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Dense coding capacity in correlated noisy channels with weak measurement*

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Capacity of dense coding via correlated noisy channel is greater than that via uncorrelated noisy channel. It is shown that the weak measurement and reversal measurement need to further improve their quantum dense coding capacity in correlated amplitude damping channel, but this improvement is very small in correlated phase damping channel and correlated depolarizing channel.

Keywords: correlated noise channel, quantum dense coding, weak measurement, reversal measurement

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1. Introduction

Since the concept of quantum dense coding (sometimes called superdense coding) was first introduced by Bennett and Wiesner,[1] this field has been discussed widely. To realize this communication protocol, it is essential to share an entangled state between the sender (Alice) and receiver (Bob) initially. The entangled state has the property that can be transformed by the sender into another state via a local operation by taking some sets of operations. Then Alice transmits her qubit to Bob, who performs an measurement on the global state together with the received qubit and his original one. The signal state that Alice sent is distinguished unambiguously by the measurement. Thus, the sending of one single qubit transmits two bits of classical information. This is absolutely impossible without entanglement; the amount of information conveyed by an isolated qubit cannot exceed one bit. Mattle *et al.*[2] have experimentally realized quantum dense coding in optical experiments on polarization-entangled photons.

It is straightforward to focus on the amount of classical information that a quantum state can reliably transmit, *i.e.*, the subsequent intensive dence coding capacity. It has been proved that for noiseless channels and unitary encoding, the dense coding capacity is given by

$$\chi = S(\bar{\rho}) - S(\rho_0), \tag{1}$$

where ρ_0 is the initial resource state shared between Alice and Bob, $\bar{\rho}$ represents the density matrix after quantum dense coding, S is the von Neumann entropy, and χ is the capacity of quantum dense coding. For the density matrix ρ_0, the von Neumann entropy is written explicitly is expressed as $S(\rho_0) = -\sum_Z \lambda_Z \log_2 \lambda_Z$, with λ_Z being the eigenvalue of the density matrix ρ_0. In recent decades, attention has been paid to many scenarios of super dense coding over noiseless channels.[3,4] Barenco and Ekert discussed the maximally entangled state and more general initial state of the two particles as well.[5,6] More general case for higher-dimensional[7] entanglement have been argued, which provides more capacity of quantum dense coding than conventional qubit entanglement. Dense coding for continuous variables has been argued by Braunstein and Kimbl.[8]

In addition, the case of uncorrelated noisy channel (*i.e.*, memoryless channel) has also been discussed.[9] Because noise exists unavoidably in reality, optical fibers and an unmodulated spin chain[10] are practically applied to such quantum noisy channels which are appropriate for long- and short-distance quantum communication, respectively. Physically, noise is a process that arises through the interaction with the environment. Mathematically, a noisy quantum channel can be described as a completely positive trace preserving (CPTP) linear map Λ, acting on the quantum state. For uncorrelated cases (channels and states) where the von Neumann entropy fulfills a specific condition, the superdense coding capacity was derived.

The above studies discussed the sequence of qubits passing through an uncorrelated channel by neglecting the correlations between multiple uses of quantum channels. However, neither the correlation effects of the quantum channel nor the transmission rate raising in quantum channel can be neglected, as it has been practically explored in the solid-state implementation of fiber[11] or quantum hardware that suffers low-frequency noise.[12] The quantum correlated channels have thus received a lot of attention recently.

In this work, we consider the quantum dense coding in correlated noisy channels. Three common noise sources are taken into account: the amplitude damping channel, phase damping channel, and depolarizing channel. Additionally, we

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analyze the effects of weak measurement and reversal measurement on the capacity of dense coding. The particular plan without weak measurement or reversal measurement is that the initial state directly passes through the correlated channel and then we make a dense coding. As an improvement in the former, the plan with weak measurement and reversal measurement is "weak measurement + correlated noise channel + reversal measurement + dense coding".

The rest of this paper is organized as follows. In Section 2, the correlated quantum channel and the quantum dense coding capacity are briefly introdiced. In Section 3, are discussed the influence of correlation strength of the channel and the initial entanglement on the capacity of quantum dense coding for different noisy models. The effects of weak measurement and reversal measurement on capacity are also discussed in this section. Finally, the conclusions are drawn from the present study in Section 4.

2. Correlated quantum channel and optimal dense coding capacity

Correlated noisy channel is the further approximation to actual transmission of memoryless noisy channel.[13] The general memoryless noise channel is first discussed. The effect of transmission channels is described by using the Kraus operator E_k, satisfing $\sum_k E_k^\dagger E_k = 1$. That means that sending one qubit in density operator ρ_0 through the channel the output state is determined by the map $\rho_0 \to \Lambda(\rho_0) = \sum_k E_k \rho_0 E_k^\dagger$. It is helpful to think about the example of Pauli channels in which the Kraus operators on individual qubit can be described by the Pauli operators $\sigma_{x,y,z}$, *i.e.*, $E_k = \sqrt{p_k}\sigma_k$, with $\sum_k p_k = 1$, $k = 0, x, y, z$ and $\sigma_0 = I$, *i.e.*, the identity operator of second order, where p_x, p_y, and p_z are the probabilities of the rotation of an angle π around the corresponding axes $\hat{x}$, $\hat{y}$, $\hat{z}$ in the qubit state, respectively, and p_0 is the probability of identity. A channel is memoryless when its effect on arbitrary state ρ, consisting of n qubits (including entangled ones), is described by $\Lambda(\rho) = \sum_{k_1\cdots k_n} \left(E_{k_n} \otimes \cdots \otimes E_{k_1}\right) \rho \left(E_{k_1}^\dagger \otimes \cdots \otimes E_{k_n}^\dagger\right)$. Pauli channels are given by Kraus operators in the following form: $E_{k_1\cdots k_n} = \sqrt{p_{k_1\cdots k_n}}\sigma_{k_1}\cdots\sigma_{k_n}$, with $\sum_{k_1\cdots k_n} p_{k_1\cdots k_n} = 1$, $k_i = 0, x, y, z$. The quantity $p_{k_1\cdots k_n}$ can be understood as the probability that a given random sequence of rotations of an angle π along axes $k_1\cdots k_n$ is applied to the sequence of n qubits sent through the channel. For a memoryless channel, $p_{k_1\cdots k_n} = p_{k_1} p_{k_2|k_1} \cdots p_{k_n|k_{n-1}}$, where $p_{k_n|k_{n-1}}$ can be explained as the conditional probability that a π rotation around the k_n axis is applied to the n-th qubit given that a π rotation around the k_{n-1} axis is exerted on the $(n-1)$-th qubit. Here we consider the case of two consecutive uses of a channel with partial memory, where the following equation is assumed:

$$p_{k_n|k_{n-1}} = (1-\mu)p_{k_n} + \mu\delta_{k_n k_{n-1}}. \tag{2}$$

This means that after the $(n-1)$-th qubit rotates an angle π around axis k_{n-1}, the n-th qubit implements the same rotation with probability μ or rotates an angle π around axis k_n with probability $(1-\mu)p_{k_n}$. Thus, $\mu \in [0,1]$ can be understood as the degree of classical correlation in the channel. When $\mu = 0$, the model depicts an uncorrelated channel, while for $\mu = 1$, the model depicts a fully correlated channel.

This so-called correlated noise channel can describe such situations in which time correlations cannot be neglected in the system. For example, μ is a function of time lapsing between the two channel uses. If two qubits are sent almost at the same time, the properties of the channel, which controls the rotation, will be unchanged, and it is rational to assume that the operators on both qubits will be in the manner of $E_k^{\rm c} = \sqrt{p_k}\sigma_k\sigma_k$, where the superscript "c" means "correlated", which means two qubits perform the same rotation of an angle π around axis k with probability p_k. It is easy to check that the Bell states $|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}\{|01\rangle \pm |10\rangle\}$, $|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}\{|00\rangle \pm |11\rangle\}$ are eigenstates of the operators $E_k^{\rm c}$, where $|0\rangle$ and $|1\rangle$ are eigenstates of the operator σ_z. As a result, they will pass undisturbedly through the correlated Pauli noisy channel.

If the time interval between the channel uses is so long that the channel properties have changed, then the operator will be $E_{k_1k_2}^{\rm u} = \sqrt{p_{k_1}}\sqrt{p_{k_2}}\sigma_{k_1}\sigma_{k_2}$, where the superscript "u" means "uncorrelated", which means that the first qubit rotates an angle π around axis k_1 with probability p_{k_1} while the second qubit rotates π around axis k_2 with probability p_{k_2}.

An intermediate case is described in the form of $E_{k_1k_2}^{\rm i} = \sqrt{p_{k_1}\left[(1-\mu)\,p_{k_2} + \mu\,\delta_{k_1k_2}\right]}\sigma_{k_1}\sigma_{k_2}$, where the superscript "i" means intermediate, which refers to "after the first qubit rotates an angle π around axis k_1 with probability p_{k_1}", and the second qubit rotates an angle π around axis $k_1$with probability μ or rotates an angle π around axis k_2 with probability $(1-\mu)p_{k_2}$. The Kraus operator for two consecutive uses of a general partial correlated channel (not only Pauli channel), is in the form of

$$E_{k_1k_2} = \sqrt{p_{k_1}[(1-\mu)p_{k_2} + \mu\,\delta_{k_1k_2}]}B_{k_1} \otimes B_{k_2}, \tag{3}$$

where B_{k_1} and B_{k_2} are the Kraus operators for a single qubit of the corresponding channel.

With the above description, we can obtain the final state of the system through a correlated noise channel as follows:

$$\rho_1 = (1-\mu)\sum_{k_1k_2} E_{k_1k_2}\rho_0 E_{k_1k_2}^\dagger + \mu\sum_k E_{kk}\rho_0 E_{kk}^\dagger, \tag{4}$$

where $E_{k_1k_2}$ represents the Kraus operator in uncorrelated noise channel and E_{kk} refers to that in the correlated noise channel.

Then we introduce the capacity for dense coding. Reference [14] showed that the d^2 signal state ($i_{\max} = d^2 - 1$) generated by mutually orthogonal unitary transformations with

equal probabilities yields a maximum value χ. The set of mutually orthogonal unitary transformations of dense coding for two qubits are

$$\begin{aligned} U_{00}|x\rangle &= |x\rangle, \\ U_{10}|x\rangle &= \mathrm{e}^{\mathrm{i}\pi x}|x\rangle, \\ U_{01}|x\rangle &= |x+1(\mathrm{mod}\ 2)\rangle, \\ U_{11}|x\rangle &= \mathrm{e}^{\mathrm{i}\pi x}|x+1(\mathrm{mod}\ 2)\rangle, \end{aligned} \tag{5}$$

where $|x\rangle$ is the single qubit computational basis. The average state of the ensemble of signal states generated by above equation is

$$\bar{\rho} = \frac{1}{4}\sum_{0}^{3}(U_i \otimes I)\rho\left(U_i^{\dagger}\otimes I\right), \tag{6}$$

where I is the second-order identity matrix and ρ represents the density matrix that needs to make dense coding. For simplicity, we suppose $0 \to 00,\ 1 \to 01,\ 2 \to 10$, and $3 \to 11$.

3. Capacity in three correlated noise channels

The capacity of dense coding through correlated noisy channel is given by

$$\chi = S(\bar{\rho}_1) - S(\rho_1), \tag{7}$$

where ρ_1 is given by Eq. (4) and $\bar{\rho}_1$ is given by Eq. (6). The initial entangled state is changed after having passed through the correlated noisy channel. So the density matrices that we use are different from Eq. (1). Below we mainly discuss the above problems in three common correlated noise channels: amplitude damping channel, phase damping channel, and depolarizing channel.

3.1. Capacity in correlated amplitude damping channel

The amplitude damping (AD) channel can also be used to describe the spontaneous emission of a photon by a two-level atom at low or zero temperature.[15,16] The Kraus operators for one qubit are

$$B_{k_1} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}, \quad B_{k_2} = \begin{pmatrix} 0 & \sqrt{1-\lambda} \\ 0 & 0 \end{pmatrix}, \tag{8}$$

where $\lambda = \mathrm{e}^{-\Gamma t}$ is the decay of the excited population and Γ is the dissipation rate. Two qubits transmitting through the partial correlated AD channel can be described by Eq. (4). We should pay attention to the Kraus operators $E_{k_1k_2}$ in uncorrelated AD channel for two-qubit transmission, which are considered in the following form:

$$E_{k_1k_2} = B_{k_1} \otimes B_{k_2}, \quad (k_1, k_2 = 0, 1), \tag{9}$$

while the Kraus operators E_{kk} in full-memory amplitude damping channel introduced in Ref. [17], different from $E_{k_1k_2}$ are given by

$$E_{00} = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&\sqrt{\lambda} \end{pmatrix}, \quad E_{11} = \begin{pmatrix} 0&0&0&\sqrt{1-\lambda} \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}. \tag{10}$$

In this paper, we take into account the initial state $\rho_0 = |\Phi\rangle\langle\Phi|$, where $\Phi = \alpha|00\rangle + \beta|11\rangle$ corresponds to the Bell-like state with $0 \le \alpha \le 1$. Since this communication protocol is based on quantum entanglement, let α and β be $\sqrt{2}/2$ in discussion in this paper, thereby being able to take advantage of maximum entanglement. According to Eq. (4), in the correlated AD channel, the evolved density matrix ρ_1 of the two-qubit system, whose elements in the standard computational basis $\{|1\rangle = |11\rangle, |2\rangle = |10\rangle, |3\rangle = |01\rangle, |4\rangle = |00\rangle\}$ are

$$\begin{aligned} \rho_{11} &= \alpha^2 - (-1+\lambda)\beta^2[1+\lambda(-1+\mu)], \\ \rho_{22} &= (-1+\lambda)\lambda\beta^2(-1+\mu), \\ \rho_{33} &= (-1+\lambda)\lambda\beta^2(-1+\mu), \\ \rho_{44} &= \lambda\beta^2(\lambda+\mu-\lambda\mu), \\ \rho_{14} &= \rho_{41} = -\lambda\alpha\beta(-1+\mu) + \sqrt{\lambda}\alpha\beta\mu. \end{aligned} \tag{11}$$

Based on Eq. (6), we can obtain the density matrix of $\bar{\rho}_1$. Then according to Eq. (7), the capacity of quantum dense coding can be obtained analytically, but it is too long to be written here. By confirming the capacity χ, the influences of the correlation strength of channel μ on capacity are depicted in Fig. 1. The blue line with empty squares in Fig. 1 corresponds to $\mu = 0$, which means that the correlation strength of channel is zero, and what is more, it corresponds to the uncorrelated channel. The purple line with empty circles, the yellow line with empty rhombuses, the green line with empty triangles and the light blue with empty inverted triangles represent $\mu = 0.25$, 0.5, 0.75, 1, respectively. From this figure, we can reasonably infer that the capacity of quantum dense coding increases monotonically with the increase of correlation strength μ when the damping coefficient λ of AD channel is fixed. So we can conclude that the correlated channel increases the capacity of quantum dense coding in comparison with the uncorrelated channel.

It is indicated in Ref. [18] that the weak measurement and reversal measurement improve the capacity of dense coding. So we consider the influence of weak measurement on capacity in correlated noise channel.

We know that standard quantum measuring procedure brings an initial state collapse to an eigenstate of the observable. Unlike standard measurement, weak measurement[19] makes so little influence on initial state that almost no initial state collapse happens to its eigenstate. Weak measurement and reversal measurement have been studied theoretically[20] and experimentally.[21]

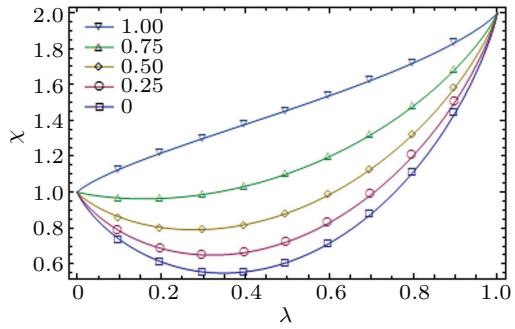


Fig. 1. Variations of capacity of quantum dense coding with damping coefficient λ in correlated amplitude damping channel, where $\alpha=\beta=\sqrt{2}/2$, and blue line with empty squares, purple line with empty circles, yellow line with empty rhombuses, green line with empty triangles, and light blue line with empty inverted triangles correspond to correlation strength of channel $\mu=0$, 0.25, 0.5, 0.75, 1, respectively.

Weak measurement operator and reversal measurement operator for two qubits can be written as

$$M_{\rm w}(m_1,m_2)=\begin{pmatrix}1&0\\0&\sqrt{1-m_1}\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&\sqrt{1-m_2}\end{pmatrix},$$
$$M_{\rm rev}(n_1,n_2)=\begin{pmatrix}\sqrt{1-n_1}&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}\sqrt{1-n_2}&0\\0&1\end{pmatrix}. \quad (12)$$

Here, $M_{\rm w}(m_1,m_2)$ is a weak measurement operator, m_1 and m_2 are the weak measurement strengths. In the same presentation, $M_{\rm rev}(n_1,n_2)$ is the reversal measurement operator, n_1 and n_2 are reversal measurement strengths. For simplicity, we assume that $m_1=m_2=m$ and $n_1=n_2=n$.

The capacity of quantum dense coding with weak measurement and reversal measurement is

$$\chi=S(\bar{\rho}_2)-S(\rho_2), \quad (13)$$

where ρ_2 is the density matrix in the "with weak measurement and reversal measurement" plan and $\bar{\rho}_2$ is that the matrix after dense coding given by Eq. (6). The density matrix can be described as

$$\rho_2=\frac{1}{T}\begin{pmatrix}\rho_{11}&0&0&\rho_{14}\\0&\rho_{22}&0&0\\0&0&\rho_{33}&0\\\rho_{41}&0&0&\rho_{44}\end{pmatrix}, \quad (14)$$

where T is the normalized factor. In this equation,

$$\begin{aligned}\rho_{11}&=(-1+n)^2(\alpha^2+(-1+\lambda)\\&\quad\times(-1+m)^2\beta^2(-1+\lambda-\lambda\mu)),\\\rho_{22}&=\rho_{33}=-(-1+\lambda)\lambda(-1+m)^2(-1+n)\beta^2(-1+\mu),\\\rho_{44}&=\lambda(\beta-m\beta)^2(\lambda+\mu-\lambda\mu),\\\rho_{14}&=\rho_{41}=\sqrt{\lambda}(1-m)(1-n)\alpha\beta\\&\quad\times(-\sqrt{\lambda}(-1+\mu)+\mu). \quad (15)\end{aligned}$$

According to Eq. (13), the capacity χ of quantum dense coding with weak measurement and reversal measurement is shown in Fig. 2. As for the research of the capacity of quantum dense coding under the influence of weak measurement and reversal measurement, we learn that no matter how the damping coefficient λ changes, the maximum value can be obtained by adjusting the weak measurement strength and reversal measurement strength. For simplicity, assume that the damping coefficient λ is a certain value, then we will discuss the effects of weak measurement and reversal measurement. To make it easier to understand, we use a point to analyze. For example, the capacity of quantum dense coding with weak measurement and reversal measurement is 1.7494 for weak measurement strength $m=0.9$ and reversal measurement strength $n=0.95$ when the damping coefficient of AD channel $\lambda=0.5$ and the correlation strength of channel $\mu=0.5$. Meanwhile, the capacity of quantum dense coding without weak measurement or reversal measurement is 0.8842. In this case, the capacity of quantum dense coding under the weak measurement and reversal measurement is greater than that without weak measurement or reversal measurement. Furthermore, the method for weak measurement and reversal measurement can be used for different values of damping coefficient λ, which can make dense coding successful and improve the capacity of dense coding in AD channel.

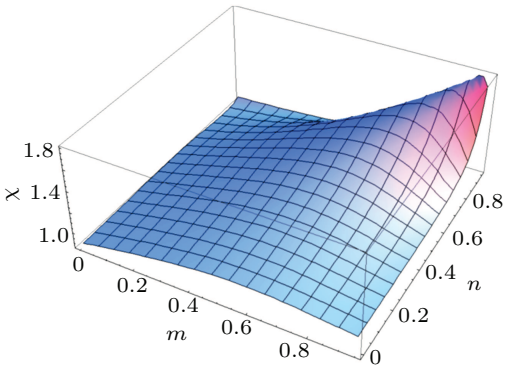


Fig. 2. Capacity of quantum dense coding *versus* weak measurement strength m and reversal measurement strength n, with damping coefficient λ being 0.5, correlation strength μ being 0.5, and $\alpha=\beta=\sqrt{2}/2$.

3.2. Capacity in correlated phase damping channel

The phase damping channel describes a decoherencing process without exchanging energy with the environment. The Kraus operators of a single qubit in the phase damped channel are represented as the Pauli operators $\sqrt{p_0}\sigma_0$ and $\sqrt{p_z}\sigma_z$. As we discussed in Section 2, the Kraus operators for two qubits passing through a memoryless phase damping channel can be written as

$$E_{k_1k_2}=\sqrt{p_{k_1}p_{k_2}}\sigma_{k_1}\otimes\sigma_{k_2}, \quad (16)$$

where k_1, $k_2=(0,z)$, $p_0=(1+\lambda)/2$, $p_z=(1-\lambda)/2$, and $\lambda=\exp(-\Gamma t)$. For the partial correlated phase damping channel, the Kraus operators E_{kk} are given as

$$E_{kk}=\sqrt{p_k}\sigma_k\otimes\sigma_k,\quad (k=0,z). \quad (17)$$

Substituting Eqs. (14) and (15) into Eq. (4), the density matrix elements of the two qubits in the correlated phase damped channel can be expressed in the following form:

$$\begin{aligned}\rho_{11} &= \alpha^2,\\ \rho_{22} &= \beta^2,\\ \rho_{14} &= \rho_{41} = \alpha\beta\left(-\lambda^2(-1+\mu)+\mu\right).\end{aligned} \tag{18}$$

Based on Eq. (7), the capacity of quantum dense coding can be obtained analytically. The influences of the correlation strength of channel μ on capacity are depicted in Fig. 3. The blue line with empty squares in Fig. 3 corresponds to the case $\mu = 0$, which refers to the uncorrelated channel. The purple line with empty circles, the yellow line with empty rhombuses, the green line with empty triangles and the light blue line with empty inverted triangles refers to $\mu = 0.25$, 0.5, 0.75, 1, respectively. On the other hand, as discussed in Section 2, the Bell states are eigenstates of the Kraus operators, E_k^c, in correlated Pauli channel. As a result, they can pass through undisturbed channel via the full-correlated channel. So when $\mu = 1$, *i.e.*, the channel becomes full-correlated, the capacity of dense coding χ is kept at 2. Based on the above discussion, we can reasonably infer that the capacity of quantum dense coding increases monotonically with the correlation strength μ increasing when the phase damping coefficient λ is fixed. So we can conclude that the correlated channel increases the capacity of quantum dense coding in comparison with the uncorrelated channel.

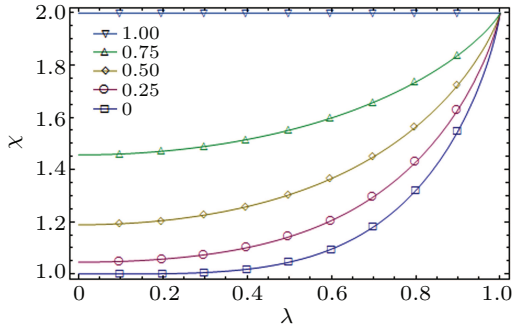


Fig. 3. Variations of capacity of quantum dense coding with phase damping coefficient λ, where $\alpha = \beta = \sqrt{2}/2$. The blue line with solid squares, purple line with empty circles, yellow line with empty rhombuses, green line with empty triangles, and light blue line with empty inverted triangles correspond to the correlation strength of channel $\mu = 0$, 0.25, 0.5, 0.75, 1, respectively.

We discuss how the weak measurement and reversal measurement affect the capacity of quantum dense coding in phase damped channel as we did in the above section. Besides the normalized factor, the density matrix is

$$\begin{aligned}\rho_{11} &= (-1+n)^2\alpha^2,\\ \rho_{44} &= (-1+m)^2\beta^2,\\ \rho_{14} &= \rho_{41} = (-1+m)(1-n)\alpha\beta(\lambda^2(-1+\mu)-\mu).\end{aligned} \tag{19}$$

According to Eq. (13), the capacity χ of quantum dense coding with weak measurement and reversal measurement can be obtained analytically, but it is too long to be written here. Because the weak measurement strength $m = 1$ and reversal measurement strength $n = 1$ is a singular point, we are difficult to find the maximum value of the capacity of quantum dense coding in phase damped channel. For simplicity, we use a point to analyze it. For example, the capacity with weak measurement is larger than that without weak measurement for $m = 0.95$ and $n = 0.95$ when the phase damping coefficient $\lambda = 0.5$ and correlation parameter $\mu = 0.5$. But we find the influence of weak measurement and reversal measurement on the capacity to be so weak that it can be neglected.

3.3. Capacity in correlated depolarizing channel

The depolarizing noise is the quantum operation that depolarizes the state into a completely mixed state. The Kraus operators for a single qubit are $B_k = \sqrt{p_k}\sigma_k$ $(k = 0, x, y, z)$, where $p_0 = (1+\lambda)/2$, $p_x = p_y = p_z = (1-\lambda)/6$, and $\lambda = \exp(-\Gamma t)$. Assuming the time interval between channel consecutive applications to the two qubits is very small, the correlated depolarizing channel model is applicable. At this point, the Kraus operators $E_{i_1 i_2}$ can be written as

$$E_{k_1 k_2} = \sqrt{p_{k_1} p_{k_2}}\sigma_{k_1} \otimes \sigma_{k_2}, \quad (k_1, k_2 = 0, x, y, z). \tag{20}$$

The Kraus operators E_{kk} are given as

$$E_{kk} = \sqrt{p_k}\sigma_k \otimes \sigma_k, \quad (k_1, k_2 = 0, x, y, z). \tag{21}$$

According to Eq. (4), the density matrix elements of the two qubits are

$$\begin{aligned}\rho_{11} &= \frac{1}{9}\big(-(2+\lambda)\alpha^2(-2+\lambda(-1+\mu)-\mu)\\ &\quad -(-1+\lambda)\beta^2(1+\lambda(-1+\mu)+2\mu)\big),\\ \rho_{22} &= \rho_{33} = \frac{1}{9}\left(-2+\lambda+\lambda^2\right)(-1+\mu),\\ \rho_{44} &= \frac{1}{9}\big(-(-1+\lambda)\alpha^2(1+\lambda(-1+\mu)+2\mu)\\ &\quad +\beta^2\big((2+\lambda)^2-\left(-2+\lambda+\lambda^2\right)\mu\big)\big),\\ \rho_{14} &= \rho_{41} = \frac{1}{9}\alpha\beta(1-4\lambda(1+\lambda)(-1+\mu)+8\mu).\end{aligned} \tag{22}$$

Based on Eq. (7), the capacity of quantum dense coding can be obtained analytically, but the expression is too long to be written here. The influences of the correlation strength of channel μ on capacity are depicted in Fig. 4. The blue line in Fig. 4 corresponds to $\mu = 0$, which corresponds to the uncorrelated channel. The purple line with empty circles, the yellow line with empty rhombuses, the green line with empty triangles, and the light blue line with empty inverted triangles refer to $\mu = 0.25$, 0.5, 0.75, 1, respectively. This correlated noise channel also adapts to the situation in Section 2, *i.e.*, the Bell states are eigenstates of the Kraus operators E_k^c in the correlated Pauli channel. As a result, they can pass through the undisturbed channel via the full-correlated channel. So when

$\mu = 1$, *i.e.*, the channel becomes full-correlated, the capacity of dense coding χ is kept at 2. Like the above part, we can reasonably infer that the capacity of quantum dense coding increases monotonically with correlation strength μ increasing when the phase damping coefficient λ is fixed. So we can conclude that the correlated channel increases the capacity of quantum dense coding in comparison with the uncorrelated channel.

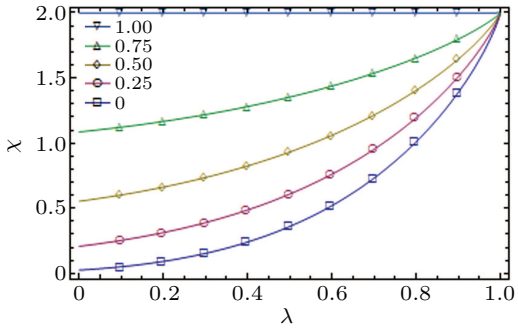


Fig. 4. Variations of capacity of quantum dense coding with depolarizing damping coefficient λ, where $\alpha = \beta = \sqrt{2}/2$, blue line with empty squares, purple line with empty circles, yellow line with empty rhombuses, green line with empty triangles, and light blue line with empty inversed triangle correspond to the correlation strength of channel $\mu = 0, 0.25, 0.5, 0.75, 1$, respectively.

Then we discuss the influence of weak measurement and reversal measurement on the capacity.

$$\begin{aligned}
\rho_{11} &= (-1+n)^2\Big(-\frac{1}{9}(2+\lambda)\alpha^2(-2+\lambda(-1+\mu)-\mu)\\
&\quad -\frac{1}{9}(-1+\lambda)(-1+m)^2\beta^2(1+\lambda(-1+\mu)+2\mu)\Big),\\
\rho_{22} &= \rho_{33} = -\frac{1}{9}(-2+\lambda+\lambda^2)(-1+n)\\
&\quad \times(\alpha^2+(-1+m)^2\beta^2)(-1+\mu),\\
\rho_{44} &= \frac{1}{9}((1-\lambda)\alpha^2(1+\lambda(-1+\mu)+2\mu)\\
&\quad +(-1+m)^2\beta^2((2+\lambda)^2-(-2+\lambda+\lambda^2)\mu)),\\
\rho_{14} &= \rho_{41} = -\frac{1}{9}(1-m)(1-n)\alpha\beta\\
&\quad \times(-1+4\lambda(1+\lambda)(-1+\mu)-8\mu). \qquad (23)
\end{aligned}$$

We find that the capacity with weak measurement is larger than without weak measurement for $m = 0.99$ and $n = 0.99$ when the depolarizing damping coefficient $\lambda = 0.5$ and correlation strength $\mu = 0.5$. But the influence of weak measurement and reversal measurement on the capacity is very weak, specifically on the order of 10^{-2}.

4. Conclusions

In this paper, we studied the dense coding capacity in correlated noise channels and the influence of weak measurement and reversal measurement on the capacity. We find two conclusions: the first one is that the correlated noisy channel can improve the capacity of dense coding in comparison with uncorrelated noisy channel. It is easy to understand in physics. The correlated noisy channel is the situation where time correlations cannot be neglected. It means that two qubits are sent almost at the same time, and the channel properties will be unchanged. But time lapse is larger in uncorrelated noisy channel. So the capacity of the correlated noisy channel is larger than that of uncorrelated noisy channel. The second conclusion is that the weak measurement and reversal measurement can further improve the capacity of dense coding in correlated amplitude damping channel, but this improvement is very small in correlated phase damping channel and correlated depolarizing channel. The explanations about this are correlated with the amplitude damping channel that is non-Pauli channel, but correlated phase damping channel and correlated depolarizing channel are Pauli channel, so the improvements by weak measurement and reversal measurement are different.

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