\lambda_1}^2\mu_5$ represent the third order dispersion coefficient, the self-frequency shift effect coefficient, and the self-steepening complex coefficient, respectively.

3.2. Conditions for generating rogue waves in blood vessels

Rogue waves (RW) were first found in the ocean, appearing out of nowhere and disappearing without a trace. It can occasionally rise up to surprising heights above the background wave field and form a huge amount of energy in a short time, which is a disastrous natural physical phenomenon. Thus, rogue wave phenomena have received wide attraction in several fields of nonlinear systems representing ubiquitous physical phenomena these years. Because of the destructive power of rogue waves, it is of great practical significance to study the problem in elastic tubes filled with fluid. Here, we focus on the conditions under which the rogue wave occurs in the fluid-filled elastic tubes. By finding a solution that describes the flow of blood through the blood vessels, we can explore which parameters in the blood vessels of the human body have an important influence on rogue waves, and then control them to gain a better understanding of the flow characteristics of blood through the blood vessels. At the same time, we hope to provide a potential reference for medical clinical research.

When the variable coefficients are chosen to satisfy the following constraints:

$$\mu_5 = 2\mu_4, \quad 3\mu_3 = \frac{2\mu_1\mu_4}{\mu_2}, \tag{41}$$

equation (40) is degenerate into one of the famous existing integrable extensions of the NLS equation called Sasa–Satsuma equation (SSE)

$$\begin{aligned} & iV_T + \frac{V_{ZZ}}{2} + V|V|^2 \\ & + i\lambda\mu_3\left[V_{ZZZ} + 3\left(|V|^2\right)_Z V + 6V_Z|V|^2\right] = 0, \end{aligned} \tag{42}$$

in which the high-order terms represent nonlinear dispersion and third-order dispersion.

Here, $\lambda\mu_3$ is an arbitrary real parameter to scale the integrable perturbations of the NLSE. As an extension of the nonlinear Schrodinger equation, the SSE contains additional terms explaining the third-order dispersion, the self-steepening, and the self-frequency shift as often found in many important physical applications. When $\lambda\mu_3 = 0$, Eq. (42) reduces to the standard NLSE which has only the terms describing the lowest order dispersion and self-phase modulation. As we all know, the rational solutions of the NLSE are always located on a background plane wave. Hence, we start directly with the plane-wave solution serving as a background for rogue waves[18]

$$V_0(Z,T) = \frac{c}{2\lambda\mu_3}\exp\left[-\frac{\mathrm{i}}{2\lambda\mu_3}\left(kZ-\frac{\omega}{4\lambda\mu_3}T\right)\right], \quad (43)$$

the dispersion relation among the amplitude c ($c>0$), the wave number k, and the frequency ω reads

$$\omega = 6c^2k - k^3 + 2c^2 - k^2, \quad 4(1+3k)^2 > 9c^2. \quad (44)$$

Under the constraints given above, the solution describing the rogue wave of Eq. (42) can be written as

$$V(Z,T) = -\frac{c}{2\lambda\mu_3}\left(1-\frac{\eta-\eta^*}{c}G\right)\exp\left[\mathrm{i}\left(\frac{k}{2\lambda\mu_3}Z+\frac{\omega}{8\lambda^2\mu_3^2}T\right)\right], \quad (45)$$

where

$$\begin{aligned}
G &= \frac{|\sigma|^2\,\mathrm{Re}[\eta]\,(\eta\sigma^* l+\eta^*\sigma h^*)+\left(\eta|l|^2+\eta^*|h|^2\right)(\eta^*\sigma^* l+\eta\sigma h^*)}{|\eta|^2\left(|\sigma|^2+|l|^2+|h|^2\right)^2-\left|\sigma^2+2hl\right|^2\mathrm{Im}[\eta]^2},\\
\eta &= \mathrm{i}\frac{\sqrt{9c^2\left(9c^2+10K^2\right)+3c\left(9c^2-4K^2\right)^{\frac{3}{2}}-2K^4}}{3\sqrt{2}K}, \quad K=1+3k, \quad a=\frac{1}{3}(1+3k)^2-12c^2-1,\\
h &= 3c\left[\frac{\sigma}{K+d-\eta}+\mathrm{i}\frac{12\lambda^2\mu_3^2}{(K+d-\eta)^2}\right], \quad l=3c\left[\frac{\sigma}{K-d+\eta}-\mathrm{i}\frac{12\lambda^2\mu_3^2}{(K-d+\eta)^2}\right],\\
\sigma &= \frac{\left[9\left(-6c^2+a\right)\eta^4+3a\left(-18c^2+a-1\right)\eta^2+a^3\right]T}{6\left(d\eta+2\eta^2+a\right)^2}-2\lambda\mu_3 Z,\\
d &= \frac{\left(-1+\mathrm{i}\sqrt{3}\right)\left(K^2\eta-9c^2\eta-\eta^3\right)^{\frac{1}{3}}}{2}+\frac{2\left(K^2+18c^2+3\eta^2\right)}{3(-1+\mathrm{i}\sqrt{3})\left(K^2\eta-9c^2\eta-\eta^3\right)^{\frac{1}{3}}}.
\end{aligned} \quad (46)$$

The lowest-order rogue wave in the SSE can feature a double-peak structure as well as an analog of the Peregrine soliton which is more complicated than the rogue wave solution of the standard NLS equation. In particular, since V involves the fourth-order polynomial, there is a significantly more complicated structure than does the Peregrine soliton, which only involves the second order polynomials. These complexities are related to the fact that the spectral problem of the SSE involves 3×3 matrices rather than 2×2 ones in the other cases.

Combining Eqs. (39) and (45), we obtain

$$U(z,t) = -\frac{c}{2\lambda\lambda_1\mu_3}\left(1-\frac{\eta-\eta^*}{c}G\right) \times\exp\left\{\mathrm{i}\left[\frac{k}{2\lambda\mu_3}z+\left(\frac{\omega}{8\lambda^2\mu_3^2}-\frac{kg}{2\mu_3}\right)t\right]\right\}. \quad (47)$$

Obviously, the solution depends on variables z and t, as well as on three real parameters $\lambda\mu_3$, k, and c. Taking all the constraints Eqs. (16), (41), and (44) into account, we can see that when the wave number k takes a fixed value, variables c, ω, λ_θ, λ_z can also be determined accordingly. This provides a reference for us to discuss the conditions under which a rogue wave appears in an elastic tube filled with fluid.

In Fig. 1, the wave number of the plane wave is chosen to be $k=0.5$. This choice of the wave number means that the circumferential stretch ratio $\lambda_\theta = 1.5$, the axial stretch ratio $\lambda_z = 1.4$, the angular frequency $\omega = 2.7$, and the growth rate $v \approx 1.4$. Similarly as indicated in Fig. 2, when the wave number $k=1$. the condition for the occurrence of the rogue wave is that the circumferential stretch ratio $\lambda_\theta = 1.8$, the axial stretch ratio $\lambda_z = 1.2$, the angular frequency $\omega = 13.2$, and the amplitude $c = 1.4$. For this choice of the plane-wave amplitude, the growth rate $v \approx 3.3$. The characteristic structure of the rogue wave when the wave number $k = 1.5$ is depicted in Fig. 3. As shown in Fig. 3, when $\lambda_\theta = 1.5$, $\lambda_z = 1.1$, $\omega = 4.9$, and $c = 1$, there is a possibility of rogue wave forming in the elastic tube filled with fluid. Notice that the three figures all display the double-peak rogue wave structure with two zero-amplitude points. As the value of k increases, the peaks of the rogue wave get closer and closer, and the crest rotates clock-

wise. In the end, at large enough values of k, the two maxima will merge and the solution has a single peak. By comparing the three figures, we find that the amplitude of the rogue wave c increases with the increase of the circumferential vascular stretch ratio λ_θ. Similarly, the decrease of the axial stretch ratio λ_z leads to a reduction in c. This suggests that the thicker the blood vessels, the more likely they are to produce the rogue wave, the larger the amplitude of the rogue wave. Thus, in actual life, more attention should be paid to preventing the rogue wave from occurring in the expanding blood vessels or larger vessel such as the aorta. When people feel nervous or afraid, their blood vessels often dilate, which increases the probability of rogue waves. Hoping the problems studied and the results obtained in this section have some reference significance in many practical applications such as diagnosing vascular disease or heart disease.

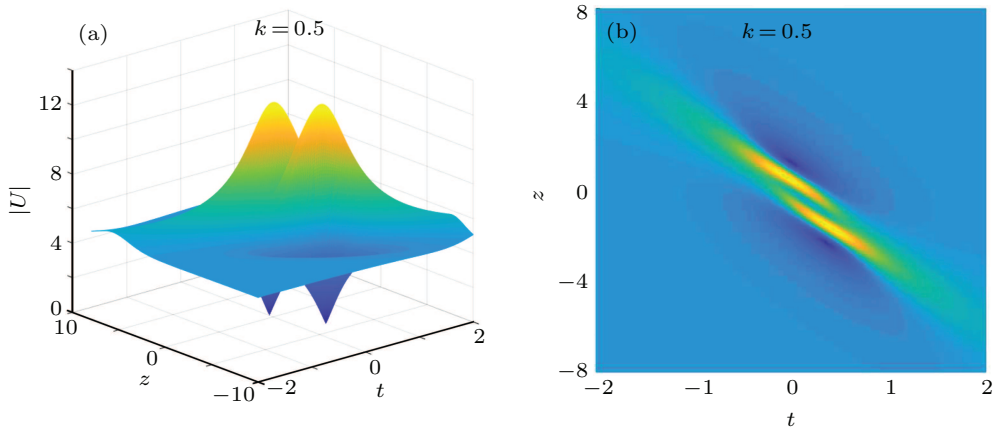


Fig. 1. Characteristic structures of the Sasa–Satsuma equation rogue wave when $\lambda_\theta = 1.5$, $\lambda_z = 1.4$, the amplitude $c = 0.8$, and the wave number $k = 0.5$. (a) The changes of the amplitude, shape, and propagation direction of RW, (b) the distribution of the crests.

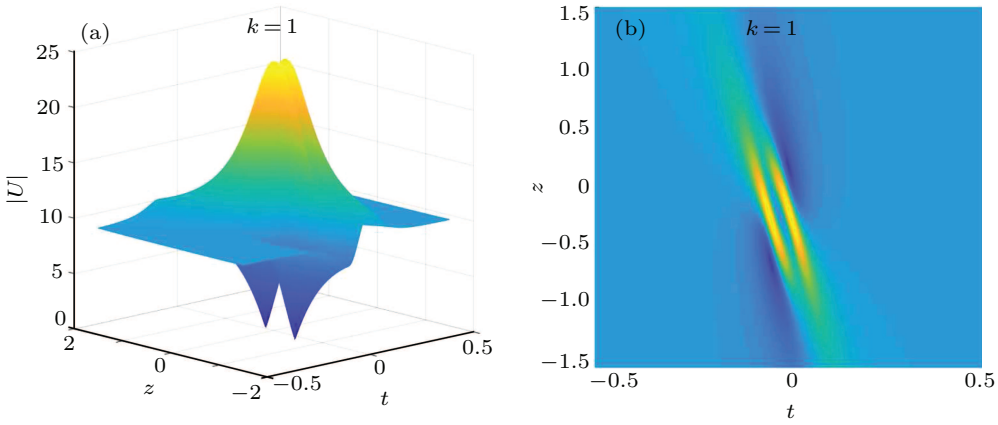


Fig. 2. Characteristic structures of the Sasa–Satsuma equation rogue wave when $\lambda_\theta = 1.8$, $\lambda_z = 1.2$, the amplitude $c = 1.4$, and the wave number $k = 1$. (a) The changes of the amplitude, shape, and propagation direction of RW, (b) the distribution of the crests.

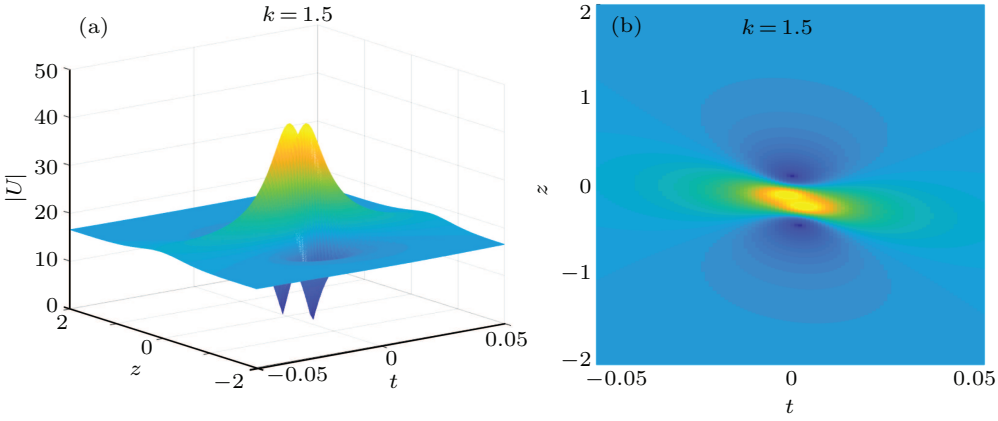


Fig. 3. Characteristic structures of the Sasa–Satsuma equation rogue wave when $\lambda_\theta = 1.5$, $\lambda_z = 1.1$, the amplitude $c = 1$, and the wave number $k = 1.5$. (a) The changes of the amplitude, shape, and propagation direction of RW, (b) the distribution of the crests.

4. The contributions of the high-order terms to the wave propagation in elastic tube

4.1. Derivation of the fourth-order NLS equation with variable coefficient

Substituting Eq. (18) into Eq. (30), and collecting the terms of $e^{2i\theta}$, $e^{3i\theta}$, and $e^{4i\theta}$, we have

$$\begin{cases}
4i\omega u_4^{(2)} - 2i\lambda_\theta k v_4^{(2)} - 2g\dfrac{\partial u_3^{(2)}}{\partial\xi} + 2\dfrac{\partial u_2^{(2)}}{\tau_1} - \dfrac{10i\omega}{\lambda_\theta k}U^* u_3^{(3)} - 3ikUv_3^{(1)} \\
\quad - ikU^* v_3^{(3)} + \lambda_\theta\dfrac{\partial v_3^{(2)}}{\partial\xi} + 2v_2^{(1)}\dfrac{\partial U}{\partial\xi} + U\dfrac{\partial v_2^{(1)}}{\partial\xi} - 2iku_2^{(0)}v_2^{(2)} - 4iku_2^{(2)}v_2^{(0)} = 0, \\
2i\omega v_4^{(2)} - g\dfrac{\partial v_3^{(2)}}{\partial\xi} + \dfrac{\partial v_2^{(2)}}{\tau_1} - \dfrac{4i\omega}{\lambda_\theta}U^* v_3^{(3)} - \dfrac{4i\omega}{\lambda_\theta}Uv_3^{(1)} + \dfrac{2\omega}{\lambda_\theta k}U\dfrac{\partial v_2^{(1)}}{\partial\xi} \\
\quad + \dfrac{2\omega}{\lambda_\theta k}v_2^{(1)}\dfrac{\partial U}{\partial\xi} + U\dfrac{\partial v_2^{(1)}}{\partial\xi} - 2ikv_2^{(0)}v_2^{(2)} - 4ik\left(v_2^{(1)}\right)^2 - 2ikp_4^{(2)} + \dfrac{\partial p_3^{(2)}}{\partial\xi} = 0, \\
6i\omega u_4^{(3)} - 3i\lambda_\theta k v_4^{(3)} - 2g\dfrac{\partial u_3^{(3)}}{\partial\xi} + \lambda_\theta\dfrac{\partial v_3^{(3)}}{\partial\xi} - \dfrac{10i\omega}{\lambda_\theta}Uu_3^{(2)} - 4ikUv_3^{(2)} \\
\quad + \dfrac{4\omega}{\lambda_\theta k}U\dfrac{\partial u_2^{(2)}}{\partial\xi} - 5iku_2^{(2)}v_2^{(1)} + 2v_2^{(2)}\dfrac{\partial U}{\partial\xi} + U\dfrac{\partial v_2^{(2)}}{\partial\xi} + \dfrac{2\omega}{\lambda_\theta k}u_2^{(2)}\dfrac{\partial U}{\partial\xi} = 0, \\
3i\omega v_4^{(3)} - g\dfrac{\partial v_3^{(3)}}{\partial\xi} - \dfrac{6i\omega}{\lambda_\theta}Uv_3^{(2)} + \dfrac{2\omega}{\lambda_\theta k}U\dfrac{\partial v_2^{(2)}}{\partial\xi} - 2ikv_2^{(1)}v_2^{(2)} + \dfrac{2\omega}{\lambda_\theta k}v_2^{(2)}\dfrac{\partial U}{\partial\xi} - 3ikp_4^{(3)} + \dfrac{\partial p_3^{(3)}}{\partial\xi} = 0, \\
8\omega u_4^{(4)} - \dfrac{14\omega}{\lambda_\theta}Uu_3^{(3)} - 6ku_2^{(2)}v_2^{(2)} - 5kUv_3^{(3)} - 4\lambda_\theta k v_4^{(4)} = 0, \\
4\omega v_4^{(4)} - \dfrac{8\omega}{\lambda_\theta}Uv_3^{(3)} - 2k\left(v_2^{(2)}\right)^2 - 4kp_4^{(4)} = 0.
\end{cases} \tag{48}$$

Introducing $p_4^{(i)}$ $(i = 2,3,4)$ into Eq. (48), we have

$$\begin{aligned}
u_4^{(2)} &= M_2U\frac{\partial^2U}{\partial\xi^2} + M_2'\left(\frac{\partial U}{\partial\xi}\right)^2 + M_2''|U|^2U^2, \quad v_4^{(2)} = N_2U\frac{\partial^2U}{\partial\xi^2} + N_2'\left(\frac{\partial U}{\partial\xi}\right)^2 + N_2''|U|^2U^2, \\
u_4^{(3)} &= iM_3U^2\frac{\partial U}{\partial\xi}, \quad v_4^{(3)} = iN_3U^2\frac{\partial U}{\partial\xi}, \quad u_4^{(4)} = M_4U^4, \quad v_4^{(4)} = N_4U^4.
\end{aligned} \tag{49}$$

Similarly, the $o(\varepsilon^5)$ equations of Eq. (7) are as follows:

$$\begin{cases}
2\left(\dfrac{\partial u_5}{\partial t} - g\dfrac{\partial u_4}{\partial\xi} + \dfrac{\partial u_3}{\partial\tau_1} + \dfrac{\partial u_2}{\partial\tau_2} + \dfrac{\partial u_1}{\partial\tau_3}\right) + 2v_1\left(\dfrac{\partial u_4}{\partial z} + \dfrac{\partial u_3}{\partial\xi}\right) + 2v_2\left(\dfrac{\partial u_3}{\partial z} + \dfrac{\partial u_2}{\partial\xi}\right) \\
\quad + 2v_3\left(\dfrac{\partial u_2}{\partial z} + \dfrac{\partial u_1}{\partial\xi}\right) + 2v_4\dfrac{\partial u_1}{\partial z} + \lambda_\theta\left(\dfrac{\partial v_5}{\partial z} + \dfrac{\partial v_4}{\partial\xi}\right) + u_1\left(\dfrac{\partial v_4}{\partial z} + \dfrac{\partial v_3}{\partial\xi}\right) \\
\quad + u_2\left(\dfrac{\partial v_3}{\partial z} + \dfrac{\partial v_2}{\partial\xi}\right) + u_3\left(\dfrac{\partial v_2}{\partial z} + \dfrac{\partial v_1}{\partial\xi}\right) + u_4\dfrac{\partial v_1}{\partial z} = 0, \\
\dfrac{\partial v_5}{\partial t} - g\dfrac{\partial v_4}{\partial\xi} + \dfrac{\partial v_3}{\partial\tau_1} + \dfrac{\partial v_2}{\partial\tau_2} + \dfrac{\partial v_1}{\partial\tau_3} + v_1\left(\dfrac{\partial v_4}{\partial z} + \dfrac{\partial v_3}{\partial\xi}\right) \\
\quad + v_2\left(\dfrac{\partial v_3}{\partial z} + \dfrac{\partial v_2}{\partial\xi}\right) + v_3\left(\dfrac{\partial v_2}{\partial z} + \dfrac{\partial v_1}{\partial\xi}\right) + v_4\dfrac{\partial v_1}{\partial z} + \dfrac{\partial p_5}{\partial z} + \dfrac{\partial p_4}{\partial\xi} = 0.
\end{cases} \tag{50}$$

Owing to the similarity of derivation process with the foregoing chapters, we will not repeat it here. We obtain $u_4^{(0)}$ and $v_4^{(0)}$ from the zeroth-order equation of Eq. (50) below:

$$\begin{aligned}
u_4^{(0)} &= M_0\left(U\frac{\partial^2U^*}{\partial\xi^2} + U^*\frac{\partial^2U}{\partial\xi^2}\right) + M_0'\frac{\partial U^*}{\partial\xi}\frac{\partial U}{\partial\xi} \\
&\quad + M_0''\left(|U|^2\right)^2, \\
v_4^{(0)} &= N_0\left(U\frac{\partial^2U^*}{\partial\xi^2} + U^*\frac{\partial^2U}{\partial\xi^2}\right) + N_0'\frac{\partial U^*}{\partial\xi}\frac{\partial U}{\partial\xi} \\
&\quad + N_0''\left(|U|^2\right)^2,
\end{aligned} \tag{51}$$

and the final result can be derived as

$$i\frac{\partial U}{\partial\tau_3} + \Lambda_1|U|^2\frac{\partial^2U}{\partial\xi^2} + \Lambda_2\frac{\partial U}{\partial\xi}\frac{\partial|U|^2}{\partial\xi}$$

$$+\Lambda_3 U^2\frac{\partial^2 U^*}{\partial\xi^2}+\Lambda_4 U^*\left(\frac{\partial U}{\partial\xi}\right)^2$$
$$+\Lambda_5 U\frac{\partial U}{\partial\xi}\frac{\partial U^*}{\partial\xi}+\Lambda_6|U|^4U$$
$$+\Lambda_7\frac{\partial^4 U}{\partial\xi^4}+\Lambda_8 U\frac{\partial^2\left(|U|^2\right)}{\partial\xi^2}=0. \tag{52}$$

Combining Eqs. (9), (25), (38), and (52), we finally derive the fourth-order NLS equation with variable coefficients in the following form:

$$\begin{aligned}&\mathrm{i}\left(U_t+gU_z\right)+\mu_1U_{zz}+\varepsilon^2\mu_2U|U|^2\\&+\mathrm{i}\left[\mu_3U_{zzz}+\varepsilon^2\left(\mu_4+\mu_5\right)U_z|U|^2+\varepsilon^2\mu_4U^2U_z^*\right]\\&+\Lambda_7U_{zzzz}+\varepsilon^4\Lambda_6U|U|^4\\&+\varepsilon^2\left[\left(\Lambda_1+\Lambda_8\right)U_{zz}|U|^2+\left(\Lambda_2+\Lambda_5+2\Lambda_8\right)U\left|U_z\right|^2\right.\\&\left.+\left(\Lambda_3+\Lambda_8\right)U^2U_{zz}^*+\left(\Lambda_2+\Lambda_4\right)U_z{}^2U^*\right]=0,\end{aligned} \tag{53}$$

where z and t are the propagation variable and time in the moving frame, respectively, with the function $U(z,t)$ being the envelope of the waves. Real constants ε, μ_i $(i=1,\ldots,5)$, Λ_j $(j=1,\ldots,8)$ are free parameters in our model. In particular, the coefficients μ_3, Λ_7, and $\varepsilon^4\Lambda_6$ control independently the values of third-order dispersion U_{zzz}, fourth-order dispersion U_{zzzz}, and the effects of fifth-order nonlinearity $U|U|^4$.

4.2. The traveling wave solutions of the fourth-order NLS equation with variable coefficients

For that purpose, we assume the solution of the following form:

$$U(z,t)=\psi(\zeta)\,\mathrm{e}^{\mathrm{i}(Kz+\Omega t)},\quad \zeta=z+nt, \tag{54}$$

where $\psi(\zeta)=\psi(z,t)$ is a real function, K,Ω, and n are constants to be determined later. Substituting the traveling wave variables (54) into Eq. (53), we obtain

$$\mathrm{Im}:\ k_1\psi'+k_2\psi^{(3)}+k_3\psi^2\psi'=0, \tag{55}$$

and

$$\begin{aligned}\mathrm{Re}:\ &k_4\psi+k_5\psi''+k_6\psi^3+k_7\psi^2\psi''+k_8\psi\left(\psi'\right)^2\\&+\Lambda_7\psi^{(4)}+\varepsilon^4\Lambda_6\psi^5=0,\end{aligned} \tag{56}$$

where

$$\begin{aligned}k_1&=m+2K\mu_1-3K^2\mu_3-4K^3\Lambda_7,\quad k_2=\mu_3+4K\Lambda_7,\\k_3&=\frac{\varepsilon^2}{3}[2\mu_4+2\mu_5+2\cdot K(\Lambda_1-\Lambda_3+\Lambda_2+\Lambda_4)],\\k_4&=-\Omega-gK-\mu_lK^2+\mu_3K^3+K^4\Lambda_7,\\k_5&=\mu_1-3K\mu_3-6K^2\Lambda_7,\\k_6&=\varepsilon^2\mu_2-K\varepsilon^2\mu_5-\varepsilon^2K^2\Lambda_1\\&\quad+\varepsilon^2K^2\Lambda_5-\varepsilon^2K^2\Lambda_3-\varepsilon^2K^2\Lambda_4,\\k_7&=\varepsilon^2(\Lambda_1+\Lambda_3+2\Lambda_8),\\k_8&=\varepsilon^2(2\cdot\Lambda_2+\Lambda_4+\Lambda_5+2\Lambda_8).\end{aligned} \tag{57}$$

Integrating Eq. (55) once and combining the result with Eq. (56), then, setting the integration constants to be zero, we get

$$f_1\psi+f_2\psi^3+f_3\psi^5+k_8\psi\left(\psi'\right)^2+\Lambda_7\psi^{(4)}=A, \tag{58}$$

where A is an integration constant which can take arbitrary constants, and f_i $(i=1,2,3)$ can be written as

$$\begin{aligned}&f_1=k_4-\frac{k_1k_5}{k_2},\quad f_2=k_6-\frac{k_3k_5}{k_2}-\frac{k_1k_7}{k_2},\\&f_3=\varepsilon^4\Lambda_6-\frac{k_7k_3}{k_2}.\end{aligned} \tag{59}$$

Considering the homogeneous balance between ψ^5 and $\psi^{(4)}$ appearing in Eq. (58), the solution of Eq. (58) can be supposed to the following form:

$$\psi(\zeta)=a_0+a_1\left(\frac{G'}{G}\right), \tag{60}$$

and $G=G(\zeta)$ satisfies the second order linear ordinary differential equation

$$G''(\zeta)+\phi G'(\zeta)+\mu G(\zeta)=0, \tag{61}$$

where a_0, a_1, μ, and λ are constants to be determined later, and $a_1\neq0$. Substituting (G'/G)-expansion (60) into Eq. (58) and utilizing linear ordinary differential equation (61), collecting all terms with the same power of G'/G together and equating their coefficients to zero, yield a system of overdetermined algebraic equations for a_0, a_1, μ, and ϕ as follows:

$$\begin{cases}{a_0}^5f_3+a_0{a_1}^2\mu^2k_8+{a_0}^3f_2+a_0f_1+\dfrac{Ak_5}{k^2}-\dfrac{Ak_7{a_0}^2}{k_2}=0,\\5{a_0}^4a_1f_3+2a_0{a_1}^2\phi\,\mu\,k_8+{a_1}^3\mu^2k_8+22a_1\phi^2\mu\,\Lambda_7+17a_1\mu^2\Lambda_7+3{a_0}^2a_1f_2+a_1f_1-\dfrac{2Aa_0a_1k_7}{k^2}=0,\\10{a_0}^3{a_1}^2f_3+a_0{a_1}^2k_8\phi^2+2{a_1}^3k_8\phi,\mu+15\Lambda_7a_1\phi^3+2a_0{a_1}^2k_8\mu+60\Lambda_7a_1\phi,\mu+3a_0{a_1}^2-\dfrac{Ak_7{a_1}^2}{k_2}=0,\\10{a_0}^2{a_1}^3f_3+{a_1}^3\phi^2k_8+2a_0{a_1}^2\phi,k_8+2{a_1}^3\mu\,k_8+50a_1\phi^2\Lambda_7+{a_1}^3f_2+40a_1\mu\,\Lambda_7=0,\\5a_0{a_1}^4f_3+2{a_1}^3k_8\phi+a_0{a_1}^2k_8+60\Lambda_7a_1\phi=0,\\{a_1}^5f_3+{a_1}^3k_8+24\Lambda_7a_1=0.\end{cases} \tag{62}$$

Solving the algebraic equations above by Maple16, we get one of the results

$$\lambda=0,\quad a_0=0,\quad a_1=\pm\sqrt{\frac{1}{k_8}\left(17\Lambda_7-\frac{f_1}{\mu^2}\right)},\quad A=0,$$
$$f_2=-\frac{2\mu k_8\left(-3\Lambda_7\mu^2+f_1\right)}{17\Lambda_7\mu^2+f_1},$$
$$f_3=\frac{\left(-7\Lambda_7\mu^2+f_1\right)\mu^2k_8^2}{\left(17\Lambda_7\mu^2+f_1\right)^2}. \tag{63}$$

1) When $\mu<0$, Eq. (58) admits a solution known as hyperbolic function solution in the form

$$U(z,t)=\pm\sqrt{\frac{1}{k_8}\left(17\Lambda_7-\frac{f1}{\mu^2}\right)}\times\frac{A_1\sqrt{-\mu}\sinh(\sqrt{-\mu}\zeta)+A_2\sqrt{-\mu}\cosh(\sqrt{-\mu}\zeta)}{A_1\cosh(\sqrt{-\mu}\zeta)+A_2\sinh(\sqrt{-\mu}\zeta)}\times e^{i(Kz+\Omega t)},\quad \zeta=z+nt. \tag{64}$$

2) When $\mu>0$, Eq. (58) admits a solution known as periodic traveling wave solution in the form

$$U(z,t)=\pm\sqrt{\frac{1}{k_8}\left(17\Lambda_7-\frac{f1}{\mu^2}\right)}\times\frac{-A_3\sqrt{\mu}\sin(\sqrt{\mu}\zeta)+A_4\sqrt{\mu}\cos(\sqrt{\mu}\zeta)}{A_3\cos(\sqrt{\mu}\zeta)+A_4\sin(\sqrt{\mu}\zeta)}\times e^{i(Kz+\Omega t)},\quad \zeta=z+nt. \tag{65}$$

Here, A_1, A_2, A_3, and A_4 are arbitrary constants, and when they take different values, the traveling wave solutions for different situations of Eq. (58) can be obtained.

In particular, if taking $A_1=0$ and $A_2=0$, we have

$$U(z,t)=\pm\sqrt{\frac{1}{k_8}\left(17\Lambda_7-\frac{f1}{\mu^2}\right)}\times\sqrt{-\mu}\coth(\sqrt{-\mu}\zeta)\times e^{i(Kz+\Omega t)},\quad \zeta=z+nt, \tag{66}$$

and if taking $A_1=0$ and $A_2=0$, we have

$$U(z,t)=\pm\sqrt{\frac{1}{k_8}\left(17\Lambda_7-\frac{f1}{\mu^2}\right)}\times\sqrt{-\mu}\tanh(\sqrt{-\mu}\zeta)\times e^{i(Kz+\Omega t)},\quad \zeta=z+nt, \tag{67}$$

equations (66) and (67) above are known as singular wave solution and kink wave solution, respectively.

As seen in Fig. 4, (a) depicts the variation of $|U|$ with respect to z, t for different initial axial stretching ratios when $\lambda_\theta=1.2$, $\lambda_z=1.2$; (b) shows the comparison of $|U|$ with respect to z for different λ_θ and λ_z. Obviously, under different initial arterial deformation, the waveform of $|U|$ is a bell curve. Besides, $|U|$ depends on the values of λ_θ and λ_z, which are specifically expressed as follows. With the increase of λ_θ and λ_z, the wave amplitude increases, the waveform becomes steeper and the wavelength becomes shorter. The deformation displacement of the blood vessel is proportional to the circumferential stretching ratio as well as to the axial stretching ratio. In the other word, the greater the degree of the vessel initial deformation, the greater the influence of the wave on the vessel wall displacement.

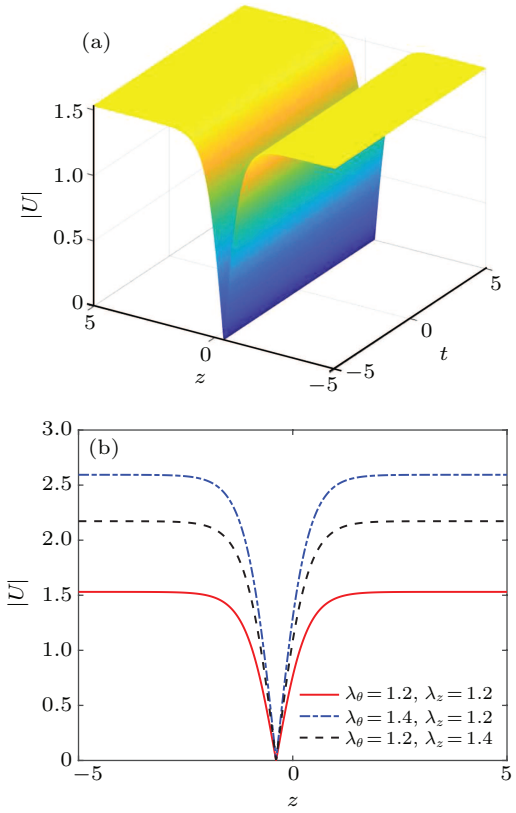


Fig. 4. The waveform of $|U|$ when $\mu=-2$, $A_1=2$, $A_2=1$, $\phi=0$, $K=2$, and $\Omega=2$. (a) Waveform of $|U|$, (b) the comparison of $|U|$ with respect to z for different λ_θ and λ_z.

Figure 5 shows the different solutions to the amplitude of the nonlinear Schrodinger equation when A_i $(i=1,2,3,4)$ take different values. Panel (a) represents the solitary wave solution when $\mu=-2$, $\lambda_\theta=1.2$, $\lambda_z=1.2$, $A_1=0$, $A_2=1$, $\phi=0$, $K=2$, and $\Omega=2$. Panel (b) draws the comparison of ψ with respect to z for different λ_θ and λ_z. We can see that with increasing λ_θ and λ_z, the wave amplitude increases and the waveform becomes steeper. When $\mu=2$, $\lambda_\theta=1.2$, $\lambda_z=1.2$, $A_3=2$, $A_4=1$, $\phi=0$, $K=2$, and $\Omega=2$, the periodic traveling wave solution can be obtained as shown in Fig. 5(c). Taking $A_1=2$, $A_2=0$, we get the singular wave solution to the amplitude of the fourth-order NLS equation as depicted in Fig. 5(d).

We analyze the effects of the higher order terms and the initial deformation of vessels on wave propagation as seen in Figs. 6 and 7. The results show that the high-order nonlinear and dispersion terms lead to the distortion of the wave, while the initial deformation of the tube wall will influence the wave amplitude, wave length, and wave steepness. Compared with the standard NLS equation, the fourth-order NLS equation with variable coefficients can more truly reflect the propagation characteristics of the pulse wave.

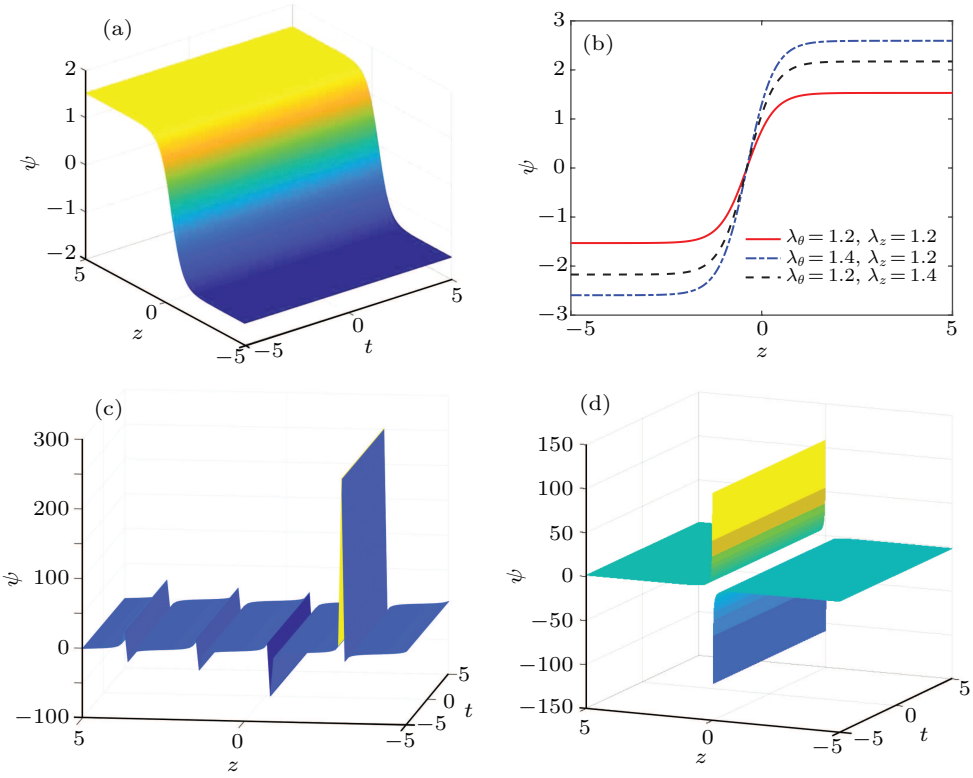


Fig. 5. The exact traveling wave solutions of the fourth-order NLSE. (a) The solitary wave solution, (b) the comparison of ψ with respect to z for different λ_θ and λ_z, (c) the periodic traveling wave solution, (d) the singular wave solution.

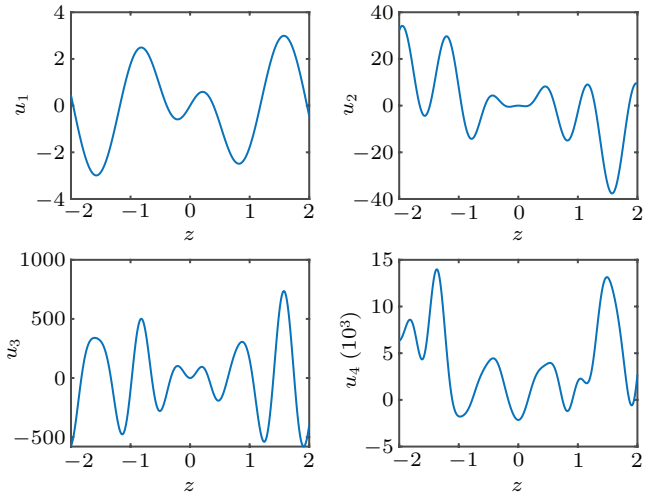


Fig. 6. The variation of u_i $(i = 1, 2, 3, 4)$ with respect to z for $\mu = -2$, $\lambda_\theta = 1.2$, $\lambda_z = 1.2$, $A_1 = 2$, $A_2 = 1$, $\phi = 0$, $K = 2$, and $\Omega = 2$.

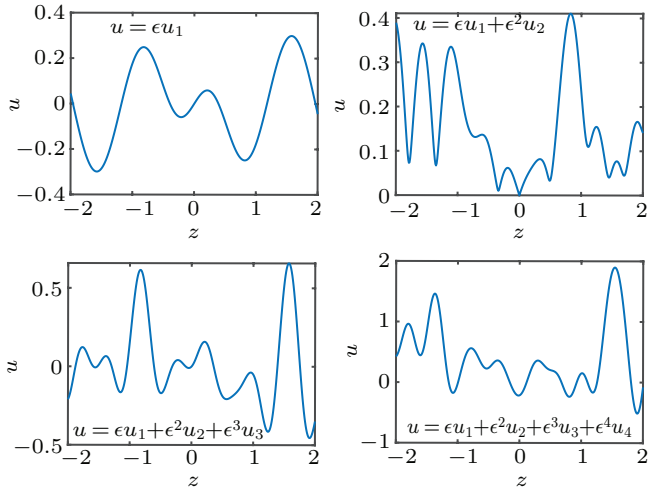


Fig. 7. The influence of higher order terms to u when $\mu = -2$, $\lambda_\theta = 1.2$, $\lambda_z = 1.2$, $A_1 = 2$, $A_2 = 1$, $\phi = 0$, $K = 2$, $\Omega = 2$, and $\varepsilon = 0.1$.

5. Conclusion

On the basis of existing studies, ignoring the viscosity effect of blood, this paper considers that blood is an incompressible ideal fluid, while the blood vessel wall is regarded as an incompressible soft biological tissue material. Using the reduced perturbation and long wave approximation, the basic equations describing the wall motion and blood flow are transformed into nonlinear Schrödinger equation (NLSE) and high-order nonlinear Schrödinger equations. Under particular conditions the third-order NSL equation can degenerate into SSE which may exist the rogue wave solution. We further discuss the conditions under which the rogue waves are generated in the blood vessels and the effect of some parameter changes on the rogue wave. Similarly, we derive a fourth-order NSL equation with variable coefficients and obtains various solutions forms by the (G'/G)-expansion method. Finally, the effects of higher order terms and the initial deformation of vessels on wave propagation are discussed. Through theoretical analysis, we obtain the following conclusions:

(1) The amplitude of rogue wave is proportional to the initial stretching ratios of elastic tubes. The bigger radius of the elastic tube, the higher possibility of developing the rogue waves, and the larger the amplitude of the rogue wave.

(2) The high-order nonlinear and dispersion terms lead to the distortion of the wave, while the initial deformation of the tube wall will influence the wave amplitude and wave steepness.

Appendix A: Coefficients

$$
\begin{aligned}
\mu_3 &= \frac{1}{\left(\frac{4\omega}{\lambda_\theta k}+\frac{2m\omega k}{\lambda_\theta\lambda_z}\right)}\left[-\left(\frac{2m\omega}{\lambda_\theta\lambda_z}+\frac{2mgk}{\lambda_\theta\lambda_z}-\frac{4\omega}{\lambda_\theta k^2}+\frac{4g}{\lambda_\theta k}\right)\mu_1+\frac{mg^2}{\lambda_\theta\lambda_z}-\frac{2g^2}{\lambda_\theta k^2}-\frac{2\omega^2}{\lambda_\theta k^4}+\frac{4\omega g}{\lambda_\theta k^3}-\alpha_0\right],\\
\mu_4 &= \frac{1}{\left(\frac{4\omega}{\lambda_\theta k}+\frac{2m\omega k}{\lambda_\theta\lambda_z}\right)}\left[\left(\frac{2\omega^2}{{\lambda_\theta}^2k^2}+\frac{4\omega g}{{\lambda_\theta}^2k}+\frac{4g^2}{{\lambda_\theta}^2}+\frac{m\omega^2}{{\lambda_\theta}^2\lambda_z}+4\alpha_1k^2-\frac{\alpha_0k^2}{\lambda_\theta}+2\beta_2\right)\Phi_1\right.\\
&\quad+\left(\frac{5m\omega^2}{{\lambda_\theta}^2\lambda_z}-\frac{2m\omega gk}{{\lambda_\theta}^2\lambda_z}+\frac{4\omega g}{{\lambda_\theta}^2k}+\frac{6\omega^2}{{\lambda_\theta}^2k^2}+6\alpha_1k^2-\frac{3\alpha_0k^2}{\lambda_\theta}+2\beta_2\right)\Phi_0\\
&\quad-\frac{4\omega}{\lambda_\theta}Q_0-\left(\frac{m\omega^2k}{{\lambda_\theta}^2\lambda_z}+\frac{2\omega^2}{{\lambda_\theta}^2k}+2\alpha_1k^3-\frac{\alpha_0k^3}{\lambda_\theta}+2\beta_2k\right)P_0-\left(\frac{2m\omega}{\lambda_\theta\lambda_z}+\frac{2mgk}{\lambda_\theta\lambda_z}-\frac{4\omega}{\lambda_\theta k^2}+\frac{4g}{\lambda_\theta k}\right)\mu_2\\
&\quad\left.+\frac{2m\omega gk}{\lambda_\theta^3\lambda_z}-\frac{6\omega^2}{\lambda_\theta^3k^2}-\frac{12\omega g}{\lambda_\theta^3k}-\frac{3m\omega^2}{\lambda_\theta^3\lambda_z}+2\alpha_2k^2-\frac{3\alpha_1k^2}{\lambda_\theta}+\frac{\alpha_0k^2}{\lambda_\theta^2}+21\left(\gamma_1-\frac{\alpha_0}{2}\right)k^4+3\beta_3\right],\\
\mu_5 &= \frac{1}{\left(\frac{4\omega}{\lambda_\theta k}+\frac{2m\omega k}{\lambda_\theta\lambda_z}\right)}\left[\left(\frac{5m\omega^2k}{{\lambda_\theta}^2\lambda_z}+\frac{6\omega^2}{{\lambda_\theta}^2k}+6\alpha_1k^3-\frac{5\alpha_0k^3}{\lambda_\theta}+2\beta_2k\right)P_2+\frac{8\omega}{\lambda_\theta k}Q_0+\frac{2\omega}{\lambda_\theta k}Q_2\right.\\
&\quad+\left(\frac{4\omega^2}{{\lambda_\theta}^2k}+\frac{2m\omega^2k}{{\lambda_\theta}^2\lambda_z}-\frac{2\alpha_0k^3}{\lambda_\theta}+4\alpha_1k^3+4\beta_2k\right)P_0-\left(\frac{2m\omega}{\lambda_\theta\lambda_z}+\frac{2mgk}{\lambda_\theta\lambda_z}-\frac{4\omega}{\lambda_\theta k^2}+\frac{4g}{\lambda_\theta k}\right)\mu_2\\
&\quad+\left(\frac{6\omega^2}{{\lambda_\theta}^2k^2}+\frac{m\omega^2}{{\lambda_\theta}^2\lambda_z}+\frac{2m\omega gk}{{\lambda_\theta}^2\lambda_z}-\frac{3\alpha_0k^2}{\lambda_\theta}+6\alpha_1k^2+\frac{4\omega g}{{\lambda_\theta}^2k}+\frac{8g^2}{{\lambda_\theta}^2}+2\beta_2\right)\Phi_1\\
&\quad+\left(\frac{8\omega g}{{\lambda_\theta}^2k}-\frac{6\omega^2}{{\lambda_\theta}^2k^2}+\frac{10m\omega gk}{{\lambda_\theta}^2\lambda_z}+\frac{5m\omega^2}{{\lambda_\theta}^2\lambda_z}-\frac{15\alpha_0k^2}{\lambda_\theta}+18\alpha_1k^2+2\beta_2\right)\Phi_0\\
&\quad\left.-\frac{6m\omega gk}{{\lambda_\theta}^3\lambda_z}-\frac{15\alpha_1k^2}{\lambda_\theta}+\frac{9\alpha_0k^2}{{\lambda_\theta}^2}+6\alpha_2k^2-\frac{24\omega g}{{\lambda_\theta}^3k}-3\gamma_1k^4-\frac{3}{2}\alpha_0k^4-\frac{3m\omega^2}{\lambda_\theta^3\lambda_z}+3\beta_3\right],\qquad\text{(A1)}\\
M_2 &= \frac{1}{8\alpha_0k^3-\frac{8m\omega^2k}{\lambda_\theta\lambda_z}-\frac{4\omega^2}{\lambda_\theta k}+2\beta_1k}\left[\left(12\alpha_0k^2-\frac{4mg^2k}{\lambda_\theta\lambda_z}-\frac{8m\omega^2g}{\lambda_\theta\lambda_z}\right)\Phi_0+\left(\frac{\omega}{k}-g\right)Q_2\right.\\
&\quad-\left(\frac{4m\omega^2}{\lambda_\theta\lambda_z}+\frac{8m\omega gk}{\lambda_\theta\lambda_z}+\frac{2\omega g}{\lambda_\theta k}-12\alpha_0k^2-\beta_1\right)P_2-\left(\frac{12\omega}{\lambda_\theta^2k}+\frac{4m\omega k}{\lambda_\theta^2\lambda_z}\right)\mu_1+\left(\frac{8\omega}{\lambda_\theta k}+\frac{16m\omega k}{\lambda_\theta\lambda_z}\right)\mu_1\Phi_0\\
&\quad\left.+\frac{2mg^2k}{\lambda_\theta^2\lambda_z}+\frac{2m\omega g}{\lambda_\theta^2\lambda_z}+\frac{2\omega g}{\lambda_\theta^2k^2}+\frac{6\omega^2}{\lambda_\theta^2k^3}+10\alpha_1k-\frac{4\alpha_0k}{\lambda_\theta}\right],\\
M_2' &= \frac{1}{12\alpha_0k^3-\frac{8m\omega^2k}{\lambda_\theta\lambda_z}-\frac{4\omega^2}{\lambda_\theta k}+2\beta_1k}\left[+\left(12\alpha_0k^2-\frac{4mg^2k}{\lambda_\theta\lambda_z}-\frac{8m\omega^2g}{\lambda_\theta\lambda_z}\right)\Phi_0+\left(\frac{\omega}{k}-g\right)Q_2\right.\\
&\quad\left.\times\left(8\alpha_0k^2-\frac{4m\omega^2}{\lambda_\theta\lambda_z}-\frac{2\omega g}{\lambda_\theta k}-\frac{8m\omega gk}{\lambda_\theta\lambda_z}+\beta_1\right)P_2+\frac{2m\omega g}{\lambda_\theta^2\lambda_z}+\frac{4\omega g}{\lambda_\theta^2k^2}-\frac{8\omega^2}{\lambda_\theta^2k^3}+\frac{4g^2}{\lambda_\theta^2k}+8\alpha_1k-\frac{2\alpha_0k}{\lambda_\theta}\right],\\
M_2'' &= \frac{1}{8\alpha_0k^3-\frac{8m\omega^2k}{\lambda_\theta\lambda_z}-\frac{4\omega^2}{\lambda_\theta k}+2\beta_1k}\left[\frac{24\omega^2}{\lambda_\theta^4k}-\frac{8m\omega^2k}{\lambda_\theta^4\lambda_z}+8\alpha_1k^5-8\beta_4k-\frac{4\omega}{\lambda_\theta}Q_3\right.\\
&\quad-\left(\frac{20m\omega^2k}{\lambda_\theta^2\lambda_z}+\frac{10\omega^2}{\lambda_\theta^2k^2}+28\alpha_1k^3-\frac{20\alpha_0k^3}{\lambda_\theta}+4\beta_2k\right)P_3+\left(\frac{8\omega}{\lambda_\theta k}\Phi_0+\frac{16m\omega k}{\lambda_\theta\lambda_z}\Phi_0-\frac{12\omega}{\lambda_\theta^2k}-\frac{4m\omega k}{\lambda_\theta^2\lambda_z}\right)\mu_2\\
&\quad\left.+\left(\frac{24\omega g}{\lambda_\theta^3}+\frac{12\omega^2}{\lambda_\theta^3k}-4\alpha_2k^3+\frac{8\alpha_1k^3}{\lambda_\theta}-\frac{4\alpha_0k^3}{\lambda_\theta^2}-6\beta_3k\right)\Phi_1\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{16m\omega^2k}{\lambda_\theta^3\lambda_z}+\frac{84\omega^2}{\lambda_\theta^3k}-22\alpha_2k^3+\frac{46\alpha_1k^3}{\lambda_\theta}-\frac{24\alpha_0k^3}{\lambda_\theta^2}-48\gamma_1k^5+24\alpha_0k^5\right)\Phi_0\\
&-\left(\frac{8m\omega^2k}{\lambda_\theta^2\lambda_z}-\frac{16\omega g}{\lambda_\theta^2}+\frac{4\omega^2}{\lambda_\theta^2k}+\frac{8g^2}{\lambda_\theta^2}+16\alpha_1k^3-\frac{8\alpha_0k^3}{\lambda_\theta}+4\beta_2k\right)\Phi_0\Phi_1\Big],\\
N_2=&\frac{2\omega}{\lambda_\theta k}M_2-\frac{g}{\lambda_\theta k}P_2+\frac{1}{2k}Q_2+\left(\frac{2}{\lambda_\theta k}\Phi_0-\frac{2}{\lambda_\theta^2k}\right)\mu_1+\frac{3g}{\lambda_\theta^2k^2}+\frac{\omega}{\lambda_\theta^2k^3},\\
N_2'=&\frac{2\omega}{\lambda_\theta k}M_2'-\frac{g}{\lambda_\theta k}P_2+\frac{1}{2k}Q_2+\frac{4g}{\lambda_\theta^2k^2}-\frac{4\omega}{\lambda_\theta^2k^3},\\
N_2''=&\frac{2\omega}{\lambda_\theta k}M_2''+\left(\frac{2}{\lambda_\theta k}\Phi_0-\frac{2}{\lambda_\theta^2k}\right)\mu_2-\frac{5\omega}{\lambda_\theta^2k^2}P_3-\frac{1}{2\lambda_\theta}Q_3-\left(\frac{8g}{\lambda_\theta^2}+\frac{2\omega}{\lambda_\theta^2k}\right)\Phi_0\Phi_1+\frac{30\omega}{\lambda_\theta^3k}\Phi_0\\
&+\left(\frac{4g}{\lambda_\theta^3k}+\frac{2\omega}{\lambda_\theta^3k}\right)\Phi_1-\frac{12\omega}{\lambda_\theta^4k},\\
M_3=&\frac{1}{\dfrac{6\omega^2}{\lambda_\theta k}+\dfrac{27m\omega^2k}{\lambda_\theta\lambda_z}-27\alpha_0k^3-3\beta_1k}\Big[\left(\frac{3\omega}{k}-3g\right)Q_3+\frac{10\omega}{\lambda_\theta}Q_2\\
&+\left(\frac{12\omega^2}{\lambda_\theta^2k^2}+\frac{30m\omega gk}{\lambda_\theta^2\lambda_z}+\frac{15m\omega^2}{\lambda_\theta^2\lambda_z}+\frac{18\omega g}{\lambda_\theta^2k}+126\alpha_1k^2-\frac{45\alpha_0k^2}{\lambda_\theta}+6\beta_2\right)\Phi_0\\
&+\left(-\frac{6\omega g}{\lambda_\theta k}-\frac{54m\omega gk}{\lambda_\theta\lambda_z}-\frac{27m\omega^2}{\lambda_\theta\lambda_z}+81\alpha_1k^3+3\beta_1\right)P_3\\
&+\left(\frac{10\omega^2}{\lambda_\theta^2k}+\frac{15m\omega^2k}{\lambda_\theta^2\lambda_z}+42\alpha_1k^3-\frac{15\alpha_0k^3}{\lambda_\theta}+6\beta_2k\right)P_2-\frac{12m\omega gk}{\lambda_\theta^3\lambda_z}-\frac{3m\omega^2}{\lambda_\theta^3\lambda_z}\\
&+30\alpha_2k^2-\frac{45\alpha_1k^2}{\lambda_\theta}+\frac{15\alpha_0k^2}{\lambda_\theta^2}-81\gamma_1k^4-\frac{81\alpha_0k^4}{2}+3\beta_3\Big],\\
N_3=&\frac{2\omega}{\lambda_\theta k}M_3+\frac{2g}{\lambda_\theta k}P_3-\frac{10\omega}{\lambda_\theta^2k}P_2-\frac{1}{k}Q_3-\frac{4}{\lambda_\theta}Q_2-\left(\frac{8\omega}{3\lambda_\theta^2k^2}+\frac{10g}{3\lambda_\theta^2k}\right)\Phi_0,\\
M_4=&\frac{1}{64\alpha_0k^3+4\beta_1k-\dfrac{8\omega^2}{\lambda_\theta k}-\dfrac{64m\omega^2k}{\lambda_\theta\lambda_z}}\Big[-16\alpha_1k^5+\frac{6\alpha_0k^5}{\lambda_\theta}-4\beta_4k\\
&\times\left(-\frac{14\omega^2}{\lambda_\theta^2k}-\frac{5\omega}{\lambda_\theta}-\frac{40m\omega^2k}{\lambda_\theta^2\lambda_z}-104\alpha_1k^3+\frac{40\alpha_0k^3}{\lambda_\theta}-8\beta_2k\right)P_3\\
&+\left(\frac{24m\omega^2k}{\lambda_\theta^3\lambda_z}-36\alpha_2k^3+\frac{52\alpha_1k^3}{\lambda_\theta}-\frac{16\alpha_0k^3}{\lambda_\theta^2}+96\gamma_1k^5-48\alpha_0k^5-4\beta_3k\right)\Phi_0\\
&-\frac{8\omega}{\lambda_\theta}Q_3-\frac{4m\omega^2k}{\lambda_\theta^4\lambda_z}-52\alpha_3k^3+\frac{8\alpha_2k^3}{\lambda_\theta}-\frac{12\alpha_1k^3}{\lambda_\theta^2}+\frac{4\alpha_0k^3}{\lambda_\theta^3}-12\gamma_1k^5\\
&-\left(\frac{16\omega^2}{\lambda_\theta^2k}+\frac{16m\omega^2k}{\lambda_\theta^2\lambda_z}+48\alpha_1k^3-\frac{16\alpha_0k^3}{\lambda_\theta}+4\beta_2k\right)\Phi_0^2\Big],\\
N_4=&\frac{2\omega}{\lambda_\theta k}M_4-\frac{7\omega}{2\lambda_\theta^2k}P_3-\frac{3\omega}{\lambda_\theta^2k}\Phi_0^2-\frac{5}{4\lambda_\theta}Q_3,\\
M_0=&\frac{\lambda_\theta}{2g^2-\beta_1\lambda_\theta}\Big[\left(\frac{4\omega}{\lambda_\theta^2k^2}+\frac{2g}{\lambda_\theta^2k}-\frac{2g}{\lambda_\theta}P_0-Q_2\right)\mu_1+\left(\frac{6\omega}{\lambda_\theta^2k}-\frac{2g}{\lambda_\theta^2}-\frac{4g}{\lambda_\theta}\Phi_1\right)\mu_3\\
&-\left(\frac{mg^2}{\lambda_\theta\lambda_z}+\alpha_0\right)\Phi_1-\frac{mg^2}{\lambda_\theta^2\lambda_z}-2\alpha_1+\frac{\alpha_0}{\lambda_\theta}-\frac{gk}{\lambda_\theta}N_1+\frac{2\omega g}{\lambda_\theta^2k^3}+\frac{4g^2}{\lambda_\theta^2k^2}-\frac{4\omega^2}{\lambda_\theta^2k^4}\Big],\\
M_0'=&\frac{\lambda_\theta}{2g^2-\beta_1\lambda_\theta}\left[\left(\frac{4\omega}{\lambda_\theta^2k^2}+\frac{4g}{\lambda_\theta^2k}+\frac{2g}{\lambda_\theta}P_0+Q_2\right)\mu_1-\left(\frac{2mg^2}{\lambda_\theta\lambda_z}+2\alpha_0\right)\Phi_1-2\alpha_1+\frac{8\omega g}{\lambda_\theta^2k^3}-\frac{8\omega^2}{\lambda_\theta^2k^4}\right]-M_0,\\
M_0''=&\frac{\lambda_\theta}{2g^2-\beta_1\lambda_\theta}\Big[\left(\frac{6\omega g}{\lambda_\theta^2k}+\frac{8m\omega^2}{\lambda_\theta^2\lambda_z}-\frac{8\alpha_0k^2}{\lambda_\theta}+2\beta_2\right)\Phi_0^2\\
&-\left(\frac{2g}{\lambda_\theta}\mu_4+\frac{g}{\lambda_\theta}\mu_5+\frac{42\omega g}{\lambda_\theta^3k}+\frac{10g^2}{\lambda_\theta^3}+\frac{8\omega^2}{\lambda_\theta^3k^2}-2\alpha_2k^2+\frac{2\alpha_1k^2}{\lambda_\theta}+2\beta_3\right)\Phi_1-\left(\frac{3\omega}{\lambda_\theta^2k}-\frac{g}{\lambda_\theta}\Phi_1\right)(2\mu_4+\mu_5)
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
&-\left(\frac{24\omega^2}{\lambda_\theta^3k^2}-\frac{15\omega g}{\lambda_\theta^3k}+\frac{2\omega g}{\lambda_\theta^2k}P_2+kQ_2-\frac{6m\omega^2}{\lambda_\theta^3\lambda_z}+2\alpha_2k^2-\frac{26\alpha_1k^2}{\lambda_\theta}+\frac{8\alpha_0k^2}{\lambda_\theta^2}+2\beta_3\right)\Phi_0\\
&+\left(\frac{8\omega}{\lambda_\theta^2k^2}+\frac{5g}{\lambda_\theta^2k}-\frac{2g}{\lambda_\theta}P_0-Q_2\right)\mu_2+\left(\beta_2+\frac{5g^2}{\lambda_\theta^2}\right)\Phi_1^2-\frac{2g}{\lambda_\theta^2}\mu_5-\frac{gk}{\lambda_\theta}\left(N_1'+N_1''\right)+\frac{6m\omega^2}{\lambda_\theta^4\lambda_z}-18\alpha_3k^2-\frac{4\alpha_2k^2}{\lambda_\theta}\\
&+\frac{10\alpha_1k^2}{\lambda_\theta^2}-\frac{6\alpha_0k^2}{\lambda_\theta^3}-6\gamma_1k^4+\frac{3\alpha_0k^4}{\lambda_\theta}+6\beta_4+\frac{66\omega^2}{\lambda_\theta^4k^2}+\frac{18\omega g}{\lambda_\theta^4k}\Bigg],\\
N_0=&\frac{2g}{\lambda_\theta}M_0-\frac{1}{\lambda_\theta}\left[\left(\frac{2}{\lambda_\theta k}-2P_0\right)\mu_1-\left(2\Phi_1+\frac{2}{\lambda_\theta}\right)\mu_3-kN_1+\frac{4g}{\lambda_\theta k^2}-\frac{2\omega}{\lambda_\theta k^3}\right],\\
N_0'=&\frac{2g}{\lambda_\theta}M_0'-\frac{1}{\lambda_\theta}\left[\left(-\frac{2}{\lambda_\theta k}-4P_0\right)\mu_1-\left(2\Phi_1+\frac{2}{\lambda_\theta}\right)\mu_3-kN_1+\frac{2\omega}{\lambda_\theta k^3}\right],\\
N_0''=&\frac{2g}{\lambda_\theta}M_0''-\frac{1}{2\lambda_\theta}\left[\left(\frac{10}{\lambda_\theta k}-4P_0\right)\mu_2-\left(4\mu_4+2\mu_5+\frac{28\omega}{\lambda_\theta^2k}+\frac{20g}{\lambda_\theta^2}\right)\Phi_1\right.\\
&\left.-\frac{2}{\lambda_\theta}\mu_5+\frac{6g}{\lambda_\theta}\Phi_1^2+\frac{12\omega}{\lambda_\theta k}\Phi_0^2-\frac{30\omega}{\lambda_\theta^2k}\Phi_0+\frac{4\omega}{\lambda_\theta k}\Phi_0P_2-2kN_1'-2kN_1''+\frac{36\omega}{\lambda_\theta^3k}\right],
\end{aligned}
\tag{A3}
$$

$$
\begin{aligned}
\Lambda_1=&-\frac{1}{\dfrac{4\omega}{\lambda_\theta k}+\dfrac{2m\omega k}{\lambda_\theta\lambda_z}}\left\{\left(\frac{4\omega^2}{\lambda_\theta^2k^2}+\frac{5m\omega^2}{\lambda_\theta^2\lambda_z}+\frac{4m\omega gk}{\lambda_\theta^2\lambda_z}+12\alpha_1k^2-\frac{\alpha_0k^2}{\lambda_\theta}+2\beta_2\right)P_2\right.\\
&+\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+\frac{4\omega^2}{\lambda_\theta^2k^2}+4\alpha_1k^2-\frac{\alpha_0k^2}{\lambda_\theta}+2\beta_2\right)P_0-\frac{2\omega}{\lambda_\theta k}N_2-\frac{4\omega}{\lambda_\theta}N_0+\left(\frac{2m\omega g}{\lambda_\theta^2\lambda_z}-\frac{4\omega g}{\lambda_\theta^2k^2}+\frac{mg^2k}{\lambda_\theta^2\lambda_z}+6\alpha_1k-\frac{\alpha_0k}{\lambda_\theta}\right)\Phi_1\\
&+\left(\frac{8m\omega g}{\lambda_\theta^2\lambda_z}+\frac{mg^2k}{\lambda_\theta^2\lambda_z}+16\alpha_1k-\frac{6\alpha_0k}{\lambda_\theta}\right)\Phi_0-\left(\frac{2mg}{\lambda_\theta\lambda_z}+\frac{2\omega}{\lambda_\theta k^3}-\frac{2g}{\lambda_\theta k^2}\right)\mu_2\\
&-\left(\frac{4m\omega^2k}{\lambda_\theta^2\lambda_z}+\frac{6\omega^2}{\lambda_\theta^2k^2}+6\alpha_1k^3-\frac{5\alpha_0k^3}{\lambda_\theta}+2\beta_2k\right)M_2\\
&-\left(\frac{m\omega^2k}{\lambda_\theta^2\lambda_z}+\frac{2\omega^2}{\lambda_\theta^2k^2}+2\alpha_1k^3-\frac{\alpha_0k^3}{\lambda_\theta}+2\beta_2k\right)M_0+\frac{4\omega}{\lambda_\theta k}\left(Q_2+Q_0\right)\\
&-\left[\left(\frac{8m\omega k}{\lambda_\theta^2\lambda_z}+\frac{14\omega}{\lambda_\theta^2k}\right)\Phi_0+\left(\frac{2m\omega k}{\lambda_\theta^2\lambda_z^2}+\frac{12g}{\lambda_\theta^2}+\frac{4\omega}{\lambda_\theta^2k}\right)\Phi_1-\frac{36\omega}{\lambda_\theta^3k}-\frac{2m\omega k}{\lambda_\theta^3\lambda_z}\right]\mu_1\\
&+\left(\frac{6mk}{\lambda_\theta\lambda_z}+\frac{6}{\lambda_\theta k}\right)\mu_1\mu_2-\left(\frac{2mgk}{\lambda_\theta\lambda_z}+\frac{2m\omega}{\lambda_\theta\lambda_z}-\frac{4\omega}{\lambda_\theta k^2}+\frac{4g}{\lambda_\theta k}\right)\mu_5+\left(\frac{\omega}{k}-g\right)N_1{}'\\
&\left.-\frac{2mg^2k}{\lambda_\theta^3\lambda_z}-\frac{4m\omega g}{\lambda_\theta^3\lambda_z}+\frac{4\omega g}{\lambda_\theta^3k^2}+6\alpha_2k-\frac{12\alpha_1k}{\lambda_\theta}+\frac{6\alpha_0k}{\lambda_\theta^2}+18\gamma_1k^3-9\alpha_0k^3\right\},\\
\Lambda_2=&-\frac{1}{\dfrac{4\omega}{\lambda_\theta k}+\dfrac{2m\omega k}{\lambda_\theta\lambda_z}}\left[\left(\frac{5m\omega^2}{\lambda_\theta^2\lambda_z}+6\alpha_1k^2-\frac{5\alpha_0k^2}{\lambda_\theta}+2\beta_2\right)P_2\right.\\
&+\left(\frac{\omega}{k}-g\right)\left(N_1'+N_1''\right)+\left(\frac{2m\omega g}{\lambda_\theta^2\lambda_z}-\frac{4\omega^2}{\lambda_\theta^2k^3}+\frac{4g^2}{\lambda_\theta^2k}+6\alpha_1k-\frac{2\alpha_0k}{\lambda_\theta}\right)\Phi_1\\
&+\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+2\alpha_1k^2-\frac{\alpha_0k^2}{\lambda_\theta}+2\beta_2\right)P_0+\left(\frac{8m\omega g}{\lambda_\theta^2\lambda_z}+12\alpha_1k^2-\frac{8\alpha_0k^2}{\lambda_\theta}\right)\Phi_0\\
&\left.+\frac{12\omega^2}{\lambda_\theta^3k^3}-\frac{4m\omega g}{\lambda_\theta^3\lambda_z}-\frac{12\omega g}{\lambda_\theta^3k^2}+4\alpha_2k-\frac{8\alpha_1k}{\lambda_\theta}+\frac{4\alpha_0k}{\lambda_\theta^2}+12\gamma_1k^3-6\alpha_0k^3\right],\\
\Lambda_3=&-\frac{1}{\dfrac{4\omega}{\lambda_\theta k}+\dfrac{2m\omega k}{\lambda_\theta\lambda_z}}\left[\left(\frac{16\omega^2}{\lambda_\theta^2k^3}-\frac{16\omega g}{\lambda_\theta^2k^2}-\frac{2m\omega g}{\lambda_\theta^2\lambda_z}+\frac{mg^2k}{\lambda_\theta^2\lambda_z}+2\alpha_1k+\frac{3\alpha_0k}{\lambda_\theta}\right)\Phi_0\right.\\
&-\frac{3\omega}{\lambda_\theta k}Q_0-\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+\frac{4\omega^2}{\lambda_\theta^2k^2}+4\alpha_1k^2-\frac{\alpha_0k^2}{\lambda_\theta}+2\beta_2\right)P_0\\
&+\left[\left(\frac{4g}{\lambda_\theta^2}+\frac{2\omega}{\lambda_\theta^2k}\right)\Phi_1-\left(\frac{4\omega}{\lambda_\theta^2k}+\frac{2m\omega k}{\lambda_\theta^2\lambda_z}\right)\Phi_0+\frac{2m\omega k}{\lambda_\theta^3\lambda_z}\right]\mu_1-\left(\frac{6}{\lambda_\theta k}+\frac{2mk}{\lambda_\theta\lambda_z}\right)\mu_1\mu_2\\
&\left.-\left(\frac{m\omega^2k}{\lambda_\theta^2\lambda_z}+\frac{2\omega^2}{\lambda_\theta^2k^2}+2\alpha_1k^3-\frac{\alpha_0k^3}{\lambda_\theta}\right)M_0-\frac{4\omega}{\lambda_\theta}N_0-\frac{m\omega^2k}{\lambda_\theta^2\lambda_z}+\frac{2m\omega g}{\lambda_\theta^3\lambda_z}-\frac{mg^2k}{\lambda_\theta^3\lambda_z}+\alpha_2k-\frac{\alpha_0k}{\lambda_\theta^2}+15\gamma_1k^3-\frac{21}{2}\alpha_0k^3\right],
\end{aligned}
\tag{A4}
$$

$$
\begin{aligned}
\Lambda_4 = &-\frac{1}{\dfrac{4\omega}{\lambda_\theta k}+\dfrac{2m\omega k}{\lambda_\theta\lambda_z}}\left[\left(\frac{4m\omega gk}{\lambda_\theta^2\lambda_z}+\frac{4\omega^2}{\lambda_\theta^2k^2}+6\alpha_1k^2+\frac{4\alpha_0k^2}{\lambda_\theta}\right)P_2-\frac{2\omega}{\lambda_\theta}N_2'+\left(\frac{2m\omega gk}{\lambda_\theta^2\lambda_z}+\frac{2\omega g}{\lambda_\theta^2k}+4\alpha_1k^2+\frac{2\alpha_0k^2}{\lambda_\theta}\right)P_0\right.\\
&+\frac{3\omega}{\lambda_\theta k}Q_2+\left(\frac{2\omega}{\lambda_\theta k}+\frac{2g}{\lambda_\theta}\right)Q_0-\left(\frac{5m\omega^2k}{\lambda_\theta^2\lambda_z}+\frac{6\omega^2}{\lambda_\theta^2k^2}+6\alpha_1k^3-\frac{5\alpha_0k^3}{\lambda_\theta}+2\beta_2k\right)M_2'\\
&\left.+\left(\frac{mg^2k}{\lambda_\theta^2\lambda_z}+4\alpha_1k+\frac{2\alpha_0k}{\lambda_\theta}\right)\Phi_0+\alpha_2k-\frac{\alpha_1k}{\lambda_\theta}-12\gamma_1k^3+6\alpha_0k^3\right],\\
\Lambda_5 = &-\frac{1}{\dfrac{4\omega}{\lambda_\theta k}+\dfrac{2m\omega k}{\lambda_\theta\lambda_z}}\left[\left(\frac{10\omega g}{\lambda_\theta^2k}-\frac{10\omega^2}{\lambda_\theta^2k^2}-\frac{2m\omega gk}{\lambda_\theta^2\lambda_z}-4\alpha_1k^2+\frac{2\alpha_0k^2}{\lambda_\theta}\right)P_2\right.\\
&-\left(\frac{2m\omega gk}{\lambda_\theta^2\lambda_z}+\frac{2m\omega^2}{\lambda_\theta^2\lambda_z}+12\alpha_1k^2-\frac{4\alpha_0k^2}{\lambda_\theta}+\frac{2\omega g}{\lambda_\theta^2k}+4\beta_2\right)P_0-\frac{4\omega}{\lambda_\theta}N_0'+\left(\frac{18\omega^2}{\lambda_\theta^2k^3}-\frac{18\omega g}{\lambda_\theta^2k^2}-\frac{4m\omega g}{\lambda_\theta^2\lambda_z}+\frac{4\alpha_0k}{\lambda_\theta}\right)\Phi_0\\
&+\left(\frac{6g}{\lambda_\theta}-\frac{2\omega}{\lambda_\theta k}\right)Q_2+\left(\frac{2\omega}{\lambda_\theta k}-\frac{2g}{\lambda_\theta}\right)Q_0-\left(\frac{m\omega^2k}{\lambda_\theta^2\lambda_z}+\frac{2\omega^2}{\lambda_\theta^2k}+2\alpha_1k^3-\frac{\alpha_0k^3}{\lambda_\theta}+2\beta_2k\right)M_0'\\
&\left.+\frac{4m\omega g}{\lambda_\theta^3\lambda_z}+\alpha_2k+\frac{3\alpha_1k}{\lambda_\theta}-\frac{4\alpha_0k}{\lambda_\theta^2}+24\gamma_1k^3-12\alpha_0k^3\right],\\
\Lambda_6 = &-\frac{1}{\dfrac{4\omega}{\lambda_\theta k}+\dfrac{2m\omega k}{\lambda_\theta\lambda_z}}\left\{-\left(\frac{m\omega^2k}{\lambda_\theta^2\lambda_z}+\frac{2\omega^2}{\lambda_\theta^2k}+2\alpha_1k^3-\frac{\alpha_0k^3}{\lambda_\theta}+2\beta_2k\right)M_0''\right.\\
&-\left(-\frac{11m\omega^2k}{\lambda_\theta^3\lambda_z}+6\alpha_2k^3-\frac{17\alpha_1k^3}{\lambda_\theta}+\frac{11\alpha_0k^3}{\lambda_\theta^2}-9\gamma_1k^5+\frac{9\alpha_0k^5}{2}+3\beta_3k\right)P_3\\
&-\left(\frac{5m\omega^2k}{\lambda_\theta^2\lambda_z}+\frac{6\omega^2}{\lambda_\theta^2k^2}+6\alpha_1k^3-\frac{5\alpha_0k^3}{\lambda_\theta}+2\beta_2k\right)M_2''-\left(\frac{48\omega g}{\lambda_\theta^4}+\frac{24\omega^2}{\lambda_\theta^4k}\right)\Phi_1\\
&-\left(-\frac{m\omega^2k}{\lambda_\theta^3\lambda_z}-\frac{8g^2k}{\lambda_\theta^3}-\frac{8\omega g}{\lambda_\theta^3}-\frac{2\omega^2}{\lambda_\theta^3k}+\alpha_2k^3-\frac{2\alpha_1k^3}{\lambda_\theta}+\frac{\alpha_0k^3}{\lambda_\theta^2}+3\beta_3k\right)\Phi_1^2\\
&-\left(-\frac{18m\omega^2k}{\lambda_\theta^3\lambda_z}-\frac{48\omega^2}{\lambda_\theta^3k}+14\alpha_2k^3-\frac{32\alpha_1k^3}{\lambda_\theta}+\frac{18\alpha_0k^3}{\lambda_\theta^2}+24\gamma_1k^5-12\alpha_0k^5+6\beta_3k\right)\Phi_0^2\\
&-\left(\frac{22m\omega^2k}{\lambda_\theta^2\lambda_z}+16\beta_4k-68\alpha_3k^3-\frac{20\alpha_2k^3}{\lambda_\theta}+\frac{48\alpha_1k^3}{\lambda_\theta^2}-\frac{28\alpha_0k^3}{\lambda_\theta^3}-24\gamma_1k^5+10\alpha_1k^5+\frac{3\alpha_0k^5}{\lambda_\theta}+\frac{96\omega^2}{\lambda_\theta^4k}+\frac{5\omega}{\lambda_\theta}Q_3\right)\Phi_0\\
&+\left(\frac{2}{\lambda_\theta k}+\frac{2mk}{\lambda_\theta\lambda_z}\right)\mu_2^2-\left(-\frac{10m\omega^2k}{\lambda_\theta^3\lambda_z}-\frac{32\omega g}{\lambda_\theta^3}-\frac{16\omega^2}{\lambda_\theta^3k}+6\alpha_2k^3-\frac{16\alpha_1k^3}{\lambda_\theta}+\frac{10\alpha_0k^3}{\lambda_\theta^2}+6\beta_3k\right)\Phi_0\Phi_1\\
&-\left[\left(\frac{18\omega}{\lambda_\theta^2k}+\frac{10m\omega k}{\lambda_\theta^2\lambda_z}\right)\Phi_0+\left(\frac{8g}{\lambda_\theta^2}+\frac{2\omega}{\lambda_\theta^2k}+\frac{2m\omega k}{\lambda_\theta^2\lambda_z}\right)\Phi_1+\frac{24\omega}{\lambda_\theta^3k}\right]\mu_2-\frac{4\omega}{\lambda_\theta}N_0''\\
&+\left(-\frac{13m\omega^2k}{\lambda_\theta^2\lambda_z}-\frac{12\omega^2}{\lambda_\theta^2k}+4\alpha_1k^3+\frac{4\alpha_0k^3}{\lambda_\theta}-2\beta_2k\right)\Phi_0P_3-\frac{2\omega}{\lambda_\theta}N_2''\\
&+\frac{72\omega^2}{\lambda_\theta^5k}+\frac{10m\omega^2k}{\lambda_\theta^5\lambda_z}+2\alpha_4k^3+\frac{2\alpha_3k^3}{\lambda_\theta}-\frac{6\alpha_2k^3}{\lambda_\theta^2}-\frac{16\alpha_1k^3}{\lambda_\theta^3}-\frac{10\alpha_0k^3}{\lambda_\theta^4}\\
&\left.-\frac{6\gamma_1k^5}{\lambda_\theta^2}+\alpha_2k^5-\frac{7\alpha_1k^5}{\lambda_\theta}+\frac{9\alpha_0k^5}{\lambda_\theta^2}+\gamma_1k^7-\frac{15\alpha_0k^7}{4}-10\gamma_2k^7-10\beta_5k\right\},\\
\Lambda_7 = &-\frac{1}{\dfrac{4\omega}{\lambda_\theta k}+\dfrac{2m\omega k}{\lambda_\theta\lambda_z}}\left[\left(\frac{2g}{\lambda_\theta k^2}-\frac{2\omega}{\lambda_\theta k^3}-\frac{2mg}{\lambda_\theta\lambda_z}\right)\mu_1+\left(\frac{4\omega}{\lambda_\theta k^2}-\frac{2m\omega}{\lambda_\theta\lambda_z}-\frac{4g}{\lambda_\theta k}-\frac{2mgk}{\lambda_\theta\lambda_z}\right)\mu_3\right.\\
&\left.+\left(\frac{2}{\lambda_\theta k}+\frac{2mk}{\lambda_\theta\lambda_z}\right)\mu_1{}^2+\left(\frac{\omega}{k}-g\right)N_1\right],\\
\Lambda_8 = &-\frac{1}{\dfrac{4\omega}{\lambda_\theta k}+\dfrac{2m\omega k}{\lambda_\theta\lambda_z}}\left[\left(\frac{2g}{\lambda_\theta k^2}-\frac{2\omega}{\lambda_\theta k^3}-\frac{2mg}{\lambda_\theta\lambda_z}\right)\mu_2+\left(\frac{4\omega}{\lambda_\theta k^2}-\frac{2m\omega}{\lambda_\theta\lambda_z}-\frac{4g}{\lambda_\theta k}-\frac{2mgk}{\lambda_\theta\lambda_z}\right)\mu_4\right.\\
&\left.+\left(\frac{2}{\lambda_\theta k}+\frac{2mk}{\lambda_\theta\lambda_z}\right)\mu_1\mu_2+\left(\frac{\omega}{k}-g\right)N_1'+\left(\frac{mg^2k}{\lambda_\theta^2\lambda_z}+4\alpha_1k+\frac{\alpha_0k}{\lambda_\theta}\right)\Phi_1\right].
\end{aligned}
\tag{A5}
$$

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