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Distributed optimization for discrete-time multiagent systems with nonconvex control input constraints and switching topologies*

Xiao-Yu Shen(沈小宇)[1], Shuai Su(宿帅)[2,3,†], and Hai-Liang Hou(侯海良)[1]

[1] School of Automation, Central South University, Changsha 410083, China
[2] National Engineering Research Center of Rail Transportation Operation and Control System, Beijing Jiaotong University, Beijing 100044, China
[3] State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University, Beijing 100044, China

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This paper addresses the distributed optimization problem of discrete-time multiagent systems with nonconvex control input constraints and switching topologies. We introduce a novel distributed optimization algorithm with a switching mechanism to guarantee that all agents eventually converge to an optimal solution point, while their control inputs are constrained in their own nonconvex region. It is worth noting that the mechanism is performed to tackle the coexistence of the nonconvex constraint operator and the optimization gradient term. Based on the dynamic transformation technique, the original nonlinear dynamic system is transformed into an equivalent one with a nonlinear error term. By utilizing the nonnegative matrix theory, it is shown that the optimization problem can be solved when the union of switching communication graphs is jointly strongly connected. Finally, a numerical simulation example is used to demonstrate the acquired theoretical results.

Keywords: multiagent systems, nonconvex input constraints, switching topologies, distributed optimization

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1. Introduction

The distributed optimization problem of multiagent systems has already been one of the most popular problems in the field of control.[1–26] The aim of distributed optimization is to minimize the global objective function in a distributed manner, where the global objective function is the sum of the local objective functions of all agents. Primary studies about distributed optimization mostly focused on the unconstrained case. For example, in Refs. [1–4], distributed optimization problems with the convex or strongly convex objective functions with locally or globally Lipschitz gradients were considered for optimal distribution convergence. Recent studies about distributed optimization paid more and more attention to the constrained case. For example, in Refs. [5–8], distributed discrete-time and continuous-time optimization problems with convex position-state constraints were investigated, where the gradient stepsizes all are uniform, and those in Refs. [7,8] are diminishing. Also, in Refs. [9–11], distributed optimization problems with inequality constraints, positivity constraints and coupled-constraints were studied. The constraints in the these existing works were usually assumed to be convex. In reality, the agent systems are often subject to nonconvex constraints rather than convex constraints. For example, the quadrotors can move in every direction but the region formed by the driving forces is nonconvex. To deal with the nonconvex constraints, some works have been performed in Ref. [12]. However, the results in Ref. [12] were on the consensus problem and the optimization problem was not considered. In Ref. [13], while distributed optimization with nonconvex constraints was studied, the nonconvex constraints were only on the velocities of the agents.

Inspired by Refs. [12,13], in this paper, we address the distributed optimization problem of discrete-time multiagent systems with nonconvex control input constraints, switching topologies and nonuniform stepsizes, where local objective functions are general differentiable and convex. We first introduce a novel distributed optimization algorithm with a switching mechanism to guarantee that all agents eventually converge to an optimal solution point, while their control inputs are constrained in their own nonconvex region. It is worth noting that the mechanism is performed to tackle the coexistence of the nonconvex constraint operator and the optimization gradient term. Based on the dynamic transformation technique, the original nonlinear dynamic system was transformed into an equivalent one with a nonlinear error term. By utilizing the nonnegative matrix theory, it is shown that the optimization problem can be solved when the union of switching communication graphs is jointly strongly connected.

In contrast to Ref. [13], where the nonlinearities are caused by the position constraints, the velocity constraints and the local objective functions, in this paper the nonlinearities are caused by the input constraints and the local objective functions. As is well known, there are no general rules to de-

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†Corresponding author. E-mail: shuaisu123@163.com

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termine whether a certain nonlinear system is stable, and any small difference might lead to completely different dynamics. Moreover, different from most of the existing works, e.g., Refs. [1–5,9], nonuniform stepsizes were taken into account, which together with nonconvex input constraints destroy the balance of the optimal convergence of the objective functions. Hence, all the existing approaches based on the graph balance cannot be applied in this paper.

Notation: $\mathbb{R}^r$ represents the real column vectors set with r dimensions; $\mathbb{R}^{r\times n}$ represents the real matrices set with $r\times n$ dimensions; I_r represents the unit matrix with r dimensions; x^{T} represents the transpose of the vector x; $||x||$ represents the standard Euclidean norm of the vector x; $P_H(h)$ represents the projection of the vector h on the closed convex set H, i.e., $P_H(h)=\arg\min_{\bar{h}\in H}\|h-\bar{h}\|$; $\nabla f(x)$ represents the gradient of the function $f(x)$ at x; the symbol / represents the division sign; $\lfloor A\rfloor_{ij}$ represents the ijth entry of matrix A; and $\mathbf{1}$ represents a column vector whose dimension is compatible and elements all are one.

2. Preliminaries

Let $\mathcal{H}(\mathcal{N},\mathcal{V})$ represent a directed graph, where $\mathcal{N}=\{1,\ldots,n\}$ denotes the set of nodes, and $\mathcal{V}\subseteq\mathcal{N}\times\mathcal{N}$ represents the set of edges. The edge $(j,i)\in\mathcal{V}$ means that agent j can transmit information to agent i. We use a_{ij} to denote the weight of edge (j,i), $a_{ij}\geq\bar{\mu}$ if $(j,i)\in\mathcal{V}$, and $a_{ij}=0$ otherwise, where $\bar{\mu}$ is a positive constant. Let $N_i=\{j\in\mathcal{N}:(j,i)\in\mathcal{V}\}$ be the set of neighbors of agent i and L denote the Laplacian of the directed graph $\mathcal{H}$. A sequence of ordered edges $(i_1,i_2),(i_2,i_3),\ldots$ represents a directed path. A graph $\mathcal{H}$ is strongly connected if there exists a directed path from every node to every other nodes.

A matrix $G=[g_{ij}]\in\mathbb{R}^{r\times n}$ is a stochastic matrix if all elements of G are nonnegative and each row sum of G is $\mathbf{1}$, i.e., $g_{ij}\geq 0$ and $G\mathbf{1}=\mathbf{1}$.

3. Model and problem statement

The multi-agent system under consideration consists of n first-order discrete-time agents. Let $\mathcal{H}(kT)$ represent the communication graph of the system at time kT, where T is the sampling time and each agent is regarded as a node in $\mathcal{H}(kT)$. When no confusion arises, we use (k) instead of (kT) for simplicity of expression throughout this paper.

Suppose that the dynamics of each agent is

$$x_i(k+1)=x_i(k)+\mathrm{S}_{U_i}(u_i(k))T, \tag{1}$$

where $x_i(k)\in\mathbb{R}^r$ represents the position state of agent i, $u_i(k)\in\mathbb{R}^r$ represents the control input, and $\mathrm{S}_{U_i}(\cdot)$ denotes the constraint operator of control input of agent i such that

$$\mathrm{S}_{U_i}(u)=\begin{cases}\frac{u}{\|u\|}\max_{0\leq b\leq\|u\|}\{b\mid\frac{abu}{\|u\|}\in U_i,\ \forall 0\leq a\leq 1\},\\ \text{if } u\in\mathbb{R}^r \text{ and } u\neq 0,\\ 0,\ \text{otherwise.}\end{cases}$$

It should be noted that the operator S_{U_i} can be used to describe the nonconvex constraints while the existing convex operators including the saturation operator and the projection operator can only be used to describe the hyperplane constraints or the convex constraints. In practical applications, the systems are often subject to nonconvex constraints.

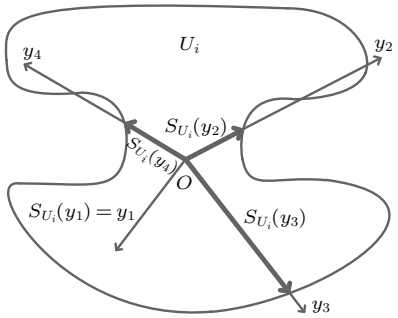


Fig. 1. Some examples of the constraint operator.

Assumption 1[12] Suppose that $U_i\subseteq\mathbb{R}^r$, $0\in U_i$ is a bounded nonempty closed set for agent $i\in\mathcal{N}$, and $\inf_{u\notin U_i}\|\mathrm{S}_{U_i}(u)\|=\underline{\gamma}_i>0$ and $\max_{u\in U_i}\|\mathrm{S}_{U_i}(u)\|=\bar{\gamma}_i>0$ for all i, where $\bar{\gamma}_i>\underline{\gamma}_i>0$.

In this paper, we aim to design a distributed control algorithm to guarantee all constrained agents eventually achieve the consensus, and find the optimal solution of the following optimization problem:

$$\text{minimize}\ \sum_{i=1}^{n}f_i(x), \tag{2}$$

where $f_i(x):\mathbb{R}^r\to\mathbb{R}$ represents the local objective function of agent i and is only known by agent i. Let $f_i(x)$ be convex and differentiable. Hence, $\sum_{i=1}^n f_i(x)$ is a convex differentiable function.

For the following analysis, we use $X_i\triangleq\{x|\nabla f_i(x)=0\}$ and $X\triangleq\{x|\sum_{i=1}^n\nabla f_i(x)=0\}$ to represent the optimal set of the local objective functions $f_i(x)$ and the optimal set of the global objective function $\sum_{i=1}^n f_i(x)$ respectively. Then, we make the following assumption.

Assumption 2[14] Suppose that each set X_i is nonempty and bounded for all $i\in\mathcal{N}$.

4. Main results

4.1. Algorithm and model transformations

(1) Algorithm

To solve the optimization problem, the proposed distributed control algorithm for each agent has the form given

by

$$\begin{aligned}
u_i(k) &= e_i(k)-v_i(k),\\
e_i(k) &= \sum_{j\in N_i(k)} a_{ij}(k)(x_j(k)-x_i(k)),\\
v_i(k) &= \begin{cases} 0, \text{ if } e_i(k)-\frac{\nabla f_i(r_i(k))}{\sqrt[m]{z_i(k)}} \neq S_{U_i}\left[e_i(k)-\frac{\nabla f_i(r_i(k))}{\sqrt[m]{z_i(k)}}\right],\\ 0, \text{ if } \sqrt[m]{z_i(k)} < \|\nabla f(r_i(k))\|^2,\\ \frac{\nabla f_i(r_i(k))}{\sqrt[m]{z_i(k)}}, \text{otherwise,}\end{cases}\\
r_i(k) &= x_i(k)+e_i(k)T,\\
z_i(k+1) &= z_i(k)+\arctan(e^{\|x_i(k)\|})T,\ z_i(0)>0,
\end{aligned} \tag{3}$$

for all $k \geq 0$, where $a_{ij}(k)$ is the edge weight such that $a_{ij}(k) \geq \bar{\mu}$ if agent i can receive information from agent j, and $a_{ij}(k) = 0$ otherwise. In this paper, the stepsizes are adopted in a general form $1/\sqrt[m]{z_i(k)}$, where $0 < m < M$ and M is a finite positive constant.

The role of $e_i(k)$ is to guarantee the consensus of all agents through local interactions. The role of the switching mechanism in the update of $v_i(k)$ is to ensure all agents to stay in a bounded region and to optimize the objective function. The variable $z_i(k)$ is updated completely based on the states of agent i, which is used to construct the stepsizes of agent i. It can be proved by using the approach of Lemma 10 in Ref. [13] that the stepsizes $\lim_{k\to+\infty} \frac{\sqrt[m]{z_j(k)}}{\sqrt[m]{z_i(k)}} = 1$, which guarantees the balance of the optimal convergence rate of all agents.

Remark 1 Note that $\frac{\pi}{2} \geq \arctan(e^{\|x_i(k)\|}) \geq \frac{\pi}{4}$ for all i, k. Hence, there must exist a positive constant T_0 such that $8kT > z_i(k) > \frac{kT}{8}$ for all $k \geq T_0$, which means that there are two bounded decreasing functions $0 < \omega_*(k) < \omega^*(k)$ with regard to k such that $\omega_*(k) < \frac{1}{\sqrt[m]{z_i(k)}} < \omega^*(k)$ for all $k \geq T_0$ and $0 < m < M$.

In the following analysis, we only discuss the case of $r = 1$ for simplicity of expression, and the general case of $r > 1$ can be analyzed in a similar way.

(2) Model transformations

Due to the coexistence of nonconvex control input constraints and nonuniform gradient stepsizes, it is very hard to analyze the system stability directly. To tackle this problem, we first make model transformations for system (1) with Eq. (3) so as to fully explore the convexity of the closed-loop system.

Let $\tilde{p}_i(k)$ be the scaling operator, which is defined as

$$\tilde{p}_i(k) = \frac{\|S_{U_i}(u_i(k))\|}{\|u_i(k)\|}, \tag{4}$$

for all k. Under Assumption 1, it is clear that $\tilde{p}_i(k) = 1$ if $u_i(k) = 0$ or $S_{U_i}(u_i(k)) = u_i(k)$, $0 < \tilde{p}_i(k) < 1$ if $u_i(k) \neq 0$ and $S_{U_i}(u_i(k)) < u_i(k)$.

Then, the system (1) with Eq. (3) can be converted into the following equivalent system:

$$\begin{aligned}
x_i(k+1) &= x_i(k)+\tilde{p}_i(k)u_i(k)T\\
&= x_i(k)+\tilde{p}_i(k)[e_i(k)-v_i(k)]T.
\end{aligned} \tag{5}$$

Define $\psi(k) = [x_1(k)^{\mathrm{T}},\ldots,x_n(k)^{\mathrm{T}}]^{\mathrm{T}}$, $P(k) = \mathrm{diag}[\tilde{p}_1(k),\ldots,\tilde{p}_n(k)]$ and $F(k) = [\tilde{p}_1(k)\cdot v_1(k)^{\mathrm{T}},\ldots,\tilde{p}_n(k)\cdot v_n(k)^{\mathrm{T}}]^{\mathrm{T}}$. Then, the system (4) can be rewritten in a compact form as

$$\psi(k+1) = \Theta(k)\psi(k) - F(k)T \tag{6}$$

for all k, where $\Theta(k) = I_n - P(k)L(k)T$. For the following analysis, we use $\Xi(k,s) = \prod_{i=s}^{k}\Theta(i) = \Theta(k)\ldots\Theta(s)$ to represent the transition matrix of the system (5).

4.2. Main theorem

Before giving the main theorem, the following necessary assumptions and lemmas are first introduced:

Assumption 3[13] Suppose that there exists a positive constant ρ such that $1 > \rho > \sum_{j\in N_i(k)} a_{ij}(k)T$ for all i and $k \geq 0$.

Assumption 4[13] Suppose that there is an infinite time sequence $k_0, k_1, k_2, \ldots$ such that the union of the graphs $\mathcal{H}(k_l), \mathcal{H}(k_l+1), \ldots, \mathcal{H}(k_{l+1}-1)$ is strongly connected for all l, where $k_0 = 0$, $0 < k_{l+1} - k_l \leq \alpha$, α is a positive integer.

Assumption 5[13] Suppose that $\mathcal{H}(k)$ are balanced for all k, i.e., $\sum_{j=1}^{n} a_{ij}(k) = \sum_{j=1}^{n} a_{ji}(k)$ for all i, j and k.

Remark 2 The role of Assumption 3 is to ensure the system matrix to be nonnegative. The role of Assumption 4 is to ensure the persistence of the information flow between agents. In Assumption 4, the union of the graphs is only required to be strongly connected every α time and the graph at each time might not be strongly connected. The role of Assumption 5 is to ensure the balance of the optimal convergence of all agents.

Lemma 1[27] Let $f_0(x) : \mathbb{R}^r \to \mathbb{R}$ be a differentiable convex function. Then, $f_0(x)$ is minimized if and only if $\nabla f_0(x) = 0$.

Lemma 2[28] Let $H \neq \emptyset$ be a convex closed set in $\mathbb{R}^r$,

(1) For any $h \in \mathbb{R}^r$, $\|h - P_H(h)\|$ is continuous with regard to h and $\nabla\frac{1}{2}\|h - P_H(h)\|^2 = h - P_H(h)$.

(2) For any $h, y \in \mathbb{R}^r$ and all $\zeta \in H$, $[h - P_H(h)]^{\mathrm{T}}(h - \zeta) \geq 0$, $\|P_H(h) - \zeta\|^2 \leq \|h - \zeta\|^2 - \|P_H(h) - h\|^2$ and $\|P_H(h) - P_H(y)\| \leq \|h - y\|$.

Lemma 3[14] Suppose that $f(x) : \Omega \mapsto \mathbb{R}$ is a convex differentiable function and H is the minimum set of $f(x)$ in Ω, where $\Omega \subseteq \mathbb{R}^r$ is a convex closed set and $H \subseteq \Omega$ is closed and bounded. There exists a constant $\eta \in (0,1)$ for any $z = \eta P_H(h) + (1-\eta)h$ such that

$$0 < \nabla f(z)^{\mathrm{T}}\frac{h - P_H(h)}{\|h - P_H(h)\|} \leq \nabla f(h)^{\mathrm{T}}\frac{h - P_H(h)}{\|h - P_H(h)\|}$$

for any $h \in \Omega - H$.

Lemma 4[14] Under Assumptions 1 and 2:

(1) X and all X_i for $i \in \mathcal{N}$ are all closed nonempty convex bounded sets.

(2) $\lim_{\|x\|\to+\infty} f_i(x) = +\infty$ and $\lim_{\|x\|\to+\infty} \sum_{i=1}^{n} f_i(x) = +\infty$ for all i.

Lemma 5[13] From the proposed algorithm $z_i(k+1) = z_i(k) + \arctan(e^{\|x_i(k)\|})T$ for all i. If $\lim_{k\to+\infty}[x_i(k) - x_j(k)] = 0$, we have

$$\lim_{k\to+\infty} \frac{z_i(k)}{z_j(k)} = 1,$$

for all i and j.

Lemma 6 Let Assumptions 1 and 2 hold, the position state of each agent is constrained in a bounded region.

Proof Consider the following Lyapunov candidate function

$$Y_1(k) = \max_i\{\|x_i(k) - \sigma\|\},$$

where $\sigma \in X$. Consider the case that $v_i(k) \neq 0$ for some i. According to the definitions of $v_i(k)$ and $\tilde{p}_i(k)$, we have $\tilde{p}_i(k) = 1$. Then, it follows from Eq. (5) that

$$x_i(k+1) = r_i(k) - \frac{\nabla f_i(r_i(k))}{\sqrt[m]{z_i(k)}}T. \tag{7}$$

Since $\tilde{p}_i(k) = 1$ and $\|\nabla f_i(r_i(k))\|^2 \leq \sqrt[m]{z_i(k)}$, it can be deduced from Eq. (7) that

$$\begin{aligned}
&\|x_i(k+1) - \sigma\|^2 \\
=\ &\|r_i(k) - \sigma\|^2 + \left\|\frac{\nabla f_i(r_i(k))T}{\sqrt[m]{z_i(k)}}\right\|^2 \\
&-2(r_i(k) - \sigma)^{\mathrm{T}}\frac{\nabla f_i(r_i(k))T}{\sqrt[m]{z_i(k)}} \\
\leq\ &\|r_i(k) - \sigma\|^2 + \frac{T^2}{\sqrt[m]{z_i(k)}} - 2(r_i(k) - \sigma)^{\mathrm{T}}\frac{\nabla f_i(r_i(k))T}{\sqrt[m]{z_i(k)}}.
\end{aligned}$$

According to the convexity of function $f_i(\cdot)$, i.e., $-(r_i(k) - \sigma)^{\mathrm{T}}\nabla f_i(r_i(k)) \leq f_i(\sigma) - f_i(r_i(k))$, it follows that

$$\begin{aligned}
&\|x_i(k+1) - \sigma\|^2 \\
\leq\ &\|r_i(k) - \sigma\|^2 + \frac{T^2}{\sqrt[m]{z_i(k)}} \\
&+2\frac{f_i(\sigma) - f_i(\sigma_i) + f_i(\sigma_i) - f_i(r_i(k))}{\sqrt[m]{z_i(k)}}T,
\end{aligned}$$

where $\sigma_i \in X_i$.

Note that $\omega_*(k) < \dfrac{1}{\sqrt[m]{z_i(k)}} < \omega^*(k)$ and $f_i(\sigma) - f_i(\sigma_i) \geq 0$, $f_i(\sigma_i) - f_i(r_i(k)) \leq 0$. It follows that

$$\begin{aligned}
&\|x_i(k+1) - \sigma\|^2 \\
\leq\ &\|r_i(k) - \sigma\|^2 + \omega_*(k)T^2 + \omega_*(k)(f_i(\sigma) - f_i(\sigma_i))T \\
&+\omega^*(k)(f_i(\sigma_i) - f_i(r_i(k)))T
\end{aligned} \tag{8}$$

for $k \geq T_0$. Since $\lim_{\|x\|\to+\infty} f_i(x) = +\infty$ for all i from Lemma 4, there must exist a convex closed bounded region $\Lambda_0 = \{y \in \mathbb{R}^m \mid \|y - \sigma\| \leq d_1\}$ for a positive constant d_1 such that $\omega^*(k)(f_i(r_i(k)) - f_i(\sigma_i)) > \omega_*(k)(\max_j[f_i(\sigma) - f_i(\sigma_i)] + T)$ for any $r_i(k) \notin \Lambda_0$. Then, if $r_i(k) \in \Lambda_0$ for all i, from Eq. (9), we have

$$\begin{aligned}
&\|x_i(k+1) - \sigma\|^2 \\
\leq\ &d_1^2 + \omega_*(k)T^2 + \omega^*(k)[f_i(\sigma) - f_i(\sigma_i)] \\
\leq\ &d_1^2 + \max_i[\omega_*(k)(T^2 + f_i(\sigma) - f_i(\sigma_i))].
\end{aligned}$$

If $r_i(k) \notin \Lambda_0$, we have

$$\omega^*(k)(f_i(r_i(k)) - f_i(\sigma_i)) > \omega_*(k)(\max_i[f_i(\sigma) - f_i(\sigma_i)] + T).$$

Therefore, together with Eq. (8), it can be obtained that $\|x_i(k+1) - \sigma\| \leq \|r_i(k) - \sigma\| \leq Y_1(k)$.

Consider the case of $v_i(k) = 0$ for some i. It follows from Eq. (4) that $0 < \tilde{p}_i(k) < 1$, then we have $x_i(k+1) = x_i(k) + \tilde{p}_i(k)e_i(k)T$. Thus,

$$\begin{aligned}
&Y_1(k+1) \\
=\ &\max_i\|x_i(k) + \tilde{p}_i(k)e_i(k)T - s\| \\
\leq\ &(1 - \tilde{p}_i(k)\textstyle\sum_{j\in N_i(k)} a_{ij}(k)T)\ \max_i\|x_i(k) - \sigma\| \\
&+\tilde{p}_i(k)\textstyle\sum_{j\in N_i(k)} a_{ij}(k)T\ \max_j\|x_j(k) - \sigma\| \\
\leq\ &\max_i\|x_i(k) - \sigma\|.
\end{aligned}$$

Summarizing the above analysis, it is clear that the position state of each agent is constrained in a bounded region.

Lemma 7 Let Assumptions 1, 2 and 3 hold:

(1) The lower boundedness of the scaling operator $\tilde{p}_i(k)$ is a positive number for all i and k.

(2) The system matrix $\Theta(k)$ is stochastic for all k.

Proof Now, based on the Lemma 6, we study the lower boundedness of $\tilde{p}_i(k)$ and the stochasticity of system matrices $\Theta(k)$ and $\Xi(k)$. Since $\|x_i(k)\|$ is bounded, it follows that $\|e_i(k)\|$ and $\|r_i(k)\|$ are also bounded. In addition, $f_i(\cdot)$ is differentiable, and thus $\|\nabla f_i(r_i(k))\|$ is bounded. Recall that $\omega_*(k) < \frac{1}{\sqrt[m]{z_i(k)}} < \omega^*(k)$ for all $k \geq T_0$ and $0 < m < M$. Therefore, there must exist an integer $T_1 > T_0$ for any positive constant $\varepsilon_0 > 0$ such that $\|e_i(k)\| < \varepsilon_0$ and $\|v_i(k)\| < \varepsilon_0$ for all i and $k > T_1$. From Assumption 1, we have $\|S_{U_i}(u_i(k))\| \geq \underline{\gamma}_i$. It can be obtained from the definition of $\tilde{p}_i(k)$ that $\tilde{p}_i(k)$ is larger than a positive constant for all i, k.

Note that $a_{ij}(k) > \underline{\mu}$ and $0 < \tilde{p}_i(k) \leq 1$. Under Assumption 3, we have $1 - \tilde{p}_i(k)\sum_{j\in N_i(k)} a_{ij}(k)T > 0$. Clearly, all nonzero entries of $\Theta(k)$ are larger than a positive constant. Moreover, $\Theta(k)\mathbf{1} = \mathbf{1}$. It follows that $\Theta(k)$ is a stochastic matrix and hence $\Xi(k)$ is also a stochastic matrix, for all $k \geq s \geq 0$.

Lemma 8 Let Assumptions 1, 2, 3 and 4 hold, all agents eventually converge to a common point.

Proof From Eq. (5) and the definition of $\Xi(k)$, we have

$$\psi(k_{l+1}) = \Xi(k_{l+1} - 1, k_l)\psi(k_l)$$

$$-\sum_{s=k_l+1}^{k_{l+1}-1}\Xi(k_{l+1}-1,s)F(s-1)T - F(k_{l+1}-1)T. \tag{9}$$

From the proof of Lemma 7, we have that each $\|r_i(k)\|$ and $\|\nabla f_i(r_i(k))\|$ are bounded for some positive constant. From the definition of $z_i(k)$ and Lemma 5, we have $\lim_{k\to+\infty}\frac{1}{\sqrt[m]{z_i(k)}}=0$. It follows that $\lim_{k\to+\infty}\|F(k)\|=0$. Therefore, there exists a constant $T_0>0$ for any positive constant $\varepsilon_1>0$ such that $\|F(k)\|<\varepsilon_1$ for all $k>T_0$. Then, it follows from Assumption 4 that

$$\begin{aligned}&\left\|\sum_{s=k_l+1}^{k_{l+1}-1}\Xi(k_{l+1}-1,s)F(s-1)T - F(k_{l+1}-1)T\right\|\\ \le\ &(\alpha+1)\varepsilon_1\end{aligned} \tag{10}$$

for all $k_l>T_1$. From Lemma 2 in Ref. [12], it can be deduced that there exists a positive number $\beta<1$ and an integer $j_0>0$ such that $\lfloor\Xi(k_{l+6n}-1,k_l)\rfloor_{ij_0}\ge\beta$ for all i and all $k_l\ge 0$. Let $k_l>T_0$. Then, from Eq. (9), we have

$$\begin{aligned}&\|\psi_i(k_{l+6n})-\sum_{j\neq j_0}\lfloor\Xi(k_{l+6n}-1,k_l)\rfloor_{ij}\psi_j(k_l)\\ &-\lfloor\Xi(k_{l+6n}-1,k_l)\rfloor_{ij_0}\psi_{j_0}(k_l)\|\\ \le\ &6n(\alpha+1)\varepsilon_1.\end{aligned} \tag{11}$$

Since $\Xi(k,s)$ is a stochastic matrix from Lemma 7, it follows that

$$\begin{aligned}&\max_i\psi_i(k_{l+6n})\\ \le\ &(1-\beta)\max_i\psi_i(k_l)+\beta\psi_{j_0}(k_l)+6n(\alpha+1)\varepsilon_1,\\ &\min_i\psi_i(k_{l+6n})\\ \ge\ &(1-\beta)\min_i\psi_i(k_l)+\beta\psi_{j_0}(k_l)-6n(\alpha+1)\varepsilon_1.\end{aligned}$$

Thus,

$$\begin{aligned}&\max_i\psi_i(k_{l+6n})-\min_i\psi_i(k_{l+6n})\\ \le\ &(1-\beta)[\max_i\psi_i(k_l)-\min_i\psi_i(k_l)]+12n(\alpha+1)\varepsilon_1.\end{aligned}$$

Let $M_i(k)=\max_i\psi_i(k)-\min_i\psi_i(k)$. Then, when $M_i(k_l)\le 12n(\alpha+1)\varepsilon_1/\beta$, we have

$$M_i(k_{l+6n})\le 12n(\alpha+1)\varepsilon_1/\beta.$$

When $M_i(k_l)>12n(\alpha+1)\varepsilon_1/\beta$, we have

$$M_i(k_{l+6n})-M_i(k_l)<0.$$

It is easy to see that for all i, $M_i(k_{l+6ni})\le 12n(\alpha+1)\varepsilon_1/\beta$ if $M_i(k_l)\le 12n(\alpha+1)\varepsilon_1/\beta$ holds. In addition, if $M_i(k_l)>18n(\alpha+1)\varepsilon_1/\beta$, then $M_i(k_{l+6n})-M_i(k_l)<-12n(\alpha+1)\varepsilon_1$. Therefore, it follows that there must exist a constant $T_1>T_0$ such that $M_i(k_{l+6nj})\le 18n(\alpha+1)\varepsilon_1/\beta$ for all $k_{l+6nj}>T_1$. In view of the arbitrariness of the constants k_l and ε_1, letting $k_l\to+\infty$ and $\varepsilon_1\to 0$, it follows that $\lim_{k_l\to+\infty}[\psi_i(k_l)-\psi_j(k_l)]=0$ for all i,j. Recall that $\|\psi(k+1)-\psi(k_l)\|\le\|\Xi(k,k_l)\psi(k_l)-\psi(k_l)\|+6n(\alpha+1)\varepsilon_1$. Since $\Xi(k,k_l)$ is stochastic for all $k\ge k_l$, we have

$$\lim_{k\to+\infty}[\psi_i(k)-\psi_j(k)]=0$$

for all i and j, which means that all agents eventually converge to a common point.

Theorem 1 Let Assumptions 1, 2, 3, 4 and 5 hold. For the system (1) with Eq. (3), all agents will reach a consensus point in the optimal set of the global objective function (2).

Proof Based on Lemmas 6–8, we will prove that all agents eventually minimize the optimal objective function (2).

Let $\bar{\psi}(k)=\frac{1}{n}\sum_{i=1}^n\psi_i(k)$. Based on the proof of Lemma 8, we have $\lim_{k\to+\infty}[\psi_i(k)-\psi_j(k)]=0$. Recall that $\lim_{k\to+\infty}F(k)=0$ and $0<\tilde{p}_i(k)\le 1$, it follows that $\lim_{k\to+\infty}u_i(k)=0$ for all i. Thus, there must exist a positive constant T_0 such that $\|u_i(k)\|<\underline{\gamma}_i$ for all i when all $k>T_0$, which implies that $\tilde{p}_i(k)=1$ for all $k>T_0$. Under Assumption 5, we have $\mathbf{1}^{\mathrm{T}}\Theta(k)=\mathbf{1}^{\mathrm{T}}$ for all k. Then, we have

$$\bar{\psi}(k+1)=\bar{\psi}(k)-\frac{1}{n}\mathbf{1}^{\mathrm{T}}F(k)T. \tag{12}$$

Consider the following Lyapunov function

$$Y_2(k)=\|\bar{\psi}(k)-P_X(\bar{\psi}(k))\|^2$$

for all $k>T_0$. According to Lemma 2, it can be obtained that $[\bar{\psi}(k+1)-P_X(\bar{\psi}(k))]^{\mathrm{T}}[P_X(\bar{\psi}(k))-P_X(\bar{\psi}(k+1))]\le 0$. From the proof of Lemma 6 and Lemma 7, $x_i(k)$ and $\|\nabla f_i(r_i(k))\|$ are bounded for all i,k. According to the definition of $z_i(k)$, it follows that $\omega_*(k)<\frac{1}{\sqrt[m]{z_i(k)}}<\omega^*(k)$ for all $k\ge T_1$ and $0<m<M$. Calculating $Y_2(k+1)$, we have

$$\begin{aligned}&Y_2(k+1)\\ =\ &\|\bar{\psi}(k+1)-P_X(\bar{\psi}(k))+P_X(\bar{\psi}(k))-P_X(\bar{\psi}(k+1))\|^2\\ =\ &\|\bar{\psi}(k+1)-P_X(\bar{\psi}(k))\|^2+\|P_X(\bar{\psi}(k))P_X(\bar{\psi}(k+1))\|^2\\ &+2[\bar{\psi}(k+1)-P_X(\bar{\psi}(k))]^{\mathrm{T}}\times[P_X(\bar{\psi}(k))-P_X(\bar{\psi}(k+1))]\\ \le\ &\|\bar{\psi}(k+1)-P_X(\bar{\psi}(k))\|^2+\|\bar{\psi}(k)-\bar{\psi}(k+1)\|^2\\ \le\ &\|\bar{\psi}(k)-\frac{1}{n}\mathbf{1}^{\mathrm{T}}F(k)T-P_X(\bar{\psi}(k))\|^2+\|\frac{1}{n}\mathbf{1}^{\mathrm{T}}F(k)T\|^2\\ \le\ &\|\bar{\psi}(k)-P_X(\bar{\psi}(k))\|^2-\frac{2}{n}(\bar{\psi}(k)-P_X(\bar{\psi}(k)))^{\mathrm{T}}\\ &\times\frac{\nabla f_i(r_i(k))}{\sqrt[m]{z_i(k)}}T+\lambda_1\omega^{*2}(k)\end{aligned}$$

for all $k>T_1$, where λ_1 is a positive constant.

Since $\lim_{k\to+\infty}[\psi_i(k)-\psi_j(k)]=0$, it can be obtained that $\lim_{k\to+\infty}[r_i(k)-\bar{\psi}(k)]=0$ and for all i and j. Then there exists a constant $T_2>T_1$ for arbitrary positive constant ε_2 such that $|\psi_i(k)-\bar{\psi}(k)|<\varepsilon_2, |r_i(k)-\bar{\psi}(k)|<\varepsilon_2$ for all $k>T_2$. Note that all functions $f_i(\cdot)$ are differentiable, we have $|f_i(r_i(k))-f_i(\bar{\psi}(k))|<\varepsilon_2$ for all $k>T_2$. Recall that $\omega_*(k)<\frac{1}{\sqrt[m]{z_i(k)}}<\omega^*(k)$ for all $k\ge T_1$ and $0<m<M$. According to the convexity of $f_i(\cdot)$, i.e., $(r_i(k)-P_X(\bar{\psi}(k)))^{\mathrm{T}}\nabla f_i(r_i(k))\le f_i(r_i(k))-f_i(P_X(\bar{\psi}(k)))$, we have

$$
\begin{aligned}
&Y_2(k+1)-Y_2(k)\\
\le\ & -\frac{2}{n}\sum_{i=1}^{n}(\bar{\psi}(k)-r_i(k)+r_i(k)-P_X(\bar{\psi}(k)))^{\mathrm{T}}\\
&\times\frac{\nabla f_i(r_i(k))}{\sqrt[m]{z_i(k)}}T+\lambda_1\omega^{*2}(k)\\
\le\ & -\frac{2}{n}\sum_{i=1}^{n}(r_i(k)-P_X(\bar{\psi}(k)))^{\mathrm{T}}\frac{\nabla f_i(r_i(k))}{\sqrt[m]{z_i(k)}}T\\
&+\lambda_1\omega^{*2}(k)+2\lambda_2\varepsilon_2\omega^{*}(k)\\
\le\ & -\frac{2}{n}\sum_{i=1}^{n}\frac{f_i(r_i(k))-f_i(\bar{\psi}(k))+f_i(\bar{\psi}(k))-f_i(P_X(\bar{\psi}(k)))}{\sqrt[m]{z_i(k)}}T\\
&+\lambda_1\omega^{*2}(k)+2\lambda_2\varepsilon_2\omega^{*}(k)\\
\le\ & -\frac{2}{n}\sum_{i=1}^{n}\frac{f_i(\bar{\psi}(k))-f_i(P_X(\bar{\psi}(k)))}{\sqrt[m]{z_i(k)}}T\\
&+\lambda_1\omega^{*2}(k)+4\lambda_2\varepsilon_2\omega^{*}(k) \qquad (13)
\end{aligned}
$$

for all $k>T_2$, where λ_2 is a positive constant.

From Lemma 5, $\lim_{k\to+\infty}\sqrt[m]{\frac{z_i(k)}{z_j(k)}}=1$ for all i,j and $m>0$. Then, there must exist a positive constant $T_3>T_2$ such that $\left|\sqrt[m]{\frac{z_i(k)}{z_j(k)}}-1\right|<\varepsilon_2$ and hence

$$
\left|[f_i(\bar{\psi}(k))-f_i(P_X(\bar{\psi}(k)))]\left(\sqrt[m]{\frac{z_1(k)}{z_i(k)}}-1\right)\right|<\varepsilon_2
$$

for all $k>T_3$. Let T_3 be large enough such that $\omega^*(k)<\varepsilon_2$ for any $k>T_3$. Thus,

$$
\begin{aligned}
&Y_2(k+1)-Y_2(k)\\
\le\ & -\frac{2T}{n}\sum_{i=1}^{n}\Big(\sqrt[m]{\frac{z_1(k)}{z_i(k)}}-1\Big)\frac{f_i(\bar{\psi}(k))-f_i(P_X(\bar{\psi}(k)))}{\sqrt[m]{z_1(k)}}\\
&-\frac{2T}{n}\sum_{i=1}^{n}\frac{f_i(\bar{\psi}(k))-f_i(P_X(\bar{\psi}(k)))}{\sqrt[m]{z_1(k)}}\\
&+\lambda_1\varepsilon_2\omega^{*}(k)+4\lambda_2\varepsilon_2\omega^{*}(k)\\
\le\ & -\frac{2T}{n}\sum_{i=1}^{n}\frac{f_i(\bar{\psi}(k))-f_i(P_X(\bar{\psi}(k)))-n\varepsilon_2}{\sqrt[m]{z_1(k)}}\\
&+\lambda_1\varepsilon_2\omega^{*}(k)+4\lambda_2\varepsilon_2\omega^{*}(k)\\
\le\ & -\frac{2\omega^*(k)T}{n}[\sum_{i=1}^{n}(f_i(\bar{\psi}(k))\\
&-f_i(P_X(\bar{\psi}(k))))-\lambda_3\varepsilon_2]
\end{aligned}
$$

for all $k>T_3$, where λ_3 is a positive constant.

Let $y=\|x-P_X(x)\|$. Suppose that there exist two positive constants $d_2>d_1+4\varepsilon_2$ such that $\min_{y=d_1}\sum_{i=1}^{n}(f_i(x)-f_i(\sigma))=\lambda_3\varepsilon_2+4\varepsilon_2$ for any $\sigma\in X$. According to Lemma 3, we have $\sum_{i=1}^{n}(f_i(x)-f_i(\sigma))>\lambda_3\varepsilon_2+4\varepsilon_2$ for any x that $\|x-P_X(x)\|>d_1$. Note that $\lim_{k\to+\infty}F(k)=0$. Let T_3 be further large enough such that $\|\bar{\psi}(k+1)-\bar{\psi}(k)\|\le\|\frac{1}{n}F(k)T\|<\varepsilon_2$ for all $k>T_3$. It is easy to see that $\|\bar{\psi}(k+1)-P_X(\bar{\psi}(k+1))\|\le d_2$ when $\|\bar{\psi}(k)-P_X(\bar{\psi}(k))\|\le d_1$ for $k>T_3$. When $d_1<\|\bar{\psi}(k)-P_X(\bar{\psi}(k))\|\le d_2$, we have $\sum_{i=1}^{n}(f_i(\bar{\psi}(k))-f_i(\sigma))>\lambda_3\varepsilon_2+4\varepsilon_2$. It follows that $Y_2(k+1)-Y_2(k)<0$ and $\|\bar{\psi}(k+1)-P_X(\bar{\psi}(k+1))\|\le\|\bar{\psi}(k)-P_X(\bar{\psi}(k))\|$, which implies that $\|\bar{\psi}(k+1)-P_X(\bar{\psi}(k+1))\|\le d_2$. Therefore, $\|\bar{\psi}(k+1)-P_X(\bar{\psi}(k+1))\|\le d_2$ holds when $\|\bar{\psi}(k)-P_X(\bar{\psi}(k))\|\le d_2$ for all $k>T_3$. Moreover, if $\|\bar{\psi}(k)-P_X(\bar{\psi}(k))\|>d_2$ for $k>T_3$, we have $\sum_{i=1}^{n}(f_i(\bar{\psi}(k))-f_i(P_X(\bar{\psi}(k))))>\lambda_3\varepsilon_2+4\varepsilon_2$, and hence $Y_2(k+1)-Y_2(k)<-\frac{4\varepsilon_2 T}{n\sqrt{kT}}$. By simple analysis, there must be an another constant $T_4>T_3$ such that $\|\bar{\psi}(k)-P_X(\bar{\psi}(k))\|\le d_1$, and we have $\sum_{i=1}^{n}[f_i(\bar{\psi}(k))-f_i(P_X(\bar{\psi}(k)))]\le\lambda_3\varepsilon_2+4\varepsilon_2$ for all $k>T_4$. In view of the arbitrariness of the constants k and ε_2, letting $k\to+\infty$ and $\varepsilon_2\to 0$, from Lemma 3, it can be deduced that

$$
\lim_{k\to+\infty}\sum_{i=1}^{n}[f_i(\bar{\psi}(k))-f_i(\sigma)]=0,
$$

where $\sigma\in X$. This implies that all agents will eventually reach a common point in the optimal set of the objective function (2).

Remark 3 The advantages of our results are mainly in three aspects. First, in our algorithm, nonuniform stepsizes are considered. In most of the existing works, the stepsizes were usually assumed to be uniform at any time. In Ref. [17], nonuniform stepsizes are considered, whereas the stepsizes are given in the form of square root which is a special case of ours. Second, the communication graphs considered are jointly strongly connected. Only a few of existing works have considered such switching graphs. Third, nonconvex input constraints are considered in this paper. In Ref. [13], nonconvex constraints have been considered, but the nonconvex constraints are on the velocity state rather than the input constraints.

5. Simulation example

Consider a multi-agent system consists of 12 first-order discrete-time agents. The communication topology of the system switches among a group of the directed graphs in Fig. 2, as long as the union of the switching communication graphs is balanced and strongly connected. The sample time T is 0.1 s and the weight of all edges is 0.5. The local objective functions for the agents are $f_1^k(x_1)=x_{11}^2+x_{12}^2$, $f_2^k(x_2)=(x_{21}+0.9+0.1k)^2+x_{22}^2$, $f_3^k(x_3)=x_{31}^2+(x_{32}+0.9+0.1k)^2$, $f_4^k(x_4)=(x_{41}+0.9+0.1k)^2+(x_{42}+0.9+0.1k)^2$ for $k=1,2,3$, where x_{i1} and x_{i2} represent two components of x_i, respectively. Suppose that the nonconvex input constraint set of all agents is $U_i=\{u|\|u-[0,0]^{\mathrm{T}}\|\le 1\}\cup\{u|\|u-[1,1]^{\mathrm{T}}\|\le 1\}\cup\{u|\|u-[1,-1]^{\mathrm{T}}\|\le 1\}\cup\{u|\|u-[-1,1]^{\mathrm{T}}\|\le 1\}\cup\{u|\|u-[-1,-1]^{\mathrm{T}}\|\le 1\}$ for all i. The only optimal solution that minimizes the global objective function (2) is $x=[-0.6,-0.6]^{\mathrm{T}}$.

Figures 3 and 4 show the simulation results for the system (1) using algorithms (3). It is clear that all agents cooperatively reach a consensus point in the optimal set and all control inputs are in the bounded nonconvex set, which is consistent with

Theorem 1. Another objective of Figs. 5 and 6 is to compare the convergence rates of the algorithm when the parameter m is adopted as different values. From the simulations, it can be observed that the convergence rate of the algorithm when $m = 5$ is faster than the case of $m = 2$. Also, we consider the impact of the initial values on the convergence rate in Figs. 6 and 7. From the simulations, it can be observed that the convergence rate is sensitive to the initial values of the stepsizes.

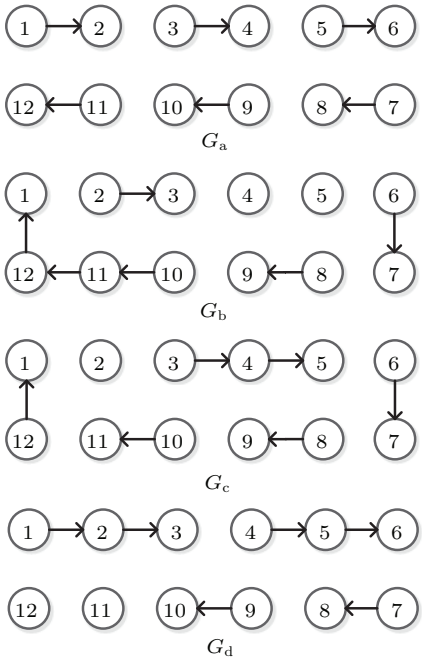


Fig. 2. A group of directed graphs.

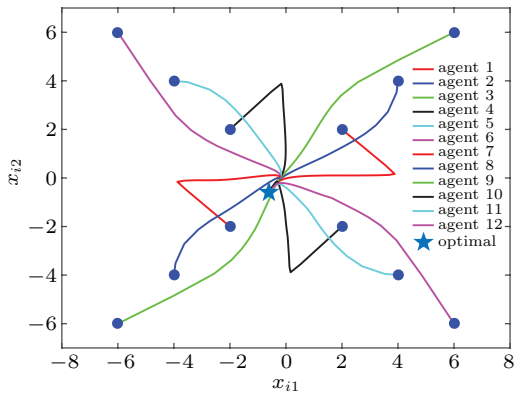


Fig. 3. The position trajectories of all agents.

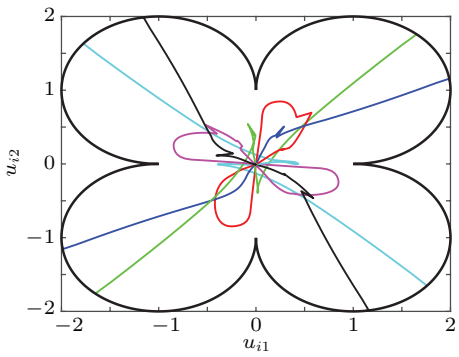


Fig. 4. The control inputs with nonconvex constraints of all agents.

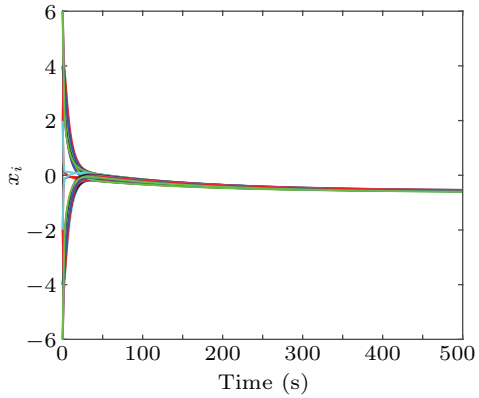


Fig. 5. The optimal convergence of agents when $m = 2$ and the initial value of the stepsizes is 0.1.

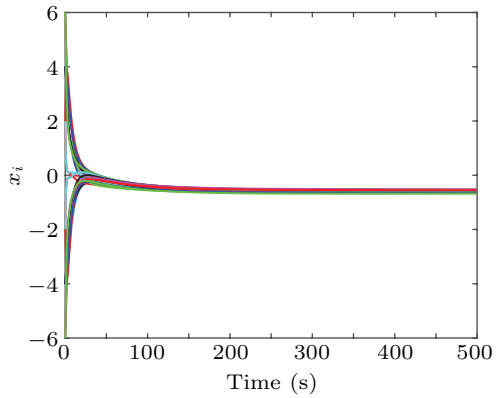


Fig. 6. The optimal convergence of agents when $m = 5$ and the initial value of the stepsizes is 0.1.

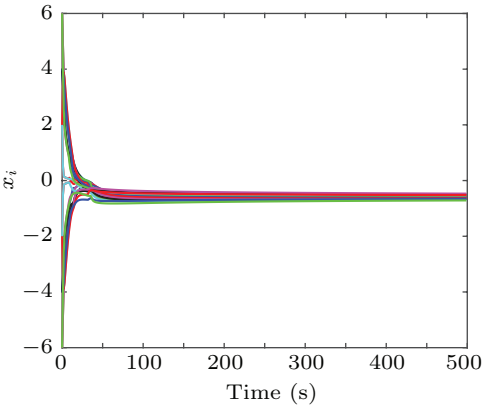


Fig. 7. The optimal convergence of agents when $m = 5$ and the initial value of the stepsizes is 10.

6. Conclusion

We have solved a distributed optimization problem of the first-order discrete-time multi-agent systems with nonconvex control input constraints, nonuniform stepsizes and switching topologies. The proposed distributed algorithms ensure that all agents cooperatively reach a consensus point in the optimal set and all the agents' inputs are constrained in their own nonconvex region. A switching mechanism was introduced to tackle the coexistence of the nonconvex constraint operator and the nonuniform gradient term. It is proved that the constrained distributed optimization problem can be solved when

the union of the switching communication graphs is balanced and jointly strongly connected.

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