

Classical-field description of Bose-Einstein condensation of parallel light in a nonlinear optical cavity

Hui-Fang Wang(王慧芳), Jin-Jun Zhang(张进军), and Jian-Jun Zhang(张建军)
Citation: Chin. Phys. B, 2021, 30 (**11**): 110301. DOI: 10.1088/1674-1056/abf4f9
Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Ultradilute self-bound quantum droplets in Bose-Bose mixtures at finite temperature

Jia Wang(王佳), Xia-Ji Liu(刘夏姬), and Hui Hu(胡辉)
Chin. Phys. B, 2021, 30 (**1**): 010306. DOI: 10.1088/1674-1056/abd2ad

Atom-pair tunneling and quantum phase transition in asymmetry double-well trap in strong-interaction regime

Ji-Li Liu(刘吉利), Jiu-Qing Liang(梁九卿)
Chin. Phys. B, 2019, 28 (**11**): 110304. DOI: 10.1088/1674-1056/ab44b6

Effects of a finite number of particles on the thermodynamic properties of a harmonically trapped ideal charged Bose gas in a constant magnetic field

Duan-Liang Xiao(肖端亮), Meng-Yun Lai(赖梦云), Xiao-Yin Pan(潘孝胤)
Chin. Phys. B, 2016, 25 (**1**): 010307. DOI: 10.1088/1674-1056/25/1/010307

Spin-orbit coupled Bose-Einstein condensates with Rydberg-dressing interaction

Lü Hao, Zhu Shao-Bing, Qian Jun, Wang Yu-Zhu
Chin. Phys. B, 2015, 24 (**9**): 090308. DOI: 10.1088/1674-1056/24/9/090308

The effect of s-wave scattering length on self-trapping and tunneling phenomena of Fermi gases in one-dimensional accelerating optical lattices

Jia Wei, Dou Fu-Quan, Sun Jian-An, Duan Wen-Shan
Chin. Phys. B, 2015, 24 (**4**): 040307. DOI: 10.1088/1674-1056/24/4/040307

Classical-field description of Bose–Einstein condensation of parallel light in a nonlinear optical cavity*

Hui-Fang Wang(王慧芳)[1], Jin-Jun Zhang(张进军)[2,†], and Jian-Jun Zhang(张建军)[2,‡]

[1] School of Chemistry and Material Science, Shanxi Normal University, Linfen 041004, China
[2] School of Physics and Information Science, Shanxi Normal University, Linfen 041004, China

(Received 19 January 2021; revised manuscript received 8 March 2021; accepted manuscript online 6 April 2021)

We study the Bose–Einstein condensation of parallel light in a two-dimensional nonlinear optical cavity, where the massive photons are converted into photon molecules (p-molecules). We extend the classical-field method to provide a description of the two-component system, and we also derive a coupled density equation which can be used to describe the conversion relation between photons and p-molecules. Furthermore, we obtain the chemical potential of the system, and we also find that the system can transform from the mixed photon and p-molecule condensate phase into a pure p-molecule condensate phase. Additionally, we investigate the collective excitation of the system. We also discuss the problem how the spontaneous decay of an atom is influenced by both the phase transition and collective excitation of the coupling system.

Keywords: parallel light, Bose–Einstein condensation, quantum phase transition, collective excitation

PACS: 03.50.–z, 03.75.Kk, 05.30.Jp, 42.65.–k **DOI:** 10.1088/1674-1056/abf4f9

1. Introduction

Bose–Einstein condensation (BEC) can be defined as macroscopic occupation of the ground state at thermal equilibrium. It is a common quantum property also a natural consequence of the statistics of many-particle boson system. Since its theoretical prediction,[1] the phenomenon of BEC has been observed in a wide variety system,[2–8] from ultracold atomic gases, to quasiparticles such as polaritions, excitons, and magnons. Photons have played a vital role for the experimental realization of BEC of these particles. We all know that photon is a boson. Considering the bosonic nature of photons, it is natural to ask if photons can also undergo a BEC.

In recent years increasing efforts have been made to investigating the equilibrium processes that lead to macroscopically occupied photon state. The difficulty is for usual photon-gas-like blackbody radiation, photons can be brought to thermal equilibrium, but the temperature and the photon number cannot be turned independently, which implies that for these systems the photon number is not conserved and the chemical potential is not well defined. However, the situation is different for the two-dimensional (2D) photon system. Recently, a photon number conserving thermalization process has been observed in a 2D optical microcavity.[9,10] In the experiment, the microcavity provides an effective mass and a nonvanishing chemical potential for photons which establish the condition for parallel light to achieve the BEC.

Recent process in research into BEC of photons has been made theoretically[11–18] and experimentally.[19,20] The statistical properties and the nonequilibrium model of photons in the BEC phase have been investigated.[21–24] Also, the superfluid state was observed in experiment based on the BEC of photons.[25,26] Furthermore, photons interact through an intermediate medium, e.g., photonic lattice and plasma have also been investigated.[27–30]

It is well known that the Kerr-like nonlinear medium is a very important "laboratory" to study the quantum properties of photon gas. In this article, we aim at describing the Bose–Einstein condensation of parallel light in the 2D nonlinear optical cavity using the classical-field method. First, the coupling model is presented to illustrate the propagation of parallel light in the nonlinear medium. Using the classical-field method, we furthermore derive a coupled density equation based on which the chemical potential and the quantum phase transition of the system are investigated. We also investigate the element excitation of the coupling system. Furthermore, we also discuss the question how the spontaneous decay of an atom is influenced by both the phase transition and collective excitation of the system.

2. Model and Hamiltonian

In this letter, the parallel light is restricted into a two-dimensional (2D) optical cavity as shown in Fig. 1, in which the field mode $\boldsymbol{E}$ has the quantum form (with $\hbar = 1$ throughout this article) as follows:

$$\boldsymbol{E} = (\varepsilon_0)^{-1/2} \left[\hat{T}^{|1/2|} \hat{\varphi}_a(\boldsymbol{r}, t) + \text{H.c.} \right]. \tag{1}$$

*Project supported by the Graduate Science and Technology Innovation Project of Shanxi Normal University (Grant No. 01053011) and the Chinese Academy of Sciences (CAS) Large-Scale Scientific Facility Program (Grant No. 1G2017IHEPKFYJO1).

†Corresponding author. E-mail: jjzhang2021@163.com
‡Corresponding author. E-mail: yfzhangphys@163.com

© 2021 Chinese Physical Society and IOP Publishing Ltd

Here ε_0 is the free space permittivity. The field operator $\hat{\varphi}_a(\boldsymbol{r},t)$ in Eq. (1) can be expended as $\hat{\varphi}_a(\boldsymbol{r},t) = 1/\sqrt{S}\sum_{\boldsymbol{k}} a_{\boldsymbol{k}}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}$, with $a_{\boldsymbol{k}}$ being the annihilation operators of photons with wave vector $\boldsymbol{k}$. In the expression, $\hat{T} = \mathrm{i}\partial/\partial t$ is the free energy operator of photons, and $|1/2|$ denotes the sqrt operation of the eigenvalue of $\hat{T}$. The free energy of a photon as a function of longitudinal wave number k_z and transversal wave number k_r is given by

$$T = \omega(k_r,k_z) = c\left(k_r^2 + k_z^2\right)^{1/2}. \tag{2}$$

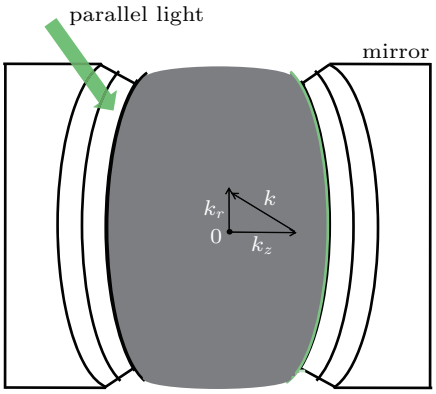


Fig. 1. Scheme of the optical micro-cavity. The micro-cavity consists of two high-reflecting curved-mirrors which provide an effective mass and a trap potential for photons.

In an optical cavity, vanishing of electric field at the reflecting surface of the mirrors also imposes a quantization condition on the z-component of the photon wave vector, $k_z = n\pi/D(r)$, where n is an integer, and $D(r) = D_0 - 2(R - \sqrt{R^2 - r^2})$ is the mirror separation at distance r from the axis with R being the radius of curvature of the mirrors. In the present model, the longitudinal size of the cavity is assumed to be very small, which causes a large frequency spacing between adjacent longitudinal modes. Thus, in the present work the corresponding longitudinal mode number can be fixed as $n = n_0$, by adjusting the frequency of the radiation reservoir. For fixed longitudinal mode number n_0, in the limit of paraxial propagation ($k_r \ll k_z$, $r \ll R$), a nonrelativistic energy-momentum relation for the 2D photons inside the microcavity can be obtained from a Taylor expansion of Eq. (2),

$$T_{\mathrm{ph}} = \frac{k_r^2}{2m_{\mathrm{ph}}} + \frac{1}{2}m_{\mathrm{ph}}\Omega^2 r^2, \tag{3}$$

where an effective photon mass

$$m_{\mathrm{ph}} = n_0\pi/cD_0 \tag{4}$$

and a harmonic trapping potential

$$\delta_{\mathrm{ph}} = m_{\mathrm{ph}}\Omega^2 r^2/2 \tag{5}$$

with the trapping frequency $\Omega = c/(D_0R/2)^{1/2}$ are introduced. To derive the above relation, we have also shifted the zero point of energy to $m_{\mathrm{ph}}c^2$. The above analysis implies that for the photon confined inside the 2D micro-cavity it is formally equivalent to a massive boson bound by a harmonic trapping potential moving in the transverse plane. Thus, in the optical cavity we need only consider the propagation of parallel light in the transverse direction. Using the definition of effective photon mass we can reexpress the field mode E_r as

$$E_r = (\varepsilon_0)^{-1/2}\left[\hat{T}_{\mathrm{ph}}^{|1/2|}\hat{\varphi}_a(r,t) + \mathrm{H.c.}\right], \tag{6}$$

where

$$\hat{T}_{\mathrm{ph}} = -\mathrm{i}\frac{\nabla_r^2}{2m_{\mathrm{ph}}} + \delta_{\mathrm{ph}} \tag{7}$$

is the free energy operator of the massive photons, and the field operator $\hat{\varphi}_a(r,t)$ is given by $\hat{\varphi}_a(r,t) = 1/\sqrt{L}\sum_{k_r} a_{k_r}\, \mathrm{e}^{\mathrm{i}(k_r\cdot r-\omega_{k_r}t)}$.

The similar cavity is first mentioned by Chiao[31] and Llaers *et al.*[32] in the theory and experiment of the BEC of photon gas. However, in our model, we have further considered that the cavity was filled with the Kerr nonlinear medium, and the nonlinear interaction was investigated. The interaction between the photon field and the medium can be described by the Hamiltonian

$$H = -\int E_r \cdot P(t)\,\mathrm{d}r, \tag{8}$$

where $P(t)$ is the polarization vector and it can be expended in powers of the field strength E_r as $P(t) = \varepsilon E_r + \chi^{(2)}E_r^2 + g^{(3)}E_r^3 + \cdots$. In the relationship, ε is the linear susceptibility, and the quantities $\chi^{(2)}$ and $g^{(3)}$ are known as the nonlinear optical susceptibilities. Substituting the expression of electric field E_r and $P(t)$ into Eq. (8), an effective Hamiltonian can be obtained and used to describe our present model:

$$\begin{aligned} H = \int \mathrm{d}r\big(&\hat{\varphi}_a^+[\hat{T}_{\mathrm{ph}} + \frac{g_{\mathrm{ph}}}{2}\hat{\varphi}_a^+\hat{\varphi}_a]\hat{\varphi}_a \\ &- \chi[\hat{\varphi}_b^+\hat{\varphi}_a\hat{\varphi}_a + \hat{\varphi}_a^+\hat{\varphi}_a^+\hat{\varphi}_b]\big). \end{aligned} \tag{9}$$

In the model, due to the second-order nonlinear interaction, two massive photons can be coupled into a photon pair whose effective mass $m_b = 2m_{\mathrm{ph}}$. The coupling of the photon pair here is very similar to the formation of the diatomic molecule by Feshbach resonance.[33–36] Thus, in this article we also name the massive photon pair as photon molecule (p-molecule). Its field operator is given by $\hat{\varphi}_b = 1/\sqrt{L}\sum_{k_r} b_{k_r}\, \mathrm{e}^{\mathrm{i}(k_r\cdot r-\omega_{k_r}t)}$, where $k_r = k_r^1 + k_r^2$, k_r^1 and k_r^2 here denote the transverse wavenumbers of the two coupled photons, and b_{k_r} is the annihilation operators of the corresponding p-molecules. The parameter χ, $\chi = \chi^{(2)}(T_{\mathrm{ph}}^0)^3/(\varepsilon_0)^{3/2}$, is used to describe the coupling interaction, with T_{ph}^0 being the free energy of photons in the condensate phase. In the present work, we also consider the four-wave fixing processes which

cause an effective photon-photon interaction. The interaction parameter $g_{\rm ph}$ is given by $g_{\rm ph} = 2g^{(3)}(T_{\rm ph}^0)^4/(\varepsilon_0)^2$. It should be emphasized that in this article we have assumed that the medium is isotropic and we have also considered that the system is in the BEC phase. Thus the differences of these parameters in the condensate phase and the general phase are ignored, and we treat them all as adjustable constants.

3. Classical-field description of the condensate phase

3.1. Description of the condensate wavefunction

It is well known that for an ideal Bose gas at absolute zero temperature, Bose–Einstein condensation is a common quantum property. According to previous analyses we know that this feature should also survive in our present coupling system. In the BEC phase, there exists a macroscopic number N_0 of particles consisting of photons and p-molecules with the relation $N_0 = N_{a0} + 2N_{b0}$ occupying the ground state at thermal equilibrium. The existence of a macroscopically populated state motivates splitting the field operator into two parts: a part with macroscopic matrix elements and a remainder, which accounts for supracondensate particles. The usual method proceeds by splitting the field operators according to

$$\hat{\varphi}_\beta(r,t) = \phi_\beta(r,t) + \delta\hat{\varphi}_\beta(r,t), \quad (\beta = a,b), \tag{10}$$

where the condensate wavefunction is obtained by taking an average $\phi_\beta(r,t) = \langle\hat{\varphi}_\beta(r,t)\rangle$ and it has the form $\phi_\beta(r,t) = \sqrt{N_{\beta 0}}\,{\rm e}^{{\rm i}\theta_\beta(r,t)}$. The expansion gives some nice properties, however it is not well suited for our present system. As we shall see below, the quantum phase transition arises in the condensate phase. In our present work, we extend the method developed by Castin and Dum,[37] and split the field operators $\hat{\varphi}_a$ and $\hat{\varphi}_b$ as the forms

$$\begin{aligned}\hat{\varphi}_a(r,t) &= \phi_a(r,t)a(t) + \delta\hat{\varphi}_a(r,t),\\ \hat{\varphi}_b(r,t) &= \phi_b(r,t)b(t) + \delta\hat{\varphi}_b(r,t).\end{aligned} \tag{11}$$

In this expression, $a(t)$ is the annihilation operator of photons in the condensate wave function $\phi_a(r,t)$, and it has matrix elements on the order of $\sqrt{N_0}$. Here, the mode operator $a(t)$ is given by

$$a(t) = \int {\rm d}r\phi_a^*(r,t)\hat{\varphi}_a(r,t). \tag{12}$$

This relation implies that the supercondensate part $\delta\hat{\varphi}_a$ is orthogonal to ϕ_a:

$$\int {\rm d}r\phi_a^*(r,t)\delta\hat{\varphi}_a(r,t) = 0. \tag{13}$$

The expression of $\delta\hat{\varphi}_a$ can be obtained by projecting the photon field operator $\hat{\varphi}_a(r,t)$ to condensate part $\phi_a(r,t)$:

$$\delta\hat{\varphi}_a(r,t) = \int {\rm d}r' \langle r|Q_a(t)|r'\rangle\hat{\varphi}_a(r',t), \tag{14}$$

where $Q_a(t)$ is given by $Q_a(t) = 1 - |\phi_a\rangle\langle\phi_a|$. To simplify the expression the operand $\circ$ is introduced, which describes the action of an operator O onto a field operator such as $\hat{\psi}(t)$: $O\circ\hat{\psi}(t) = \int {\rm d}sO|s\rangle\hat{\psi}(s,t)$. With the notation we can rewrite Eq. (12) and (14) as the simpler form

$$\begin{aligned}a(t) &= \langle\phi(t)|\circ\hat{\varphi}_a(t),\\ \delta\hat{\varphi}_a(t) &= Q_a(t)\circ\hat{\varphi}_a(t).\end{aligned} \tag{15}$$

For the p-molecules condensate part, the annihilation operator $b(t)$ and the corresponding condensate wave function $\phi_b(r,t)$ have the properties similar to $a(t)$ and $\phi_b(r,t)$. However, for p-molecule, $b(t)$ has matric element on the order of $\sqrt{N_0/2}$.

3.2. Derivation of the coupled density equation

From the description of the annihilation operator β ($\beta = a,b$) and $\delta\hat{\varphi}_\beta(r,t)$, we can obtain $\langle\beta^+\delta\hat{\varphi}_\beta(r,t)\rangle = 0$. Inspired by the result, in this section we introduce the decomposition operators $\hat{\Lambda}_a$ and $\hat{\Lambda}_b$ as follows:

$$\begin{aligned}\hat{\Lambda}_a &= \frac{1}{\sqrt{N_0}}a^+(t)\,\delta\hat{\varphi}_a(r,t),\\ \hat{\Lambda}_b &= \frac{1}{\sqrt{N_0/2}}b^+(t)\,\delta\hat{\varphi}_b(r,t),\end{aligned} \tag{16}$$

to investigate the condensate properties of the system. The expectation values of Λ_a and Λ_b vanish exactly at all time,

$$\langle\hat{\Lambda}_a(r,t)\rangle = 0, \quad \langle\hat{\Lambda}_b(r,t)\rangle = 0. \tag{17}$$

In the BEC phase, the numbers of the noncondensed photons and p-molecules are much smaller than the number of condensed particles,

$$\begin{aligned}\frac{\langle\delta N_a\rangle}{N_0} &= \frac{\int {\rm d}r\langle\delta\hat{\varphi}_a^+(r,t)\delta\hat{\varphi}_a(r,t)\rangle}{N_0} \ll 1,\\ \frac{\langle\delta N_b\rangle}{N_0/2} &= \frac{\int {\rm d}r\langle\delta\hat{\varphi}_b^+(r,t)\delta\hat{\varphi}_b(r,t)\rangle}{N_0/2} \ll 1,\end{aligned} \tag{18}$$

so that we can expand Λ_a and Λ_b in powers of $1/\sqrt{N_0}$ and $1/\sqrt{N_0/2}$, respectively,

$$\begin{aligned}\hat{\Lambda}_a &= \hat{\Lambda}_a^0 + \frac{1}{\sqrt{N_0}}\hat{\Lambda}_a^{(1)} + \frac{1}{N_0}\hat{\Lambda}_a^{(2)} + \cdots,\\ \hat{\Lambda}_b &= \hat{\Lambda}_b^0 + \frac{1}{\sqrt{N_0/2}}\hat{\Lambda}_b^{(1)} + \frac{1}{N_0/2}\hat{\Lambda}_b^{(2)} + \cdots.\end{aligned} \tag{19}$$

The first term $\hat{\Lambda}_\beta^0$ above is related to the condensate phase. As a starting point for our analysis, we need to calculate the time derivatives of the operators Λ_a and Λ_b. From the definition of Λ_a and Λ_b, we can obtain the expressions

$${\rm i}\frac{{\rm d}}{{\rm d}t}\hat{\Lambda}_a = \frac{\rm i}{\sqrt{N_0}}\left\{a^+\left[\left(\frac{\rm d}{{\rm d}t}Q_a\right)\circ\hat{\varphi}_a + Q_a\circ\left(\frac{\rm d}{{\rm d}t}\hat{\varphi}_a\right)\right]\right.$$

$$+\left(\frac{\mathrm{d}}{\mathrm{d}t}a^{+}\right)Q_a\circ\hat{\varphi}_a\Big\},$$
$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Lambda}_b = \frac{\mathrm{i}}{\sqrt{N_0/2}}\Big\{b^{+}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}Q_b\right)\circ\hat{\varphi}_b+Q_b\circ\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varphi}_b\right)\right]$$
$$+\left(\frac{\mathrm{d}}{\mathrm{d}t}b^{+}\right)Q_b\circ\hat{\varphi}_b\Big\}. \tag{20}$$

Furthermore, by taking the time derivatives of the corresponding operators and keeping only the contribution from the condensed particles we finally get (for detail see Appendix A)

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Lambda}_a = \frac{1}{\sqrt{N_0}}\Big\{a^{+}Q_a\circ\Big[(T^0_{\mathrm{ph}}+g_a a^{+}\phi_a^{*}\phi_a a)\phi_a a$$
$$-\left(2\chi a^{+}\phi_a^{*}\right)\phi_b b-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi_a a\right)\Big]\Big\}+O\left(1/\sqrt{N_0}\right),$$
$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Lambda}_b = \frac{1}{\sqrt{N_0/2}}\left\{b^{+}Q_b\circ\left[-\chi\left(\phi_a a\right)\phi_a a-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi_b b\right)\right]\right\}$$
$$+O\left(1/\sqrt{N_0/2}\right), \tag{21}$$

where $O\left(1/\sqrt{N_0}\right)$ and $O\left(1/\sqrt{N_0/2}\right)$ represent the contribution of the noncondensed particles, in the thermodynamic limit $N_0\to\infty$, they satisfy the relations $\lim\limits_{N_0\to\infty}O(1/\sqrt{N_0})=0$ and $\lim\limits_{N_0\to\infty}O(1/\sqrt{N_0/2})=0$.

In our present model, there exists two kinds of condensate consisting of photons and p-molecules. In order to understand the relation between them, we introduce two new variables $\rho_a=\phi_a a/\sqrt{N_0}$ and $\rho_b=\phi_b b/\sqrt{N_0/2}$ to describe the coupling system. According to the previous description of ϕ_a and ϕ_b, we know the complex variables ρ_a and ρ_b satisfy the following relations:

$$|\rho_a|^2=\left\langle\phi_a\left|a^{+}a\right|\phi_a\right\rangle=N_{a0}/N_0,$$
$$|\rho_b|^2=\left\langle\phi_b\left|b^{+}b\right|\phi_b\right\rangle=2N_{b0}/N_0. \tag{22}$$

Obviously, here $|\rho_a|^2$ and $|\rho_b|^2$ can be defined as the condensed particle density of photons and p-molecules and they satisfy the normalization condition $|\rho_a|^2+|\rho_b|^2=1$, ρ_a and ρ_b are the probability amplitudes of them. From the requirement of Eq. (17), $(\mathrm{d}/\mathrm{d}t)\langle\hat{\Lambda}_i\rangle=0$, we know that the condensed part contribution to Λ_a and Λ_b vanish. Furthermore, using the definitions of Q_a and Q_b and considering the orthogonally relationship $\langle\phi_a|\phi_b\rangle=0$, the coupled density equations of photons and p-molecules can be obtained as follows:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\rho_a\\ \rho_b\end{pmatrix}=L(t)\begin{pmatrix}\rho_a\\ \rho_b\end{pmatrix}, \tag{23}$$

where

$$L(t)=\begin{pmatrix}T^0_{\mathrm{ph}}+g_N|\rho_a|^2-\varepsilon_a(t) & -\sqrt{2}\chi_N\rho_a^{*}\\ -\sqrt{2}\chi_N\rho_a^{*} & -\varepsilon_b(t)\end{pmatrix}. \tag{24}$$

Here, the parameters g_N and χ_N are given by $g_N=g_{\mathrm{ph}}N_0$ and $\chi_N=\chi\sqrt{N_0}$, respectively. In Eq. (24), the real functions $\varepsilon_a(t)$, $\varepsilon_a(t)=\left\langle\phi_a\right|-\mathrm{i}\,\mathrm{d}/\mathrm{d}t+T_{\mathrm{ph}}+g_N|\rho_a|^2\left|\phi_a\right\rangle$, and $\varepsilon_b(t)$, $\varepsilon_b(t)=\langle\phi_b|-\mathrm{i}\,\mathrm{d}/\mathrm{d}t|\phi_b\rangle$, correspond to the global phases of the wave functions ϕ_a and ϕ_b. In the thermodynamic equilibrium state, they can be chosen as the chemical potentials of condensed photons and p-molecules. Considering the fact that in our model the creation and annihilation of photons are always in pairs, in this study it is appropriate to set $\varepsilon_a(t)=2\mu_a$ and $\varepsilon_b(t)=\mu_b$.

3.3. The chemical potential and the quantum phase transition

When the system is in a thermodynamic steady state, ρ_a and ρ_b are time independent and we can convert the nonlinear equation (23) into the form of an intrinsic equation of ρ_a and ρ_b:

$$L_{\mathrm{GP}}\begin{pmatrix}\rho_a\\ \rho_b\end{pmatrix}=2\mu\begin{pmatrix}\rho_a\\ \rho_b\end{pmatrix}, \tag{25}$$

where

$$L_{\mathrm{GP}}=\begin{pmatrix}T^0_{\mathrm{ph}}+g_N|\rho_a|^2 & -\sqrt{2}\chi_N\rho_a^{*}\\ -\sqrt{2}\chi_N\rho_a & 0\end{pmatrix}. \tag{26}$$

Here, we have used the relation $\mu_b=2\mu_a\equiv2\mu$. By solving Eq. (26), we can obtain the expression of the chemical potential μ as

$$2\mu=\left(T^0_{\mathrm{ph}}+g_N|\rho_a|^2\right)|\rho_a|^2-\sqrt{2}\chi_N\left[\rho_a^2\rho_b^{*}+\rho_b(\rho_a^{*})^2\right]. \tag{27}$$

In the steady state the complex variables ρ_a and ρ_b can also be written as the forms $\rho_a=|\rho_a|\,\mathrm{e}^{\mathrm{i}\theta_a}$ and $\rho_b=|\rho_b|\,\mathrm{e}^{\mathrm{i}\theta_b}$. For the present model, we are more interested in investigating the density difference and the phase difference between photons and p-molecule, they are given by $\rho=|\rho_b|^2-|\rho_a|^2$ and $\theta=\theta_b-2\theta_a$, respectively. With the help of ρ and θ, and using the relation $|\rho_a|^2+|\rho_b|^2=1$, we can rewrite μ as

$$2\mu = \frac{1}{4}g_N\rho^2-\frac{1}{2}\left(T^0_{\mathrm{ph}}+g_N\right)\rho$$
$$-\chi_N(1-\rho)\sqrt{1+\rho}\cos\theta+\xi, \tag{28}$$

where the constant term $\xi=g_N/4+T^0_{\mathrm{ph}}/2$. In the expression of $\mu(\rho,\theta)$, ρ and θ are two canonically conjugate variables. In the ground state, the optimum values of the two canonical variables are determined by the condition $\partial\mu/\partial\theta=\partial\mu/\partial\rho=0$, which are given by

$$(\rho,\theta)=\begin{cases}1,\ \theta_m; & T_{\mathrm{eff}}>1,\\ \rho_m,\ 0; & T_{\mathrm{eff}}<1,\end{cases} \tag{29}$$

where θ_m can take any suitable values in the region $0\le\theta_m\le2\pi$, and ρ_m is in the range of $-1<\rho_m<1$. In Eq. (29), we also introduce a dimensionless variable $T_{\mathrm{eff}}=T^0_{\mathrm{ph}}/\sqrt{2\chi_N^2}$, and we name it the effective free energy of photons in the optical cavity. From Eq. (29), we find that for the current system there exists a quantum phase transition in the ground state. When $T_{\mathrm{eff}}>1$, all photons in the condensate phase will be

converted into p-molecules and the system can transform from the mixed photon and p-molecule condensate phase into a pure p-molecule phase.

To better visualize the phase transition, it is useful to plot the phase-space structure evolution of the chemical potential μ. The results are shown in Fig. 2. Two fixed points $\mu_{\min}$ and $\mu_{\max}$ can be observed from Fig. 2. The fixed points on the phase space correspond to the minimum and maximum of μ. It is found that for the case of $T_{\text{eff}} < 1$, with the increase of T_{eff}, $\mu_{\min}$ moves smoothly along the line $\theta = 0$ towards the phase boundary $\rho = 1$ while the maximum approaches $\rho = -1$. Especially, across the critical point $T_{\text{eff}} = 1$ the system follows the minimum point and finally reaches $\rho = 1$, implying the entire conversion of photons into p-molecules, i.e., the system transforms into the pure p-molecules condensate phase.

Furthermore, we also find that from the figure the photon interactions do not change the above phase transition behavior but only change the conversion rate of p-molecules. We have known that the photon interactions here mainly come from the four-wave fixing process, which is determined by the three-order nonlinear optical susceptibilities $g^{(3)}$. In our present work, we mainly consider the situation that the high-order nonlinear effects are very weak compared to the second-order one. Thus, the effective interaction between photons g_{eff}, $g_{\text{eff}} = g_N/\sqrt{2\chi_N^2}$, in our present model is very small and it can be ignored.

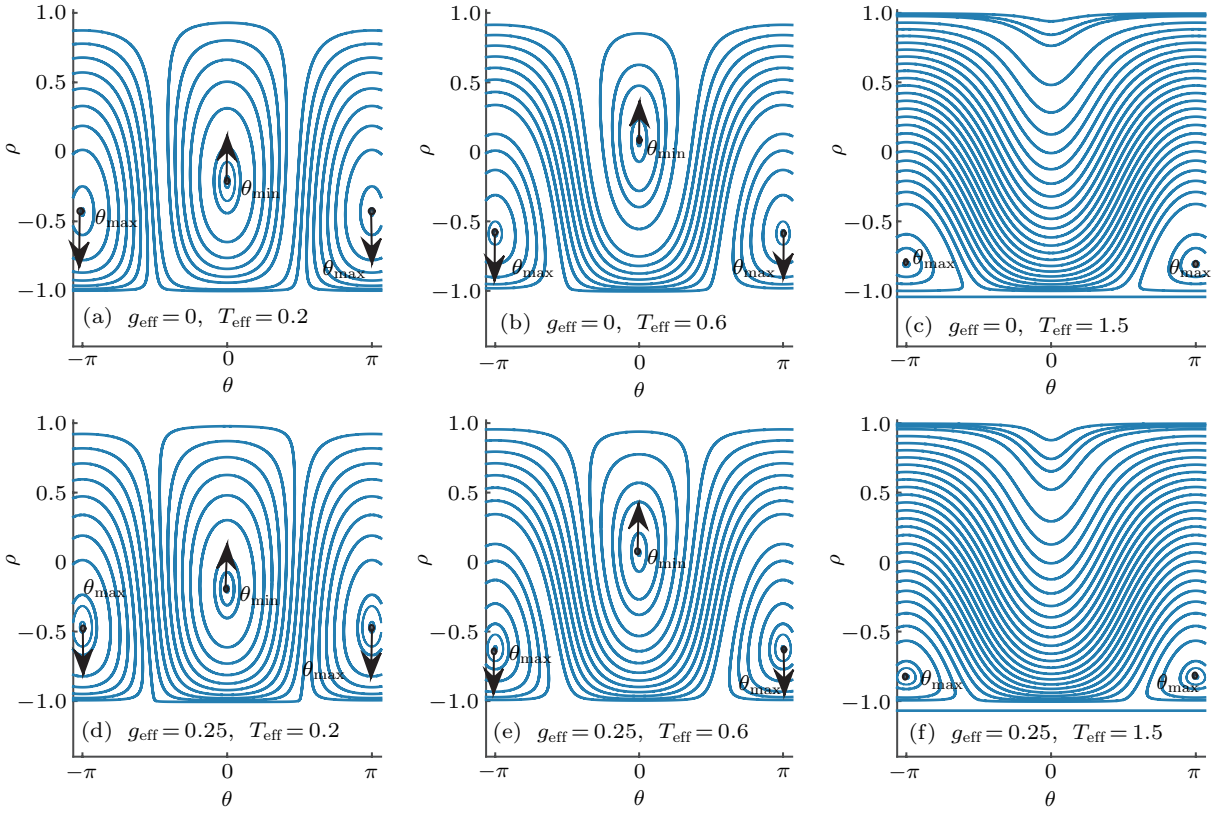


Fig. 2. The phase space evolution of the chemical potential μ as T_{eff} changes. The arrows indicate the shifting direction of the fixed points as T_{eff} increases. Here, μ is scaled by $\sqrt{2}\chi_N$, we also define $g_{\text{eff}} = g_N/\sqrt{2\chi_N^2}$.

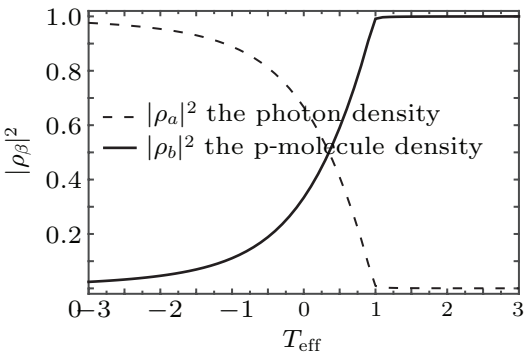


Fig. 3. Particle densities of photon and p-molecule in the ground state as a function of parameter T_{eff}. Here, we set $N = 2000$ and $g_N/\sqrt{2\chi_N^2} = 0$.

Having analyzed the chemical potential, we now investigate the state function of the system. In the large particle number limit, the coherent state (i.e., Gross–Pitaevskii states)

$$|\psi_{\text{GP}}\rangle = \frac{1}{\sqrt{M!}}\left[\rho_a c^+ + \rho_b b^+\right]^M |0\rangle \tag{30}$$

gives a good approximation of the quantum ground state,[38,39] where $M = N/2$, and c^+ is the double photon creation operator satisfying the relation $(c^+)^n = \sqrt{n!/(2n)!}(a^+)^{2n}$.

In the previous analysis, we have realized that $|\rho_a|^2$ and $|\rho_b|^2$ are the condensed particle densities of photons and p-molecules, and ρ_a and ρ_b are the probability amplitudes of them. In the ground state of the system, we have also calculated the corresponding values of $|\rho_a|^2$ and $|\rho_b|^2$ by diago-

nalizing the Hamiltonian (9) numerically in the Hillbert space with Fock basis:

$$|2(M-n),n\rangle = [n!(M-n)!]^{-1/2}(c^+)^{M-n}(b^+)^n|0\rangle, \quad n=1,2,\ldots. \tag{31}$$

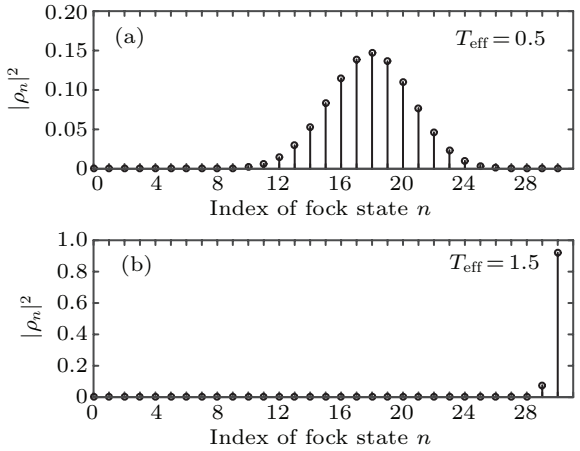


Fig. 4. The probabilities of the system occupying different Fock states. Here, we set $N=60$ and $g_N/\sqrt{2\chi_N^2}=0$.

The results are given in Fig. 3. From Fig. 3, we can see that in the pure p-molecule phase (for $T_{\rm eff}>1$) almost all photons are coupled with each other into p-molecules. In consequence, in the p-molecule phase the relationship $\lim_{M\to\infty}|\rho_b|^2=1$ should can be satisfied, and the system is in a quantum pure state with state function $|\psi_{\rm GP}\rangle=(M!)^{-1/2}[b^+]^M|0\rangle=|0,M\rangle$. In the mixed photon and p-molecule phase, the system is in a thermodynamic mixed state and the state function $|\psi_{\rm GP}\rangle$ can be further expended as the form $|\psi_{\rm GP}\rangle=\sum_{n=0}^{M}\rho_n|2(M-n),n\rangle$, where $\rho_n=(M!)^{1/2}(n!(M-n)!)^{-1/2}\rho_a^{M-n}\rho_b^n$ is the probability amplitude for state $|2(M-n),n\rangle$. The above analysis can be confirmed by numerical calculations of $|\rho_n|^2$, as plotted in Fig. 4. From this figure, we can clearly see that across the phase transition point $T_{\rm eff}=1$ the system will transform from the mixed state condensate phase into a pure state phase.

4. The spectrum of collective excitations

4.1. The photon and p-molecule condensate phase

We know that the time evolution of the small perturbations $\delta\hat{\varphi}_a$ and $\delta\hat{\varphi}_b$, that is the dynamics of the noncondensed particles, describes the elementary excitations of the system. From the definition of $\delta\hat{\varphi}_a$ and $\delta\hat{\varphi}_b$ given in Eqs. (14) and (15), we can obtain the dynamics equation of supracondensate wavefunctions

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta\hat{\varphi}_a,\delta\hat{\varphi}_b,\delta\hat{\varphi}_a^+,\delta\hat{\varphi}_b^+\right)^{\mathrm{T}} = \delta L\left(\delta\hat{\varphi}_a,\delta\hat{\varphi}_b,\delta\hat{\varphi}_a^+,\delta\hat{\varphi}_b^+\right)^{\mathrm{T}}, \tag{32}$$

where δL is given by

$$\delta L=\begin{pmatrix} \hat{T}_{\rm ph}+2g_N|\rho_a|^2 & -2\chi_N\rho_a^* & g_N\rho_a^2-\sqrt{2}\chi_N\rho_b & 0\\ -2\chi_N\rho_a & 0 & 0 & 0\\ g_N(\rho_a^*)^2-\sqrt{2}\chi_N\rho_b^* & 0 & T_{\rm ph}+2g_N|\rho_a|^2 & -2\chi_N\rho_a\\ 0 & 0 & -2\chi_N\rho_a^* & 0 \end{pmatrix}, \tag{33}$$

and we have also used the definitions of ρ_a and ρ_b.

Following Bogoliubov,[40,41] we can expend the supracondensate wavefunctions in term of quasiparticle bases as

$$\begin{aligned}\delta\hat{\varphi}_a&=\sum_{k_r}u_{ak_r}c_{k_r}\mathrm{e}^{\mathrm{i}(k_r\cdot r-\omega_{k_r}t)}-v_{ak_r}c_{k_r}^+\mathrm{e}^{-\mathrm{i}(k_r\cdot r-\omega_{k_r}t)},\\ \delta\hat{\varphi}_b&=\sum_{k_r}u_{bk_r}c_{k_r}\mathrm{e}^{\mathrm{i}(k_r\cdot r-\omega_{k_r}t)}-v_{bk_r}c_{k_r}^+\mathrm{e}^{-\mathrm{i}(k_r\cdot r-\omega_{k_r}t)},\end{aligned} \tag{34}$$

where c_{k_r} and $c_{k_r}^+$ are the quasiparticle annihilation and creation operators with frequency ω_{k_r}, and the parameters $\mu_{\beta k_r}$ and $v_{\beta k_r}$ satisfy the condition

$$u_{\beta k_r}^2-v_{\beta k_r}^2=1, \tag{35}$$

which retains the Bose commutation relation for all operators.

Unlike the standard Bogliubov expansion, in which the only mixture is between the single particle creation and annihilation term, in this expression the quasiparticle have both photon and p-molecule components.

If we substitute the Bogliubov substitations into the dynamics equations (32), and collect terms with the same phase, then the four-component Bogliubov–dGenes equation can be obtained as follows:

$$L_{\rm B-d}^M\left(u_{a,k_r},u_{b,k_r},v_{a,k_r},v_{b,k_r}\right)^{\mathrm{T}} = \omega_{k_r}\left(u_{a,k_r},u_{b,k_r},v_{a,k_r},v_{b,k_r}\right)^{\mathrm{T}}, \tag{36}$$

where $L_{\rm B-d}^M$ is given by

$$L_{\rm B-d}^M=\begin{pmatrix} H_{{\rm GP},a} & -\chi_N(\rho_a^*+\rho_a) & -(D_{{\rm GP},a}+D_{{\rm GP},a}^*)/2 & 0\\ -\chi_N(\rho_a^*+\rho_a) & 0 & 0 & 0\\ (D_{{\rm GP},a}+D_{{\rm GP},a}^*)/2 & 0 & -H_{{\rm GP},a} & \chi_N(\rho_a^*+\rho_a)\\ 0 & 0 & \chi_N(\rho_a^*+\rho_a) & 0 \end{pmatrix}, \tag{37}$$

and $D_{\mathrm{GP},a} = g_N\rho_a^2 - \sqrt{2}\chi_N\rho_b$, the condensate energy $H_{\mathrm{GP},a}$ of photons is given by $H_{\mathrm{GP},a} = T_{\mathrm{ph}} + 2g_N|\rho_a|^2$.

The eigenmodes can be obtained by diagonalizing the resulting 4×4 matrix (in this process we have used the definition of the density difference ρ and taken the corresponding phase factors $\theta_{a,b} = 0$). There are two branches of excitations with the dispersion,

$$\omega_{a,b}^2(k_r) = \frac{1}{2}\left(R_{k_r} \mp \sqrt{R_{k_r}^2 - 4S_{k_r}}\right), \tag{38}$$

where

$$R_{k_r} = \frac{3}{4}g_N^2\rho^2 - \left(\frac{3}{2}g_N^2 + 4\chi_N^2 + 2T_{\mathrm{ph}}g_N\right)\rho + L_R \tag{39}$$

with the constant term

$$L_R = \frac{3}{4}g_N^2 + T_{\mathrm{ph}}^2 + 4\chi_N^2 + 2T_{\mathrm{ph}}g_N \tag{40}$$

and

$$S_{k_r} = 2\chi_N^4(1-\rho)^2. \tag{41}$$

Figure 5 shows the quasiparticle dispersion spectra (in the limit of weak photon interaction, $g_N/\chi_N \ll 1$). As can be seen clearly in the figure, the photonic excitation energy ω_a is smaller than the excitation energy ω_b for p-molecule. We also find that the energy to create a photon-dominated quasiparticle decreases with ρ. The above results reveal that with the increase of the p-molecule conversion rate the collective excitation of the system is mainly contributed by the noncondensate photon modes.

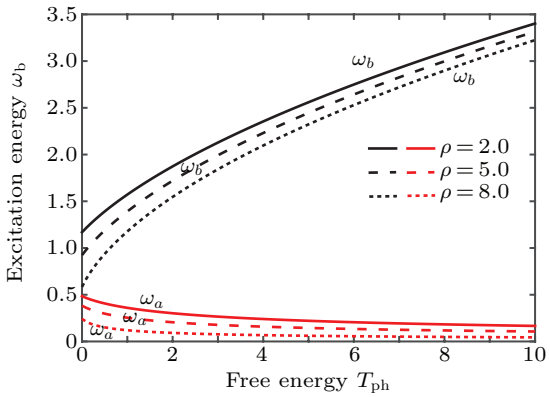


Fig. 5. Illustrating the shape of the spectrums of elementary excitations in the mixed photon and p-molecule condensate phase for different values of the density difference ρ. Here, the free energy T_{ph} of photons and the excitation energy $\omega_\beta\,(\beta = a,b)$ are scaled by $\sqrt{2}\chi_N$, and we also set $g_N/\sqrt{2\chi_N^2} = 0$.

4.2. The pure p-molecule condensate phase

In the pure p-molecule condensate phase, the density difference ρ between p-molecules and photons is given by $\rho = 1$. Therefore, by taking the phase factor $\theta_{a,b} = 0$, we can replace the condensate parts $\phi_a a$ and $\phi_b b$ of the field functions by two real-numbers, with relations $\phi_a a = 0$ and $\phi_b b = \sqrt{N_0/2}$. Making these substitutions into the system Hamiltonian (9), the supracondensate part δH can be obtained in the quadratic form of $\delta\hat{\varphi}_a$ and $\delta\hat{\varphi}_a^+$ (taking the second order approximation)

$$\delta H = \int \mathrm{d}r\left[\delta\hat{\varphi}_a^+\hat{T}_{\mathrm{ph}}\delta\hat{\varphi}_a - \frac{\sqrt{2}}{2}\chi_N\left(\delta\hat{\varphi}_a\delta\hat{\varphi}_a + \delta\hat{\varphi}_a^+\delta\hat{\varphi}_a^+\right)\right]. \tag{42}$$

Furthermore, taking the responding Bogliubov transition (34) into the Hamiltonian (42), we can obtain the following pair of coupled equations for u_{a,k_r} and v_{a,k_r}:

$$\begin{aligned}\left(T_{\mathrm{ph}} - \omega_{k_r}^a\right)u_{a,k_r} - \sqrt{2}\chi_N v_{a,k_r} = 0,\\ \left(T_{\mathrm{ph}} + \omega_{k_r}^a\right)v_{a,k_r} - \sqrt{2}\chi_N u_{a,k_r} = 0.\end{aligned} \tag{43}$$

By solving Eqs. (43), the dispersion law for the elementary excitations of the system in the pure p-molecule condensate phase can be obtained, which has the form

$$\omega_{k_r}^a = \left(T_{\mathrm{ph}}^2 - \Delta^2\right)^{1/2}, \tag{44}$$

where $\Delta = \sqrt{2}\chi_N$ is the energy gap. We have also found that the parameters u_{a,k_r} and v_{a,k_r} in the pure p-molecules condensate phase have the following form

$$\begin{aligned}u_{a,k_r}^2 &= \frac{1}{2}\left(\frac{T_{\mathrm{ph}}}{\omega_{k_r}} + 1\right),\\ v_{a,k_r}^2 &= \frac{1}{2}\left(\frac{T_{\mathrm{ph}}}{\omega_{k_r}} - 1\right).\end{aligned} \tag{45}$$

The expressions for the excitation spectrum and the factors u_{a,k_r} and v_{a,k_r} are analogous to those for a superconductor in the BCS theory, apart from some sign changes due to the fact that we are here dealing with bosons rather than fermions. The dispersion relation here also implies the existence of superfluid phase of photons.

5. Atom inside the micro-cavity

5.1. The interaction energy between the atom and the radiation field

It is well known that a single atom is an ideal "laboratory" to study the quantum properties of the radiation field. We have also known that in the experiment of the BEC of paraxial light, there exists continuous absorption and emission of photons by the dye solution to maintain the conservation of photon number. Therefore, it is necessary to investigate the atom-light coupling in the present model. In this section, we consider a two-level atom immersed inside the microcavity. The coupling interaction between the atom and the radiation field consisting of photons and p-molecules can be described by the following Hamiltonian (in the second quantization representation):

$$H_{\mathrm{c}} = H_{\mathrm{r}} + H_{\mathrm{a}} + V, \tag{46}$$

where

$$H_{\mathrm{r}} = \sum_{k_r} T_{\mathrm{ph}} a_{k_r}^{+} a_{k_r} + \frac{g_{\mathrm{ph}}}{2L} \sum_{k_r,k'_r,q} a_{k_r+q}^{+} a_{k'_r-q}^{+} a_{k_r} a_{k'_r} - \frac{\chi}{\sqrt{L}} \sum_{k_r,k'_r} \left(b_{k_r+k'_r}^{+} a_{k_r} a_{k'_r} + \mathrm{H.c.} \right) \tag{47}$$

is the Hamiltonian of the radiation field, and it is also given in Eq. (9), and

$$H_{\mathrm{a}} = \frac{1}{2} \omega_{eg} \sigma_z \tag{48}$$

is the energy of the atom. Here, $\omega_{eg} = E_e - E_g$ is the transition frequency between the excited state $|e\rangle$ and the ground state $|g\rangle$ of the atom, and σ_z is given by $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$. The interaction energy V between the atom and the radiation field can be expressed as

$$V = \sum_{k_r \neq 0} \sum_{\beta=a,b} \left(\Omega_{\beta k_r} \sigma_+ \beta_{k_r} + \Omega_{\beta k_r}^{*} \sigma_- \beta_{k_r}^{+} \right), \tag{49}$$

where $\sigma_+ = |e\rangle\langle g|$ and $\sigma_- = |g\rangle\langle e|$ are the raising and lowering operators respectively. The term associated with $\sigma_+ \beta_{k_r}$ corresponds to the process by which the atom in the ground state absorbs a photon or p-molecule and is jumped up to the excited state, the term $\sigma_- \beta_{k_r}^{+}$ describes the opposite process. The parameter $\Omega_{\beta k_r} = -|e|\langle e|r|g\rangle \varepsilon_{\beta k_k} \cos\theta$ is defined as the 2D vacuum Rabi frequency, where $\varepsilon_{\beta k_r}$ denotes the electric field per mode and using the definition of effective photon mass it can be expressed by $\varepsilon_{\beta k_r} = \left(T_{\mathrm{ph}}/2\varepsilon_0 L\right)^{1/2}$. Here θ is the angle between the position vector of the electron with $|e|$ being the electron charge and the polarization vector of the electron field (in the 2D microcavity the direction of polarization vector is fixed for parallel light). In the previous analysis, we have investigated the elementary excitations of the binary system. It is found that for the radiation field H_{r}, in the BEC phase, we can diagonal the supracondensate part and rewrite the Hamiltonian H_{r} as the following form:

$$H_{\mathrm{r}} = H_0 + \sum_{k_r \neq 0} \sum_{\beta=a,b} \omega_{\beta k_r} c_{\beta k_r}^{+} c_{\beta k_r}, \tag{50}$$

where H_0 denotes the condensate energy, and $c_{\beta k_r}$ represents the annihilation operator of the photon or p-molecule type of quasiparticles with frequency $\omega_{\beta k_r}$, which is given by the dispersion relation (38) (in the mixed photon and p-molecule condensate phase) and relation (44) (in the pure p-molecule condensate phase) respectively. Especially, it should be pointed out that in the pure p-molecule condensate phase the elementary excitation of the system is only related to the thermal vibration of photons and the vibration of p-molecule can be ignored, which means that in the pure p-molecule condensate phase we can take $\omega_{bk_r} = 0$. In order to reveal the characteristic of the collective excitations, here we have also rewritten the interaction energy V in terms of the quasiparticle operators by introducing the previous Bogliubov transition (34), the result is given by

$$V = \sum_{k_r \neq 0} \sum_{\beta=a,b} \mathrm{e}^{\theta_{\beta k_r}} \left(\Omega_{\beta k_r} \sigma_+ c_{\beta k_r} + \Omega_{\beta k_r}^{*} \sigma_- c_{\beta k_r}^{+} \right). \tag{51}$$

Compared with Eq. (49), in the present expression of V a Bogliubov factor $\mathrm{e}^{\theta_{\beta k_r}}$ is introduced, which turns the coupling between the atom and the interacting radiation field into the coupling between the atom and independent quasiparticle field. The factor $\mathrm{e}^{\theta_{\beta k_r}}$ here is given by $\mathrm{e}^{\theta_{\beta k_r}} = u_{\beta k_r} + v_{\beta k_r}$, in the expression we have set $u_{\beta k_r} = \cosh\theta_{\beta k_r}$ and $v_{\beta k_r} = \sinh\theta_{\beta k_r}$ due to the quantum relation $u_{\beta k_r}^2 - v_{\beta k_r}^2 = 1$. In the present model, it is convenient to work in the interaction picture. The interaction energy V in the interaction picture is furthermore given by

$$V^I = \sum_{k_r \neq 0} \sum_{\beta=a,b} \mathrm{e}^{\theta_{\beta k_r}} \left[\Omega_{\beta k_r} \sigma_+ c_{\alpha k_r} \mathrm{e}^{\mathrm{i}(\omega_{eg} - \omega_{\beta k_r})t} + \mathrm{H.c.} \right]. \tag{52}$$

5.2. Spontaneous decay of the atom

In the present model, we assume that the transition frequency ω_{eg} between the two atomic levels is enough large so that only the spontaneous decay of the atom needs to be considered. To analyze the process of spontaneous decay, we also assume that the atom initially is in the excited state $|e\rangle$ and the quasiparticle field is in the vacuum state. This state can be written as $|e,0,0\rangle$. As time evolves, the atom has the probability to drop down to the ground state and the system emits a quasiparticle. Thus the time evolution of the quantum state takes the form

$$\begin{aligned} |\varphi_{r-a}(t)\rangle = {} & C_e(t)|e,0,0\rangle \\ & + \sum_{k_r \neq 0} C_{g,ak_r}(t)|g,1_{ak_r},0\rangle \\ & + \sum_{k_r \neq 0} C_{g,bk_r}(t)|g,0,1_{bk_r}\rangle. \end{aligned} \tag{53}$$

By substituting the wavefunction into the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\varphi_{r-a}(t)\rangle = V^I|\varphi_{r-a}(t)\rangle, \tag{54}$$

and matching terms properly, we can obtain the equation of motion of coefficients $C_{g,ak_r}(t)$, $C_{g,bk_r}(t)$ and $C_e(t)$

$$\begin{aligned} & \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left[C_e(t), C_{g,ak_r}(t), C_{g,bk_r}(t)\right]^{\mathrm{T}} \\ = {} & M_{\mathrm{T}}\left[C_e(t), C_{g,ak_r}(t), C_{g,bk_r}(t)\right]^{\mathrm{T}}, \end{aligned} \tag{55}$$

where the transition matrix M_{T} is given by

$$M_{\mathrm{T}}=\begin{pmatrix} 0 & \sum\limits_{k_r\neq 0} \mathrm{e}^{\theta_{ak_r}}\Omega_{ak_r}\mathrm{e}^{\mathrm{i}\Delta_{ak_r}t} & \sum\limits_{k_r\neq 0} \mathrm{e}^{\theta_{bk_r}}\Omega_{bk_r}\mathrm{e}^{\mathrm{i}\Delta_{bk_r}t} \\ \sum\limits_{k_r\neq 0} \mathrm{e}^{\theta_{ak_r}}\Omega_{ak_r}\mathrm{e}^{\mathrm{i}\Delta_{ak_r}t} & 0 & 0 \\ \sum\limits_{k_r\neq 0} \mathrm{e}^{\theta_{bk_r}}\Omega_{bk_r}\mathrm{e}^{\mathrm{i}\Delta_{bk_r}t} & 0 & 0 \end{pmatrix}, \tag{56}$$

and $\Delta_{\beta k_r}$ is given by $\Delta_{\beta k_r}=\omega_{eg}-\omega_{\beta k_r}$. From Eq. (55), the self-consistently equation for $C_e(t)$ describes that the spontaneous decay of the excited state can be obtained, which has the form

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}C_e(t)\\ &=-\sum_{k_r\neq 0}\mathrm{e}^{2\theta_{ak_r}}|\Omega_{ak_r}|^2\int_0^t \mathrm{d}t'\,\mathrm{e}^{-\mathrm{i}\Delta_{ak_r}(t'-t)}C_e(t')+F_b(t),\end{aligned} \tag{57}$$

where $F_b(t)$ is given by

$$F_b(t)=-\sum_{k_r\neq 0}\mathrm{e}^{2\theta_{bk_r}}|\Omega_{bk_r}|^2\int_0^t \mathrm{d}t'\,\mathrm{e}^{-\mathrm{i}\Delta_{bk_r}(t'-t)}C_e(t').$$

In this article we define $F_b(t)$ as the damping force of photons. Equation (57) implies that in the mixed photon and p-molecule condensate phase, the spontaneous decay inducing by the coupling between the atom and the photon gas has the form of damped motion, and the damping force is related to the coupling between the atom and the p-molecules. Especially, as the system turns into the pure p-molecule condensate phase, the damping force $F_b(t)=0$. By substituting the expressions of the vacuum Rabi frequency Ω_{ak_r} and the Bogliubov modified factor $\mathrm{e}^{\theta_{\alpha k_r}}$ into Eq. (57), and transforming the summation into an integral form, the integral-differential equation for $C_e(t)$ can be obtained as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}C_e(t)=&\frac{1}{2\pi\varepsilon_0}(2m_a)^{1/2}\left|P_{eg}\right|^2\\ &\times\int_0^\infty \mathrm{d}\,\omega_{ak_r}\frac{1}{\omega_{ak_r}}\left(\omega_{ak_r}^2+\Delta^2\right)^{3/4}\\ &\times\int_0^t \mathrm{d}t'\,\mathrm{e}^{-\mathrm{i}\Delta_{ak_r}(t'-t)}C_e(t'),\end{aligned} \tag{58}$$

where $P_{eg}=|e|\langle e|r|g\rangle$ is real, and Δ is the energy gap of the photonic quasiparticle given in Eq. (44). In Eq. (58), we notice that the major contribution to the integral over ω_{ak_r} is from a small region centered around ω_{eg}, thus we can use the Weisskopf–Wigner approximation[42] and replace ω_{ak_r} in the integral by ω_{eg}. We finally find in the pure p-molecule phase Eq. (58) has a simple form

$$\frac{\mathrm{d}}{\mathrm{d}t}C_e(t)=-\frac{\Gamma}{2}C_e(t). \tag{59}$$

Here, Γ is the spontaneous decay constant of the atom, it is different from the usual spontaneous decay constant Γ_0, $\Gamma_0=\omega_{eg}^2\left|P_{eg}\right|^2/\varepsilon_0c^2$, in 2D free space. In the present model it is affected by the microcavity and the nonlinear medium and has the form

$$\Gamma=\Gamma_{\mathrm{L}}\Gamma_{\mathrm{N}}\Gamma_0, \tag{60}$$

where

$$\Gamma_{\mathrm{L}}=\frac{2c^2}{\omega_{eg}^{3/2}}\left(2m_{\mathrm{ph}}\right)^{1/2} \tag{61}$$

is the linear modified factor which comes from the introduction of the effective mass of photons. The nonlinear factor Γ_{N} has the form

$$\Gamma_{\mathrm{N}}=\left[1+\left(\frac{\Delta}{\omega_{eg}}\right)^2\right]^{3/4}. \tag{62}$$

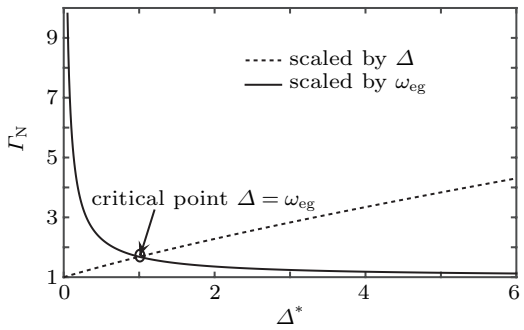


Fig. 6. Variation of the nonlinear decay factor Γ_{N} with respect to the effective energy gap Δ^*. Above, Γ_{N} are scaled by Δ and ω_{eg}, respectively.

From Eqs. (60) and (62), we find that obviously in the pure p-molecule phase the decay rate Γ is an increasing function of the energy gap Δ. The result is easily to understand. From the dispersion relation (44), we know that with the increasing energy gap Δ, the excitation of the photonic quasiparticle becomes more easily. Consequently, the interaction between the atom and the quasiparticle field enhances, and the spontaneous decay of the atom increases. If we review the definition of energy gap given in Eq. (44), then we can find that the nonlinear factor Γ_{N} mainly comes from the convertion between photons and p-moleules. Furthermore, from Eq. (62) we have known that the nonlinear effect is also influenced by the transition frequency ω_{eg}. In order to illustrate the dependency between the energy gap Δ and the transition frequency ω_{eg}, here we have also introduced the notion of effective gap $\Delta^*=\Delta/\omega_{eg}$. The function relation between Γ_{N} and Δ^* is plotted in Fig. 6. From Fig. 6, we also find that for fixed energy gap Δ, in the region of $\omega_{eg}>\Delta$ the nonlinear factor $\Gamma_{\mathrm{N}}\to 1$,

which implies that in this region the nonlinear effect disappears. Thus, we have also defined Δ as the critical transition frequency of the atom.

6. Conclusion

In our present model, photons confined inside the microcavity can be treated as usual massive bosons. Considering the propagation of the massive photon in the Kerr nonlinear medium, it is found that due to the nonlinear effect the massive photons are converted into p-molecules. In this article, we have first illustrated the coupling model consisting of photon and p-molecule. We have also considered that the coupling system is in the BEC phase. It is well known that the Bogliubov theory is a valid tool to analyze the condensate properties of the Bose gas. The theory should also be suitable to describe our present system. Using the classical-field method, we have furthermore proposed a coupled density equation that can be used to describe the conversion relation between photons and p-molecules. Based on the density equation, we have obtained the chemical potential, and we have also investigated the quantum phase transition of the coupling system. The results show that in the ground state the current system can transform from the mixed photon and p-molecule condensate phase into the pure p-molecule condensate phase. Furthermore, we have also calculated the collective mode and the spectrum function of the collective excitations.

Subsequently, we have considered the question how the spontaneous decay of an atom is influenced by both the phase transition and collective excitation of the system. By introducing the Bogliubov transition we have found that a correction factor can be obtained, which turns the interaction between the atom and the radiation field into the interaction between the atom and the quasiparticle field. Especially, in the pure p-molecule condensate phase, we also find that the spontaneous decay of the atom is influenced by the energy gap Δ of the quasiparticle, and it is an increasing function of Δ.

Appendix A

From the definitions of $\delta\hat{\varphi}_a$ and $\hat{\Lambda}_a$ we can obtain

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Lambda}_a^{\mathrm{ex}} = \mathrm{i}\frac{1}{\sqrt{N_0}}\left\{a^+\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}Q_a\right)\circ\hat{\varphi}_a + Q_a\circ\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varphi}_a\right)\right] + \left(\frac{\mathrm{d}}{\mathrm{d}t}a^+\right)Q_a\circ\hat{\varphi}_a\right\}, \tag{A1}$$

where the time derivative of a^+ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}a^+ = \left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varphi}_a^+\right)\circ|\phi_a\rangle + \hat{\varphi}_a^+\circ\frac{\mathrm{d}}{\mathrm{d}t}|\phi_a\rangle, \tag{A2}$$

and the first term inside the brackets can be rewritten as

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}Q_a\right)\circ\hat{\varphi}_a = -Q_a\left(\frac{\mathrm{d}}{\mathrm{d}t}|\phi_a\rangle\right)a - |\phi_a\rangle\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\phi_a|\right)\circ\delta\hat{\varphi}_a. \tag{A3}$$

Substituting Eqs. (A2) and (A3) into Eq. (A1), we can find the time derivative of $\hat{\Lambda}_a$ in the following form:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Lambda}_a = \frac{\mathrm{i}}{\sqrt{N_0}}\left\{a^+Q_a\circ\left[\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varphi}_a - \frac{\mathrm{d}}{\mathrm{d}t}(\phi_a a)\right]\right\} + f_1(\delta\hat{\varphi}_a), \tag{A4}$$

where

$$f_1(\delta\hat{\varphi}_a) = \frac{\mathrm{i}}{\sqrt{N_0}}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varphi}_a^+\right)\circ|\phi_a\rangle + \hat{\varphi}_a^+\circ\frac{\mathrm{d}}{\mathrm{d}t}|\phi_a\rangle\right]\delta\hat{\varphi}_a - a^+|\phi_a\rangle\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\phi_a|\right)\circ\delta\hat{\varphi}_a \tag{A5}$$

is the contribution of the noncondensate particles. Here, we have also used the relation $Q_a\circ|\phi_a\rangle\,(\mathrm{d}a/\mathrm{d}t) = 0$.

The time evolution of $\hat{\varphi}_a$ is determined by the Heisenberg equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varphi}_a = [\hat{\varphi}_a, H]. \tag{A6}$$

Using Eqs. (9) and (11) and separating the contribution of the noncondensate part, we can reach

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varphi}_a = T_{\mathrm{ph}}^0\phi_a a + g_{\mathrm{ph}}|\phi_a a|^2\phi_a a - 2\chi a^+\phi_a^*\phi_b b + f_2(\delta\hat{\varphi}_a). \tag{A7}$$

Furthermore, substituting Eq. (A7) into Eq. (A4) and keeping only the dominant contribution from the condensed particles we finally obtain

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Lambda}_a = \frac{1}{\sqrt{N_0}}\left\{a^+Q_a\circ\left[(T_{\mathrm{ph}}^0 + g_{\mathrm{ph}}a^+\phi_a^*\phi_a a)\phi_a a - \left(2\chi a^+\phi_a^*\right)\phi_b b - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}(\phi_a a)\right]\right\} + O\left(1/\sqrt{N_0}\right). \tag{A8}$$

Similarly, for $\hat{\varphi}_b$ the time derivative can be rewritten as

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Lambda}_b = \frac{1}{\sqrt{N_0/2}}\left\{b^+Q_b\circ\left[-\chi(\phi_a a)\phi_a a - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}(\phi_b b)\right]\right\} + O\left(1/\sqrt{N_0/2}\right). \tag{A9}$$

References

[1] Bose 1924 *Z. Phys.* **26** 178
[2] Fang J S and Xiang P 2011 *Chin. Phys. B* **20** 040310
[3] Einstein J P and Macdonald A H 2004 *Nature* **432** 691
[4] Balili R, Hartwell V, Snoke D, Pfeiffer L and West K 2007 *Science* **316** 1007
[5] Zhang X F, Zhang P, He W Q and Liu X X 2011 *Chin. Phys. B* **20** 020307
[6] Sun Y B, Wen P, Yoon Y, Liu G Q, Steger M, Pfeiffer L N, West K, Snoke D W and Nelso K A 2017 *Phys. Rev. Lett.* **118** 016602
[7] Lü H, Zhu S B, Qian J and Wang Y Z 2015 *Chin. Phys. B* **24** 090308
[8] Weill R, Bekker A, Levit B and Fischer B 2019 *Nat. Commun.* **10** 747
[9] Klaers J, Schmitt J, Vewinger F and Weitz M 2010 *Nature* **468** 545

[10] Klaers J, Schmitt J, Damm T, Vewinger F and Weitz M 2012 *Phys. Rev. Lett.* **108** 160403
[11] van der Wurff E C I, de leeuw A W, Duine R A and Stoof H T C 2014 *Phys. Rev. Lett.* **113** 135301
[12] Weiss C and Tempere J 2016 *Phys. Rev. E* **94** 042124
[13] Cheng Z 2016 *Phys. Rev. A* **93** 023829
[14] Mendonca J T and Tecas H 2017 *Phys. Rev. A* **95** 063611
[15] Kruchkov A J 2016 *Phys. Rev. A* **93** 043817
[16] Rajan R, Babu P R and Senthilnathan K 2016 *Front. Phys.* **11** 110502
[17] Strinati M C and Conti C 2014 *Phys. Rev. A* **90** 043853
[18] Dung D, Kurtscheid C, Damn T, Schmitt J, Vewinger F, Weitz M and Klaers J 2017 *Nat. Photon.* **11** 565
[19] Marelic J and Nyman R A 2015 *Phys. Rev. A* **91** 033813
[20] Schmitt J, Damm T, Dung D, Vewinger F, klaers J and Weitz M 2015 *Phys. Rev. A* **92** 011602
[21] de Leeuw A W, Stoof H T C and Duine R A 2013 *Phys. Rev. A* **88** 033829
[22] Chiocchetta A and Carusotto I 2014 *Phys. Rev. A* **90** 023633
[23] Sob'yania D N 2013 *Phys. Rev. E* **88** 022132
[24] kirton P and Keeling J 2015 *Phys. Rev. A* **91** 033826
[25] Fontaine Q, Bienaime T, Pigeon S, Giacobino E, Bramati A and Glorieux Q 2018 *Phys. Rev. Lett.* **121** 183604
[26] Leboeuf P and Moulieras S 2010 *Phys. Rev. Lett.* **105** 163904
[27] Sela E, Rosch A and Fleurov V 2014 *Phys. Rev. A* **89** 043844
[28] Prakapenia M A and Vereshchagin G V 2019 *Europhys. Lett.* **128** 50002
[29] Vyas V M, Panigrahi P K and Srinivasan V 2016 arXiv:1602.08280 [physics.optics]
[30] Klaers J, Schmitt J, Damm T, Dung D, Vewinger F and Weitz M 2013 *Proc. SPIE Int. Soc. Opt. Eng.* **8600** 86000L
[31] Chiao R Y and Boyce J 1999 *Phys. Rev. A* **60** 4114
[32] Klaers J, Schmitt J, Damm T, Vewinger F and Weitz M 2011 *Appl. Phys. B* **105** 17
[33] Santos G, Tonel A, Foerster A and Links J 2016 *Phys. Rev. A* **73** 023609
[34] Cheng D, Li Y, Feng E and Huang W Y 2017 *Chin. Phys. B* **26** 013402
[35] Liu Y X and Zhao B 2020 *Chin. Phys. B* **29** 023103
[36] Tanzi L, Cabrera C R, Sanz J, Cheiney P, Tomza M and Tarruell L 2018 *Phys. Rev. A* **98** 062712
[37] Castin Y and Dum R 1998 *Phys. Rev. A* **57** 3008
[38] Li S C, Liu J and Fu L B 2011 *Phys. Rev. A* **83** 042107
[39] Li S C and Fu L B 2011 *Phys. Rev. A* **84** 023605
[40] Bogoliubov N 1947 *J. Phys. (USSR)* **11** 23
[41] Duine R A and Stoof H T C 2003 *Phys. Rev. Lett.* **91** 150405
[42] Scully M O and Zubairy M S 1997 *Quantum Optics* (Cambridge: Cambridge University) p. 206