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Dynamics of bright soliton in a spin–orbit coupled spin-1 Bose–Einstein condensate*

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We have investigated the dynamics of bright solitons in a spin–orbit coupled spin-1 Bose–Einstein condensate analytically and numerically. By using the hyperbolic sine function as the trial function to describe a plane wave bright soliton with a single finite momentum, we have derived the motion equations of soliton's spin and center of mass, and obtained its exact analytical solutions. Our results show that the spin–orbit coupling couples the soliton's spin with its center-of-mass motion, the spin oscillations induced by the exchange of atoms between components result in the periodical oscillation of center-of-mass, and the motion of center of mass of soliton can be viewed as a superposition of periodical and linear motions. Our analytical results have also been confirmed by the direct numerical simulations of Gross–Pitaevskii equations.

Keywords: Bose–Einstein condensate, spin–orbit coupling, soliton

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Soliton as nonlinear wave has been one of the most important research objects in nonlinear science due to its unique physical properties and potential application. In particular, with the realization of Bose–Einstein condensation (BEC) and degenerated Fermi gases in ultracold atoms, experimental and theoretical investigations show that the nonlinear atomic interactions will also result in the excitations of matter wave solitons, which have drawn enormous research interest due to the potential applications in coherent atom optics, atom interferometry and atom transport. In BEC, dark and bright solitons have been observed successively in experiments.[1–6] Recently, the MIT group has also created dark soliton and vortex soliton in ultracold Fermi gases.[7,8]

The realization of artificial spin–orbit coupling (SOC) in ultracold atoms provides a new platform to study the nonlinear matter wave solitons inside a gauge field.[9–13] Since the SOC leads to degeneracy of single-particle ground state at finite momentum,[14–22] many novel matter wave solitons can be formed, such as stripe soliton, two-dimensional (2D) composite soliton, and half-vortex gap soliton.[23–44] The Majorana fermions inside a soliton have also been predicted in Fermi superfluids with SOC.[45] On the other hand, in multi-component BEC, SOC couples different components and induce the population transfer between components, which will affect the center of mass (CoM) motion of solitons and lead to interesting dynamical evolution features.[46–51]

To further understand the effects of SOC on the dynamics of solitons, in this letter, we systematically study the dynamics of the bright solitons with a single finite momentum in the experimentally realized one-dimensional spin–orbit coupled spin-1 BEC. By defining the soliton spin and using the hyperbolic sine function as the trial function to describe a plane wave bright soliton with a single finite momentum, we first derive the motion equations of soliton's spin and center-of-mass based on the Gross–Pitaevskii (GP) equations, which show clearly that SOC couples the soliton's spin with its center-of-mass motion. In particular, we have obtained the exact solutions of soliton's spin and CoM, and find that the spin oscillations caused by the exchange of atoms between components leading to a periodical oscillation of CoM. To confirm these analytical predictions, we have also solved the GP equations numerically, and find that the numerical results agree well with the analytical results.

We consider a spin-1 BEC of ^{87}Rb atom, where two counter-propagating Raman laser beams in z direction couple $|F,m_F\rangle = |1,-1\rangle$ to $|F,m_F\rangle = |1,0\rangle$, and couple $|F,m_F\rangle = |1,0\rangle$ level to $|F,m_F\rangle = |1,1\rangle$. Since the linear and quadratic Zeeman energies can be tuned by using the external magnetic field and microwave dressing,[52,53] here we setting the linear Zeeman energy $q = -\delta\omega$ with $\delta\omega$ being the frequency differ-

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ence of two Raman beams, and the quadratic Zeeman energy $q=-2k_{\rm R}^2$ with the wave vector $k_{\rm R}$ of Raman beams. In this case, the single-particle Hamiltonian can be written as

$$\hat{H}_0=\frac{\hat{k}_x^2+\hat{k}_y^2+\hat{k}_z^2}{2m}\hat{\boldsymbol{I}}+\frac{2\hbar k_{\rm R}}{m}\hat{k}_z\hat{f}_z+\hbar\Omega_{\rm R}\hat{f}_x, \tag{1}$$

where $\hat{k}_\alpha=-{\rm i}\hbar\partial_\alpha$ with $\alpha=x,y,z$, $\hat{f}_x$ and $\hat{f}_z$ are the 3×3 spin-1 matrices, $\hat{\boldsymbol{I}}$ represents the 3×3 identity matrix. $\Omega_{\rm R}$ is the strength of Raman beams, $k_{\rm R}$ is the strength of SOC related to the wave number of Raman beams.

Assuming the BEC is in the ground state of a tightly harmonic trap with frequency $\omega_\perp$ in xy plane, and measuring the length and time in units of $\xi_\perp=\sqrt{\hbar/m\omega_\perp}$ and $1/\omega_\perp$, respectively. Within mean-field theory at zero-temperature, the dynamics of the uniform BEC along the z direction can be described by the following dimensionless GP equations taking the atomic interactions into account

$$\begin{aligned}{\rm i}\frac{\partial\psi_{+1}}{\partial t}&=-\frac{1}{2}\frac{\partial^2\psi_{+1}}{\partial z^2}-{\rm i}\lambda\frac{\partial\psi_{+1}}{\partial z}+\frac{\Omega}{\sqrt{2}}\psi_0\\&\quad+\left[c_0n+c_2(n_{+1}+n_0-n_{-1})\right]\psi_{+1}\\&\quad+c_2\psi_{-1}^*\psi_0^2,\end{aligned} \tag{2a}$$

$$\begin{aligned}{\rm i}\frac{\partial\psi_0}{\partial t}&=-\frac{1}{2}\frac{\partial^2\psi_0}{\partial z^2}+\frac{\Omega}{\sqrt{2}}(\psi_{+1}+\psi_{-1})\\&\quad+\left[c_0n+c_2(n_{+1}+n_{-1})\right]\psi_0\\&\quad+2c_2\psi_{+1}\psi_{-1}\psi_0^*,\end{aligned} \tag{2b}$$

$$\begin{aligned}{\rm i}\frac{\partial\psi_{-1}}{\partial t}&=-\frac{1}{2}\frac{\partial^2\psi_{-1}}{\partial z^2}+{\rm i}\lambda\frac{\partial\psi_{-1}}{\partial z}+\frac{\Omega}{\sqrt{2}}\psi_0\\&\quad+\left[c_0n+c_2(n_0+n_{-1}-n_{+1})\right]\psi_{-1}\\&\quad+c_2\psi_{+1}^*\psi_0^2,\end{aligned} \tag{2c}$$

where the total density distribution reads $n=\sum_{m=-1,0,+1}n_m$ with $n_m=|\psi_m(z,t)|^2$, the wave function $\psi_m(z,t)$ satisfies the normalized condition $\int_{-\infty}^{+\infty}n{\rm d}z=1$, the superscript $*$ in GP Eq. (2) denotes the complex conjugate. The rescaled strengths of SOC and Raman coupling are $\lambda=2k_{\rm R}\xi_\perp$ and $\Omega=\Omega_{\rm R}/\omega_\perp$, respectively. The nonlinear coefficients c_0 and c_2 are the strengths of interactions in the spin-0 and spin-2 channels, respectively. Since the strengths and signs of c_0 and c_2 can be tuned using optical Feshbach resonances, we mainly concentrate the case of $c_0<0$ with positive and negative values of c_2, and study the dynamics of bright solitons under action of SOC and Raman coupling.

From the single-particle Hamiltonian, for $\Omega<\lambda^2$, the single-particle ground state is doubly degenerated at the momentum $k=\pm\lambda\sqrt{1-\Omega^2/\lambda^4}$, there are the so-called plane wave soliton and stripe stationary solitons with oscillating density modulation,[23,24] the former occupies a single momentum, while the latter can be viewed as the linear superposition between two plane wave solitons with opposite momentum. To deeply understand the effects of SOC on the dynamics of bright solitons, in what follows we will mainly investigate the dynamics of the plane wave bright soliton.

Without SOC, the GP equations have exact analytical soliton solutions under the special parameters constraint. However, for our system, the SOC breaks the integrability of GP equations, it is impossible to construct the exact analytical soliton solutions.[54,55] Therefore, we employ the variational method with all parameters being time-dependent. In general, a bright soliton can be described by the hyperbolic sine function in a uniform BEC, thus for a plane wave bright soliton carrying a finite momentum k_z, the variational wave functions are assumed to be

$$\psi_m=A_m\sqrt{\frac{\eta}{2}}{\rm sech}\left[\eta\left(z-z_{\rm c}\right)\right]{\rm e}^{{\rm i}k_zz}, \tag{3}$$

where A_m is related to the amplitude of the bright solitons in three different components and satisfying the normalized condition $\sum_m|A_m|^2=1$. k_z is the wave number corresponding to the average momentum of BEC, η and $z_{\rm c}$ are the width and CoM of solitons, respectively.

We further define the soliton spin as

$$S_{x,y,z}=\langle\hat{f}_{x,y,z}\rangle=\int_{-\infty}^{+\infty}\Psi^\dagger\hat{f}_{x,y,z}\Psi{\rm d}z$$

with $\Psi=(\psi_+,\psi_0,\psi_-)^{\rm T}$, where $\hat{f}_{x,y,z}$ is the spin operator for spin-1 BEC and the superscript $\dagger$ denotes the conjugate transpose. In this case, the soliton spin can be expressed as follows:

$$S_x=\frac{1}{\sqrt{2}}\left[(A_{+1}+A_{-1})A_0^*+\left(A_{+1}^*+A_{-1}^*\right)A_0\right], \tag{4a}$$

$$S_y=\frac{\rm i}{\sqrt{2}}\left[(A_{+1}-A_{-1})A_0^*+\left(A_{-1}^*-A_{+1}^*\right)A_0\right], \tag{4b}$$

$$S_z=|A_{+1}|^2-|A_{-1}|^2, \tag{4c}$$

which show that the dynamics of soliton spin are completely determined by the amplitude of solitons.

Since the interacting Hamiltonian $\hat{H}_{\rm int}$ conserves the spin ($[\hat{f}_{x,y,z},\hat{H}_{\rm int}]=0$, where the bracket denote the commutation operation), so from the Heisenberg equation

$$\frac{{\rm d}\hat{f}_{x,y,z}}{{\rm d}t}=-{\rm i}[\hat{f}_{x,y,z},\hat{H}]$$

with $\hat{H}=\hat{H}_0+\hat{H}_{\rm int}$ being the total Hamiltonian, we can observe that the dynamics of soliton spin should be determined completely by the single-particle Hamiltonian $\hat{H}_0$. To confirmed this point, we take the derivative of both sides of Eq. (4) with respect to time t and combine with the GP equation (2) by inserting the wave function (3), so after integrating the spatial coordinate z, the motion equations of soliton spin can be obtained

$$\frac{{\rm d}S_x}{{\rm d}t}=-\lambda k_zS_y, \tag{5a}$$

$$\frac{{\rm d}S_y}{{\rm d}t}=\lambda k_zS_x-\Omega S_z, \tag{5b}$$

$$\frac{\mathrm{d}S_z}{\mathrm{d}t} = \Omega S_y, \tag{5c}$$

where verifies that the soliton spin along x, y, and z directions are coupled by SOC and Raman coupling, unrelated to the atomic interactions.

We note that since the system is dynamical-invariant, it can be verified that the momentum k_z is conserved, thus the equation (5) can be solved exactly as

$$S_x = -\frac{a_3\lambda k_z}{\omega}\sin(\omega t) + \frac{a_2\lambda k_z}{\omega}\cos(\omega t) + a_1, \tag{6a}$$

$$S_y = a_2\sin(\omega t) + a_3\cos(\omega t), \tag{6b}$$

$$S_z = \frac{a_3\Omega}{\omega}\sin(\omega t) - \frac{a_2\Omega}{\omega}\cos(\omega t) + \frac{a_1\lambda k_z}{\Omega}, \tag{6c}$$

where $\omega = \sqrt{\Omega^2 + \lambda^2 k_z^2}$. The constants are

$$a_1 = \Omega\left(\Omega S_{x,0} + \lambda k_z S_{z,0}\right)/\omega^2,\ \ a_2 = \left(\lambda k_z S_{x,0} - \Omega S_{z,0}\right)/\omega,$$

and $a_3 = S_{y,0}$ with subscript 0 being the initial value. The solution (6) shows that the spin of soliton may oscillate periodically with the frequency related to the strengths of SOC and Raman coupling.

To proceed, we derive the motion equation of soliton's CoM by using

$$\frac{\mathrm{d}z_\mathrm{c}}{\mathrm{d}t} = \int_{-\infty}^{+\infty} z\frac{\mathrm{d}n}{\mathrm{d}t}\mathrm{d}z.$$

From the GP equations (2), one can get

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \sum_{m=-1,0,+1}\left[\mathrm{i}\frac{1}{2}\left(\psi_m^*\frac{\partial^2\psi_m}{\partial z^2} - \psi_m\frac{\partial^2\psi_m^*}{\partial z^2}\right) - \lambda m\left(\psi_m\frac{\partial\psi_m^*}{\partial z} + \psi_m^*\frac{\partial\psi_m}{\partial z}\right)\right]. \tag{7}$$

So after inserting the variational wave functions (3) into Eq. (7) and performing the integral in the motion equation of soliton's CoM, we can obtain

$$\frac{\mathrm{d}z_\mathrm{c}}{\mathrm{d}t} = k_z + \lambda S_z, \tag{8}$$

which indicates that the spin–orbit coupling couples the soliton's spin with its motion of center of mass. In particular, Equations (5) and (8) leads to $\mathrm{d}^2 z_\mathrm{c}/\mathrm{d}t^2 = \Omega\lambda S_y$, which implies that the solitons will experience a driven force provided by S_y. Consequently, we conclude that the presence of SOC will lead to interesting dynamical evolution of solitons. On the other hand, with the solution of S_z in Eq. (6), the solutions of the soliton's CoM in Eq. (8) can also be derived exactly as

$$z_\mathrm{c} = -\frac{\lambda\Omega}{\omega^2}\left[a_2\sin(\omega t) + a_3\cos(\omega t)\right] + \left(\frac{a_1\lambda^2}{\Omega} + 1\right)k_z t + \frac{a_3\lambda\Omega}{\omega^2}, \tag{9}$$

where we have assumed $z_\mathrm{c} = 0$ at $t = 0$. It is easy to see that in the absence of SOC ($\lambda = 0$), the CoM velocity is a constant, and the Raman coupling only gives rise to a well-known Rabi oscillations with frequency Ω. While for $\lambda \neq 0$, the motion of soliton's center of mass can be viewed as a superposition between the linear motion and periodical motion with frequency ω.

To confirm our analytical results and excavate the underlying physics given by the SOC, we solve the GP Eq. (2) numerically by using the time-splitting Fourier spectral method.[56] The initial wave function is chosen as the dynamically stable exact ferromagnetic bright soliton solutions of GP Eq. (2) in the absence of SOC and Raman coupling, which are $\psi_{+1} = \sqrt{\eta/2}\,\mathrm{sech}\left[\eta\left(k_z t - z\right)\right]\exp\left[\mathrm{i}\left(k_z z + \eta^2 t/2\right)\right]$ and $\psi_0 = \psi_{-1} = 0$ with $\eta = -\left(c_0 + c_2\right)/2 > 0$,[54,55] corresponding to $S_{z,0} = 1$ and $S_{x,0} = S_{y,0} = 0$. In this case, we can derive

$$S_x = \frac{2\Omega\lambda k_z}{\omega^2}\sin^2\left(\frac{\omega}{2}t\right), \tag{10a}$$

$$S_y = -\frac{\Omega}{\omega}\sin(\omega t), \tag{10b}$$

$$S_z = \frac{\lambda^2 k_z^2}{\omega^2} + \frac{\Omega^2}{\omega^2}\cos(\omega t), \tag{10c}$$

$$z_\mathrm{c} = \frac{\lambda\Omega^2}{\omega^3}\sin(\omega t) + \frac{\lambda^3 k_z + \omega^2}{\omega^2}k_z t. \tag{10d}$$

Figures 1 and 2 show the numerical results of GP simulations with the initial ferromagnetic solitons for $k_z = 0$ and $k_z \neq 0$, respectively. A common feature shown in Figs. 1 and 2 is that although the components $m = 0$ and $m = -1$ have no solitons initially, the Raman coupling will produce bright solitons in such two components, and gives rise to the Rabi oscillation with the periodical exchange of atom among the three components. More specifically, the two cases with $k_z = 0$ and $k_z \neq 0$ also show different features. For $k_z = 0$, we have $S_x = 0$ and $\omega = \Omega$. As shown in Figs. 1(e)–1(f) and Eq. (10), S_y and S_z oscillate periodically with frequency ω, the spin of solitons rotate periodically around an unit circle in S_y–S_z plane, and the solitons move periodically around $z = 0$ under the action of SOC. For the case of $k_z \neq 0$, the initial soliton carries a finite momentum, it not only oscillates but also propagates along z direction, as shown in Fig. 2 and Eq. (9), where the motion of soliton can be viewed as a superposition between periodical and linear motions. Moreover, the S_x also oscillate with frequency $\omega/4$ for $k_z \neq 0$, and the soliton spin will rotate periodically along a closed circle in three-dimensional spin space, as shown in Fig. 2(f). Last but not least, we have also chosen other initial conditions to carry out the numerical simulations, and the results show that our analytical analysis agrees well GP simulations and capture the essential physics of the dynamics of soliton in the presence of SOC.

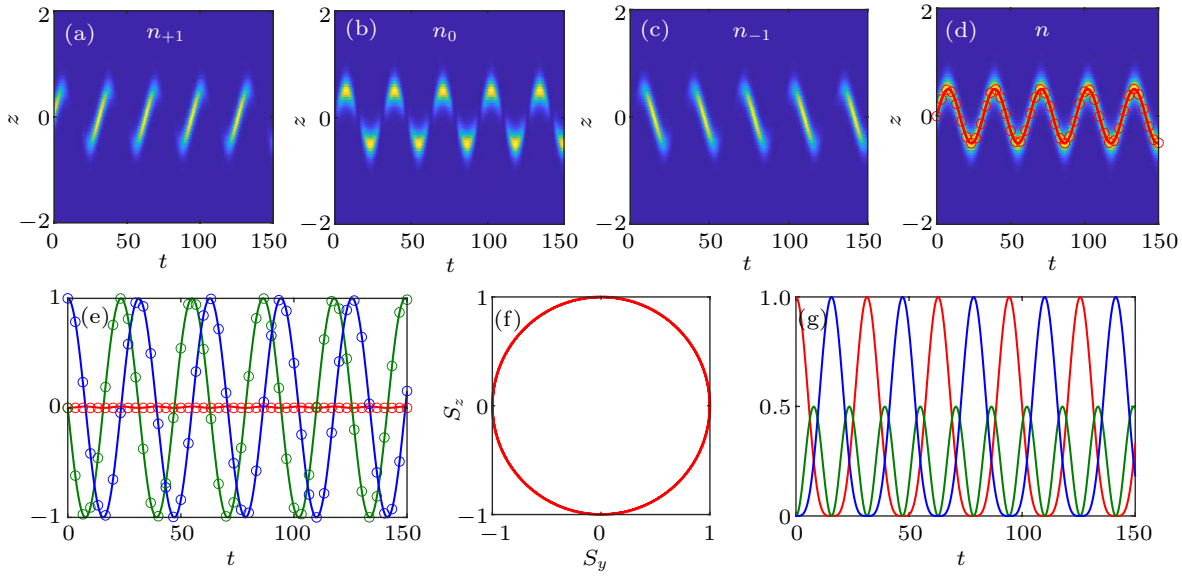


Fig. 1. The dynamical evolutions of initial ferromagnetic solitons with $k_z = 0$ under the action of SOC, where the parameters are $c_0 = -10$, $c_2 = -1$, $\lambda = 0.1$, and $\Omega = 0.2$. Panels (a)–(c) show the evolutions of densities of three components, respectively. Panel (d) shows the evolution of total density. The red solid line and circles in panel (d) are the center-of-mass of solitons given by the numerical solutions of GP equations and the exact solutions in Eq. (9), respectively. Panel (e) shows the evolutions of $S_{x,y,z}$, the solid red, blue, and green lines represent the numerical results of S_x, S_y, and S_z, respectively, and the corresponding circles are given by exact solutions of Eq. (10), respectively. Panel (f) shows the evolution of spin in the S_y–S_z plane. In panel (g), the red, green, and blue lines represent the evolutions of particle number in the components $m = +1, 0, -1$, respectively.

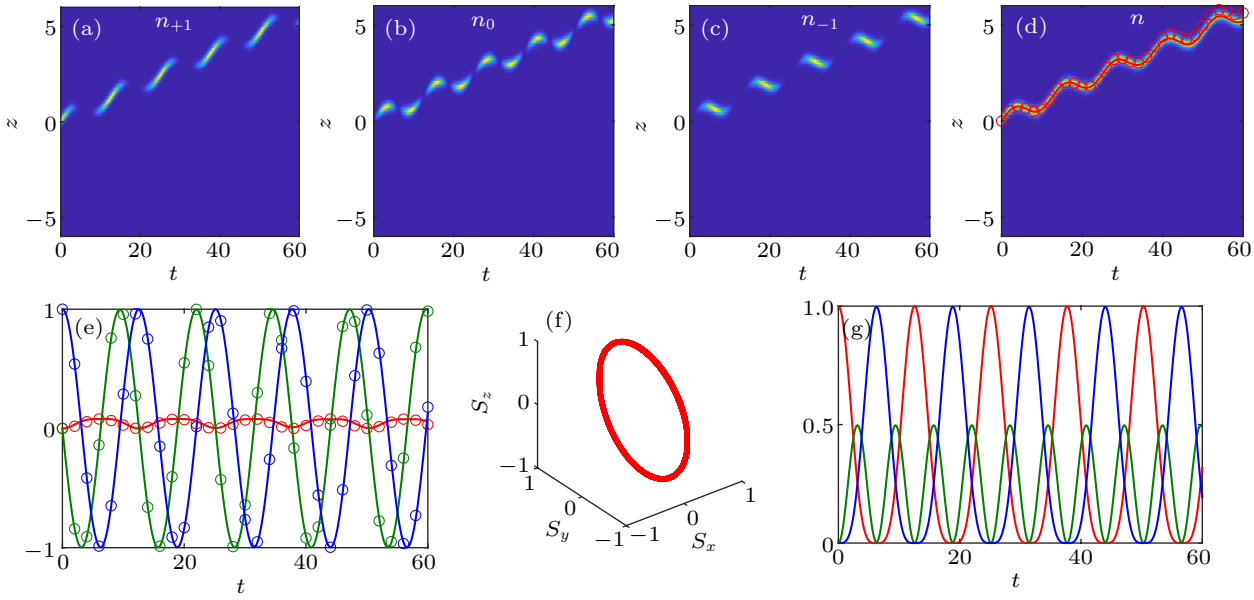


Fig. 2. The dynamical evolutions of initial ferromagnetic solitons with $k_z = 0.1$ under the action of SOC, where the parameters are $c_0 = -10$, $c_2 = -1$, $\Omega = 0.5$, and $\lambda = 0.2$. Panels (a)–(c) show the evolutions of densities of three components, respectively. Panel (d) shows the evolution of total density. The red solid line and circles in panel (d) are the center-of-mass of solitons given by the numerical solutions of GP equations and the exact solutions of Eq. (9), respectively. Panel (e) shows the evolutions of $S_{x,y,z}$, the solid red, blue, and green lines represent the numerical results of S_x, S_y, and S_z, respectively, and the corresponding circles are given by exact solutions of Eq. (10), respectively. Panel (f) shows the evolution of spin in the three-dimensional spin space. In panel (g), the red, green, and blue lines represent the evolutions of particle number in the components $m = +1, 0, -1$, respectively.

In summary, we have investigated the effects of SOC on the dynamics of plane wave bright solitons in spin-1 BEC. By using the hyperbolic sine function as the trial function to describe a plane wave bright soliton with a single finite momentum and defining the soliton's average spin, we have derived the motion equations of soliton's spin and center of mass based on the GP equations, and obtained its exact analytical solutions. Our results show that the SOC couples the soliton's spin with its CoM coordinate effectively, the spin oscillations induced by the exchange of atoms between different components results in an effective force to drive the soliton oscillate with frequency related to the strengths of SOC and Raman coupling. Moreover, the motion of CoM of soliton can be viewed as a superposition between periodical and linear motions. Finally, we have also taken the exact ferromagnetic soliton solutions of GP equation without SOC and Raman coupling as initial wave function to solve the GP equation numerically, and the results agree well with our analytical analysis.

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