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Optimized monogamy and polygamy inequalities for multipartite qubit entanglement*

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We investigate the monogamy and polygamy inequalities of arbitrary multipartite quantum states, and provide new classes of monogamy and polygamy inequalities of multiqubit entanglement in terms o f concurrence, entanglement of formation, negativity, and Tsallis-q entanglement, respectively. We show that these new monogamy and polygamy inequality relations are tighter than the existing ones with detailed examples.

Keywords: monogamy inequalities, polygamy inequalities, entanglement measures

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1. Introduction

Quantum entanglement is a quintessential manifestation of the quantum mechanics. It is an essential feature of quantum mechanics which distinguishes the quantum from the classical world. While the entanglement in bipartite quantum systems has been intensively studied with rich understanding, the entanglement of multipartite quantum systems is less known in characterization and quantification. One interesting feature of multipartite entanglement is known as the monogamy of entanglement (MOE),[1,2] that is, the impossibility of sharing entanglement unconditionally among many subsystems of a composite quantum system. For example, for three two-level quantum systems, denoted by A, B and C, if A and B are in a maximally entangled state, then A cannot be entangled with C at all.

The first mathematical characterization of the MOE was expressed as a form of inequality for three-qubit state.[3] Osborne and Verstraete proved the Coffman–Kundu–Wootters (CKW) monogamy constraints for concurrence.[4] New classes of monogamy relations have been presented for other entanglement measures.[5–8] It has been shown that the αth power of concurrence and the convex-roof extended negativity (CREN) satisfy the monogamy inequalities in multiqubit systems for $\alpha \geq 2$, and α-th power of entanglement of formation (EoF), Tsallis-q entanglement and Rényi-α entanglement show similar monogamy properties.[9–12]

Monogamy relations help people to understand the distribution of entanglement in multipartite systems. They also play an important role in the security analysis of quantum key distribution,[13] condensed-matter physics,[14] and even black-hole physics.[15] On the other hand, the assisted entanglement, which is a dual amount to bipartite entanglement measures, is also shown to have a polygamous of entanglement (POE). POE can be considered as another kind of entanglement constraints in multiparty quantum systems. A polygamy inequality was first established for arbitrary multiqubit systems by using the squared concurrence of assistance,[16,17] and it was generalized to multiqubit systems and arbitrary dimensional quantum systems in terms of various assisted entanglements.[18–20] POE is characterized by the polygamy inequality, $E_\alpha(\rho_{A|BC}) \leq E_\alpha(\rho_{AB}) + E_\alpha(\rho_{AC})$ for a tripartite quantum state ρ_{ABC}, where $E_\alpha(\rho_{A|BC})$ is the assisted entanglement[18] between A and BC. It provides an upper bound for the distribution of bipartite entanglement in a multipartite system.[16]

In recent years, tight classes of monogamy and polygamy inequalities of multiqubit entanglement using the non-negative α-th power of various entanglement measures were also proposed.[10,18,21–25] In this paper, we optimize the monogamy and polygamy inequalities of multipartite entanglement for arbitrary quantum states, based on the α-th power of the multipartite qubit entanglement. Furthermore, we show that our optimized monogamy and polygamy relations are tighter inequalities than the results in Refs. [10,16–18,21–26].

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2. Optimized relations for concurrence

For a bipartite pure state $|\psi\rangle_{AB}$ in Hilbert space $H_A \otimes H_B$, the concurrence is defined as $\mathcal{C}(|\psi\rangle_{AB}) = \sqrt{2(1-\mathrm{tr}(\rho_A^2))}$ with $\rho_A = \mathrm{tr}_B(|\psi\rangle_{AB}\langle\psi|)$.[27,28] For a bipartite mixed state ρ_{AB}, the concurrence is defined by the convex-roof extension

$$\mathcal{C}(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \mathcal{C}(|\psi_i\rangle_{AB}),$$

where the minimum is taken over all possible decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$ with $\sum_i p_i = 1$ and $p_i \geq 0$. For a dual quantity to $\mathcal{C}$, the concurrence of assistance of state ρ_{AB} is defined as the maximum average entanglement[16]

$$\mathcal{C}_a(\rho_{AB}) = \max_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \mathcal{C}(|\psi_i\rangle_{AB}),$$

where the maximum is taken over all possible pure-state decompositions of ρ_{AB}.[29]

For any N-qubit quantum state $\rho_{AB_1\cdots B_{N-1}} \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_{N-1}}$, the concurrence $\mathcal{C}(\rho_{A|B_1\cdots B_{N-1}})$ of the state $\rho_{A|B_1\cdots B_{N-1}}$ across the bipartition $A : B_1\cdots B_{N-1}$ satisfies[9]

$$\mathcal{C}^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \geq \mathcal{C}^\alpha(\rho_{AB_1}) + \mathcal{C}^\alpha(\rho_{AB_2}) + \cdots + \mathcal{C}^\alpha(\rho_{AB_{N-1}}), \quad (1)$$

for $\alpha \geq 2$, where ρ_{AB_j} denote two-qubit reduced density matrices of subsystems AB_j for $j = 1,2,\ldots,N-1$. Generally, a bipartite entanglement measure E is said to be monogamous if the following inequality is satisfied:[25]

$$E^{\alpha_c}(\rho_{A|B_1\cdots B_{N-1}}) \geq \sum_{i=1}^{N-1} E^{\alpha_c}(\rho_{AB_i}), \quad (2)$$

where $\rho_{AB_i} = \mathrm{tr}_{B_1\cdots B_{i-1}B_{i+1}\cdots B_{N-1}}(\rho_{A|B_1\cdots B_{N-1}})$, and α_c is the minimum value such that Eq. (2) holds.

In Ref. [25], the authors have shown that for $0 \leq x \leq \frac{1}{k}$, $k \geq 1$ and $\mu \geq 1$, the following inequality holds:

$$\begin{aligned}(1+x)^\mu &\geq 1 + \frac{k\mu}{k+1}x + \left[(k+1)^\mu - \left(1+\frac{\mu}{k+1}\right)k^\mu\right]x^\mu \\ &\geq 1 + [(k+1)^\mu - k^\mu]x^\mu. \end{aligned} \quad (3)$$

In order to investigate the monogamy relations of multipartite entanglement, we need the following lemma.

Lemma 2.1 For $\alpha \geq 2$, $k \geq 1$ and any $2\otimes 2\otimes 2^{N-2}$ quantum state ρ_{ABC}, if $\mathcal{C}_{AB} \geq \sqrt{k}\mathcal{C}_{AC}$, then

$$\begin{aligned}\mathcal{C}^\alpha_{A|BC} \geq\ & \mathcal{C}^\alpha_{AB} + \frac{\alpha}{2(k+1)}\mathcal{C}^2_{AC}\left(k\mathcal{C}^{(\alpha-2)}_{AB} - k^{\frac{\alpha}{2}}\mathcal{C}^{(\alpha-2)}_{AC}\right) \\ &+ \left[(k+1)^{\frac{\alpha}{2}} - k^{\frac{\alpha}{2}}\right]\mathcal{C}^\alpha_{AC}, \end{aligned} \quad (4)$$

where A and B are qubit systems, C is a 2^{N-2}-dimensional qudit system, consisting of $N-2$ qubit systems, and $\mathcal{C}_{AB} = \mathcal{C}(\rho_{AB})$.

Proof If $\mathcal{C}_{AB} = 0$, then $\mathcal{C}_{AC} = 0$, ρ_{AB} and ρ_{AC} are separable, hence Eq. (4) holds.

Otherwise, according to $\mathcal{C}^2_{A|BC} \geq \mathcal{C}^2_{AB} + \mathcal{C}^2_{AC}$,[4] and $\mathcal{C}_{AB} \geq \sqrt{k}\mathcal{C}_{AC}$, we have

$$\begin{aligned}\mathcal{C}^\alpha_{A|BC} &\geq (\mathcal{C}^2_{AB} + \mathcal{C}^2_{AC})^{\frac{\alpha}{2}} \\ &= \mathcal{C}^\alpha_{AB}\left(1 + \frac{\mathcal{C}^2_{AC}}{\mathcal{C}^2_{AB}}\right)^{\frac{\alpha}{2}} \\ &\geq \mathcal{C}^\alpha_{AB}\left[1 + \frac{k\alpha}{2(k+1)}\frac{\mathcal{C}^2_{AC}}{\mathcal{C}^2_{AB}}\right. \\ &\quad \left. + \left[(k+1)^{\frac{\alpha}{2}} - \left(1+\frac{\alpha}{2(k+1)}\right)k^{\frac{\alpha}{2}}\right]\left(\frac{\mathcal{C}^2_{AC}}{\mathcal{C}^2_{AB}}\right)^{\frac{\alpha}{2}}\right] \\ &= \mathcal{C}^\alpha_{AB} + \frac{\alpha}{2(k+1)}\mathcal{C}^2_{AC}\left(k\mathcal{C}^{(\alpha-2)}_{AB} - k^{\frac{\alpha}{2}}\mathcal{C}^{(\alpha-2)}_{AC}\right) \\ &\quad + \left[(k+1)^{\frac{\alpha}{2}} - k^{\frac{\alpha}{2}}\right]\mathcal{C}^\alpha_{AC}, \end{aligned}$$

where the second inequality is due to inequality (3).

Note Lemma 2.1 shows that there exits a new trade-off relation among subsystems A, B and C of a composite quantum system, i.e., the entanglement shared among subsystems A, B and C is restricted by inequality (4).

In term of Lemma 2.1, we get a new general monogamy relation of concurrence for $\rho_{AB_1\cdots B_{N-1}}$.

Proposition 2.1 For $\alpha \geq 2$, $k \geq 1$ and any quantum state $\rho_{AB_1\cdots B_{N-1}}$, if $\mathcal{C}_{AB_i} \geq \sqrt{k}\mathcal{C}_{A|B_{i+1}\cdots B_{N-1}}$, for $i = 1,2,\ldots,m$ and $\sqrt{k}\mathcal{C}_{AB_j} \leq \mathcal{C}_{A|B_{j+1}\cdots B_{N-1}}$, for $j = m+1,\ldots,N-2$, then

$$\begin{aligned}\mathcal{C}^\alpha_{A|B_1B_2\cdots B_{N-1}} \geq\ & \sum_{i=1}^{m} h^{i-1}(\mathcal{C}^\alpha_{AB_i} + J_{AB_i}) \\ &+ h^m \sum_{j=m+1}^{N-2}(h\mathcal{C}^\alpha_{AB_j} + \bar{J}_{AB_j}) + h^m\mathcal{C}^\alpha_{AB_{N-1}}, \end{aligned} \quad (5)$$

where

$$\begin{aligned} J_{AB_i} &= \frac{\alpha}{2(k+1)}\mathcal{C}^2_{A|B_{i+1}\cdots B_{N-1}}\left(k\mathcal{C}^{\alpha-2}_{AB_i} - k^{\frac{\alpha}{2}}\mathcal{C}^{\alpha-2}_{A|B_{i+1}\cdots B_{N-1}}\right), \\ \bar{J}_{AB_j} &= \frac{\alpha}{2(k+1)}\mathcal{C}^2_{AB_j}\left(k\mathcal{C}^{\alpha-2}_{A|B_{j+1}\cdots B_{N-1}} - k^{\frac{\alpha}{2}}\mathcal{C}^{\alpha-2}_{AB_j}\right), \\ h &= (k+1)^{\frac{\alpha}{2}} - k^{\frac{\alpha}{2}}. \end{aligned}$$

Proof For $i = 1,2,\ldots,m$, using the monogamy inequality (4), we have

$$\begin{aligned}&\mathcal{C}^\alpha_{A|B_1B_2\cdots B_{N-1}} \\ &\geq \mathcal{C}^\alpha_{AB_1} + h\mathcal{C}^\alpha_{A|B_2\cdots B_{N-1}} + \frac{\alpha}{2(k+1)}\mathcal{C}^2_{A|B_2\cdots B_{N-1}} \\ &\quad \times\left(k\mathcal{C}^{\alpha-2}_{AB_1} - k^{\frac{\alpha}{2}}\mathcal{C}^{\alpha-2}_{A|B_2\cdots B_{N-1}}\right) \\ &\geq \mathcal{C}^\alpha_{AB_1} + h\left[\mathcal{C}^\alpha_{AB_2} + h\mathcal{C}^\alpha_{A|B_3\cdots B_{N-1}} + \frac{\alpha}{2(k+1)}\mathcal{C}^2_{A|B_3\cdots B_{N-1}}\right. \\ &\quad \left.\times\left(k\mathcal{C}^{\alpha-2}_{AB_2} - k^{\frac{\alpha}{2}}\mathcal{C}^{\alpha-2}_{A|B_3\cdots B_{N-1}}\right)\right] \\ &\quad + \frac{\alpha}{2(k+1)}\mathcal{C}^2_{A|B_2\cdots B_{N-1}}\left(k\mathcal{C}^{\alpha-2}_{AB_1} - k^{\frac{\alpha}{2}}\mathcal{C}^{\alpha-2}_{A|B_2\cdots B_{N-1}}\right) \\ &\geq \cdots \\ &\geq \sum_{i=1}^{m} h^{i-1}(\mathcal{C}^\alpha_{AB_i} + J_{AB_i}) + h^m\mathcal{C}^\alpha_{A|B_{m+1}\cdots B_{N-1}}. \end{aligned} \quad (6)$$

Similarly, for $j = m+1,\ldots,N-2$, we get

$$\mathcal{C}^\alpha_{A|B_{m+1}\cdots B_{N-1}} \geq \mathcal{C}^\alpha_{A|B_{m+2}\cdots B_{N-1}} + h\mathcal{C}^\alpha_{AB_{m+1}} + \frac{\alpha}{2(k+1)}\mathcal{C}^2_{AB_{m+1}}$$

$$\times\left(kC^{\alpha-2}_{A|B_{m+2}\cdots B_{N-1}}-k^{\frac{\alpha}{2}}C^{\alpha-2}_{AB_{m+1}}\right)$$
$$\geq\sum_{j=m+1}^{N-2}(hC^{\alpha}_{AB_j}+\bar{J}_{AB_j})+C^{\alpha}_{AB_{N-1}}. \quad (7)$$

By combining inequalities (6) and (7), we come to the conclusion.

Remark In Proposition 2.1, the conditions $C_{AB_i}\geq\sqrt{k}C_{A|B_{i+1}\cdots B_{N-1}}$ and $\sqrt{k}C_{AB_j}\leq C_{A|B_{j+1}\cdots B_{N-1}}$ make those J_{AB_i} and $\bar{J}_{AB_j}$ greater than 0, thus our results can be tighter than those inequalities in Ref. [22].

In particular, if $C_{AB_i}\geq\sqrt{k}C_{A|B_{i+1}\cdots B_{N-1}}$ for $i=1,2,\ldots,N-2$, then the following monogamy relation holds.

Proposition 2.2 For any N-qubit quantum state $\rho_{AB_1\cdots B_{N-1}}$, if $C_{AB_i}\geq\sqrt{k}C_{A|B_{i+1}\cdots B_{N-1}}$, then

$$C^{\alpha}_{A|B_1B_2\cdots B_{N-1}}\geq\sum_{i=1}^{N-2}h^{i-1}(C^{\alpha}_{AB_i}+J_{AB_i})+h^{N-2}C^{\alpha}_{AB_{N-1}}, \quad (8)$$

where

$$h=(k+1)^{\frac{\alpha}{2}}-k^{\frac{\alpha}{2}},$$
$$J_{AB_i}=\frac{\alpha}{2(k+1)}C^2_{A|B_{i+1}\cdots B_{N-1}}\left(kC^{\alpha-2}_{AB_i}-k^{\frac{\alpha}{2}}C^{\alpha-2}_{A|B_{i+1}\cdots B_{N-1}}\right).$$

Proof From the inequality (4), we have

$$\begin{aligned}
&C^{\alpha}_{A|B_1B_2\cdots B_{N-1}}\\
&\geq C^{\alpha}_{AB_1}+hC^{\alpha}_{A|B_2\cdots B_{N-1}}+\frac{\alpha}{2(k+1)}C^2_{A|B_2\cdots B_{N-1}}\\
&\quad\times\left(kC^{\alpha-2}_{AB_1}-k^{\frac{\alpha}{2}}C^{\alpha-2}_{A|B_2\cdots B_{N-1}}\right)\\
&\geq C^{\alpha}_{AB_1}+h\left[C^{\alpha}_{AB_2}+hC^{\alpha}_{A|B_3\cdots B_{N-1}}+\frac{\alpha}{2(k+1)}C^2_{A|B_3\cdots B_{N-1}}\right.\\
&\quad\left.\times\left(kC^{\alpha-2}_{AB_2}-k^{\frac{\alpha}{2}}C^{\alpha-2}_{A|B_3\cdots B_{N-1}}\right)\right]\\
&\quad+\frac{\alpha}{2(k+1)}C^2_{A|B_2\cdots B_{N-1}}\left(kC^{\alpha-2}_{AB_1}-k^{\frac{\alpha}{2}}C^{\alpha-2}_{A|B_2\cdots B_{N-1}}\right)\\
&\geq\cdots\\
&\geq C^{\alpha}_{AB_1}+hC^{\alpha}_{AB_2}+\cdots+h^{N-2}C^{\alpha}_{AB_{N-1}}+\frac{\alpha h^{N-3}}{2(k+1)}C^2_{AB_{N-1}}\\
&\quad\times\left(kC^{\alpha-2}_{AB_{N-2}}-k^{\frac{\alpha}{2}}C^{\alpha-2}_{AB_{N-1}}\right)\\
&\quad+\cdots+\frac{\alpha h}{2(k+1)}C^2_{A|B_3\cdots B_{N-1}}\left(kC^{\alpha-2}_{AB_2}-k^{\frac{\alpha}{2}}C^{\alpha-2}_{A|B_3\cdots B_{N-1}}\right)\\
&\quad+\frac{\alpha}{2(k+1)}C^2_{A|B_2\cdots B_{N-1}}\left(kC^{\alpha-2}_{AB_1}-k^{\frac{\alpha}{2}}C^{\alpha-2}_{A|B_2\cdots B_{N-1}}\right).
\end{aligned}$$

Setting $J_{AB_i}=\frac{\alpha}{2(k+1)}C^2_{A|B_{i+1}\cdots B_{N-1}}\left(C^{\alpha-2}_{AB_i}-C^{\alpha-2}_{A|B_{i+1}\cdots B_{N-1}}\right)$, one gets the result.

The new monogamy relations of concurrence (5) and (8) are tighter than (16) in Ref. [22].

Example 2.1 Consider the three-qubit state $|\psi\rangle_{ABC}$ in the generalized Schmidt decomposition form[30]

$$|\psi\rangle_{ABC}=\lambda_0|000\rangle+\lambda_1 e^{i\varphi}|100\rangle+\lambda_2|101\rangle+\lambda_3|110\rangle+\lambda_4|111\rangle, \quad (9)$$

where $\sum_{m=0}^{4}\lambda_m^2=1(\lambda_m\geq 0)$, $i=\sqrt{-1}$ and $0\leq\phi\leq\pi$. According to the definition of concurrence, we have $C_{A|BC}=2\lambda_0\sqrt{\lambda_2^2+\lambda_3^2+\lambda_4^2}$, $C_{AB}=2\lambda_0\lambda_2$ and $C_{AC}=2\lambda_0\lambda_3$. In the light of $C_{AB_i}\geq\sqrt{k}C_{A|B_{i+1}\cdots B_{N-1}}$ in Proposition 2.2, we take $\lambda_0=\frac{1}{2}$, $\lambda_2=\sqrt{\frac{1}{2}}$, $\lambda_1=\lambda_4=\sqrt{\frac{5}{72}}$ and $\lambda_3=\frac{1}{3}$, then

$$\begin{aligned}
C^{\alpha}_{A|BC}&=\left(\frac{49}{72}\right)^{\frac{\alpha}{2}}\\
&\geq C^{\alpha}_{AB}+hC^{\alpha}_{AC}+\frac{\alpha}{2(k+1)}C^2_{AC}(kC^{\alpha-2}_{AB}-k^{\frac{\alpha}{2}}C^{\alpha-2}_{AC})\\
&=\left(\frac{1}{2}\right)^{\frac{\alpha}{2}}+\left(\frac{1}{3}\right)^{\alpha}h+\frac{\alpha}{18(k+1)}\\
&\quad\times\left[k\left(\sqrt{\frac{1}{2}}\right)^{\alpha-2}-k^{\frac{\alpha}{2}}\left(\frac{1}{3}\right)^{\alpha-2}\right].
\end{aligned}$$

For $k=2$, our monogamy inequality of concurrence is tighter than the formula $C^{\alpha}_{AB}+\frac{(1+r)^{\alpha/2}-1}{r^{\alpha/2}}C^{\alpha}_{AC}$ $(r=0.5)$ given in Ref. [22] for $\alpha\geq 2$. The larger the k is, the tighter the monogamy inequality of concurrence becomes (see Fig. 1).

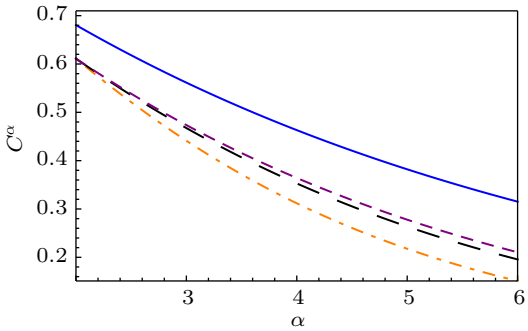


Fig. 1. The axis C^{α} stands for the α-th power concurrence of $|\psi\rangle$. The blue solid line represents the concurrence of $|\psi\rangle$ in Example 2.1. The black dotted (purple dashed) line represents the lower bound from our result when $k=2$ $(k=3)$, while the orange dotdashed line represents the lower bound from the corresponding result $r=0.5$ in Ref. [22].

In order to discuss the polygamy relations of the concurrence of assistance (CoA) $C_{aA|B_1\cdots B_{N-1}}$, we need the following inequality.[25] For $0\leq x\leq 1/k$, $k\geq 1$ and $0\leq\mu\leq 1$,

$$\begin{aligned}
(1+x)^{\mu}&\leq 1+\frac{k\mu}{(k+1)^2}[kx-(kx)^{\mu}]+[(k+1)^{\mu}-k^{\mu}]x^{\mu}\\
&\leq 1+[(k+1)^{\mu}-k^{\mu}]x^{\mu}\leq 1+\mu x^{\mu}. \quad (10)
\end{aligned}$$

Using inequality (10), we have the following polygamy relation.

Lemma 2.2 For $0\leq\alpha\leq 2$ and any $2\otimes 2\otimes 2^{N-2}$ quantum state ψ_{ABC}, if $C_{aAB}\geq\sqrt{k}C_{aAC}$ with $k\geq 1$, then

$$\begin{aligned}
C^{\alpha}_{aA|BC}&\leq C^{\alpha}_{aAB}+\frac{k\alpha}{2(k+1)^2}C^2_{aAC}\left(kC^{(\alpha-2)}_{aAB}-k^{\frac{\alpha}{2}}C^{(\alpha-2)}_{aAC}\right)\\
&\quad+\left[(k+1)^{\frac{\alpha}{2}}-k^{\frac{\alpha}{2}}\right]C^{\alpha}_{aAC}, \quad (11)
\end{aligned}$$

where $C_{aAB}=C_a(\rho_{AB})$, $\rho_{AB}=\mathrm{tr}_C|\psi\rangle_{ABC}\langle\psi|$, and $\rho_{AC}=\mathrm{tr}_B|\psi\rangle_{ABC}\langle\psi|$.

Proof If $C_{aAB}=0$, then $C_{aAC}=0$, and $C^{\alpha}_{aA|BC}=0$, hence the polygamy relation (11) holds.

Otherwise, according to $\mathcal{C}^2_{aA|BC} \le \mathcal{C}^2_{aAB} + \mathcal{C}^2_{aAC}$[16] and $\mathcal{C}_{aAB} \ge \sqrt{k}\mathcal{C}_{aAC}$, we have

$$\begin{aligned}
\mathcal{C}^\alpha_{A|BC} &\le (\mathcal{C}^2_{aAB}+\mathcal{C}^2_{aAC})^{\frac{\alpha}{2}} \\
&= \mathcal{C}^\alpha_{aAB}\left(1+\frac{\mathcal{C}^2_{aAC}}{\mathcal{C}^2_{aAB}}\right)^{\frac{\alpha}{2}} \\
&\le \mathcal{C}^\alpha_{aAB}\left[1+\frac{k\alpha}{2(k+1)^2}\frac{\mathcal{C}^2_{aAC}}{\mathcal{C}^2_{aAB}}\right. \\
&\quad \left.+\left[(k+1)^{\frac{\alpha}{2}}-\left(1+\frac{k\alpha}{2(k+1)^2}\right)k^{\frac{\alpha}{2}}\right]\left(\frac{\mathcal{C}^2_{aAC}}{\mathcal{C}^2_{aAB}}\right)^{\frac{\alpha}{2}}\right] \\
&= \mathcal{C}^\alpha_{aAB}+\frac{k\alpha}{2(k+1)^2}\mathcal{C}^2_{aAC}\left(k\mathcal{C}^{(\alpha-2)}_{aAB}-k^{\frac{\alpha}{2}}\mathcal{C}^{(\alpha-2)}_{aAC}\right) \\
&\quad +\left[(k+1)^{\frac{\alpha}{2}}-k^{\frac{\alpha}{2}}\right]\mathcal{C}^\alpha_{aAC},
\end{aligned}$$

where the second inequality is due to inequality (10).

Note Lemma 2.2 shows that there exits another trade-off relation among subsystems A, B and C of a composite quantum system, namely polygamy relation, the information shared among A, B and C is restricted by the inequality (11).

Associated with the Lemma 2.2, we obtain the following two new polygamy relations.

Proposition 2.3 For $0 \le \alpha \le 2$, $k \ge 1$ and N-qubit quantum state $|\psi\rangle_{AB_1\cdots B_{N-1}}$, if $\mathcal{C}_{aAB_i} \ge \sqrt{k}\mathcal{C}_{aA|B_{i+1}\cdots B_{N-1}}$ for $i = 1,2,\ldots,m$, and $\sqrt{k}\mathcal{C}_{aAB_j} \le \mathcal{C}_{aA|B_{j+1}\cdots B_{N-1}}$ for $j = m+1,\ldots,N-2$, then

$$\begin{aligned}
\mathcal{C}^\alpha_{aA|B_1B_2\cdots B_{N-1}} &\le \sum_{i=1}^{m} h^{i-1}(\mathcal{C}^\alpha_{aAB_i}+J_{aAB_i}) \\
&\quad +h^m\sum_{j=m+1}^{N-2}(h\mathcal{C}^\alpha_{aAB_j}+\bar{J}_{aAB_j})+h^m\mathcal{C}^\alpha_{aAB_{N-1}}, \quad (12)
\end{aligned}$$

where

$$\begin{aligned}
&h=(k+1)^{\frac{\alpha}{2}}-k^{\frac{\alpha}{2}}, \\
&J_{aAB_i}=\frac{k\alpha}{2(k+1)^2}\mathcal{C}^2_{aA|B_{i+1}\cdots B_{N-1}}\left(k\mathcal{C}^{\alpha-2}_{aAB_i}-k^{\frac{\alpha}{2}}\mathcal{C}^{\alpha-2}_{aA|B_{i+1}\cdots B_{N-1}}\right), \\
&\bar{J}_{aAB_j}=\frac{k\alpha}{2(k+1)^2}\mathcal{C}^2_{aAB_j}\left(k\mathcal{C}^{\alpha-2}_{aA|B_{j+1}\cdots B_{N-1}}-k^{\frac{\alpha}{2}}\mathcal{C}^{\alpha-2}_{aAB_j}\right).
\end{aligned}$$

Proof For $i = 1,\ldots,m$, using inequality (11), we have

$$\begin{aligned}
&\mathcal{C}^\alpha_{aA|B_1B_2\cdots B_{N-1}} \\
&\le \mathcal{C}^\alpha_{aAB_1}+h\mathcal{C}^\alpha_{aA|B_2\cdots B_{N-1}}+\frac{k\alpha}{2(k+1)^2} \\
&\quad \times\mathcal{C}^2_{aA|B_2\cdots B_{N-1}}\left(k\mathcal{C}^{\alpha-2}_{aAB_1}-k^{\frac{\alpha}{2}}\mathcal{C}^{\alpha-2}_{aA|B_2\cdots B_{N-1}}\right) \\
&\le \mathcal{C}^\alpha_{aAB_1}+h\left[\mathcal{C}^\alpha_{aAB_2}+h\mathcal{C}^\alpha_{aA|B_3\cdots B_{N-1}}+\frac{k\alpha}{2(k+1)^2}\right. \\
&\quad \left.\times\mathcal{C}^2_{aA|B_3\cdots B_{N-1}}\left(k\mathcal{C}^{\alpha-2}_{aAB_2}-k^{\frac{\alpha}{2}}\mathcal{C}^{\alpha-2}_{aA|B_3\cdots B_{N-1}}\right)\right] \\
&\quad +\frac{k\alpha}{2(k+1)^2}\mathcal{C}^2_{aA|B_2\cdots B_{N-1}}\left(k\mathcal{C}^{\alpha-2}_{aAB_1}-k^{\frac{\alpha}{2}}\mathcal{C}^{\alpha-2}_{aA|B_2\cdots B_{N-1}}\right) \\
&\le \cdots \\
&\le \sum_{i=1}^{m} h^{i-1}(\mathcal{C}^\alpha_{aAB_i}+\frac{k\alpha}{2(k+1)^2}J_{aAB_i})+h^m\mathcal{C}^\alpha_{aA|B_{m+1}\cdots B_{N-1}},
\end{aligned}$$

for $j = m+1,\ldots,N-2$,

$$\begin{aligned}
&\mathcal{C}^\alpha_{aA|B_{m+1}\cdots B_{N-1}} \\
&\le \mathcal{C}^\alpha_{aA|B_{m+2}\cdots B_{N-1}}+h\mathcal{C}^\alpha_{aAB_{m+1}}+\frac{k\alpha}{2(k+1)^2} \\
&\quad \times\mathcal{C}^2_{aAB_{m+1}}\left(k\mathcal{C}^{\alpha-2}_{aA|B_{m+2}\cdots B_{N-1}}-k^{\frac{\alpha}{2}}\mathcal{C}^{\alpha-2}_{aAB_{m+1}}\right) \\
&\le \sum_{j=m+1}^{N-2}(h\mathcal{C}^\alpha_{aAB_j}+\frac{k\alpha}{2(k+1)^2}\bar{J}_{aAB_j})+\mathcal{C}^\alpha_{aAB_{N-1}}.
\end{aligned}$$

Remark The conditions $\mathcal{C}_{aAB_i} \ge \sqrt{k}\mathcal{C}_{aA|B_{i+1}\cdots B_{N-1}}$ and $\sqrt{k}\mathcal{C}_{aAB_j} \le \mathcal{C}_{aA|B_{j+1}\cdots B_{N-1}}$ make those J_{aAB_i} and $\bar{J}_{aAB_j}$ greater than 0, thus our results can be tighter than those inequalities in Ref. [22].

In particular, if $i = 1,2,\ldots,N-2$, then we obtain following new polygamy relation.

Proposition 2.4 For $0 \le \alpha \le 2$, $k \ge 1$ and quantum state $|\psi\rangle_{AB_1\cdots B_{N-1}}$, if $\mathcal{C}_{aAB_i} \ge \sqrt{k}\mathcal{C}_{aA|B_{i+1}\cdots B_{N-1}}$, then we have

$$\mathcal{C}^\alpha_{aA|B_1B_2\cdots B_{N-1}} \le \sum_{i=1}^{N-2} h^{i-1}(\mathcal{C}^\alpha_{aAB_i}+J_{aAB_i})+h^{N-2}\mathcal{C}^\alpha_{aAB_{N-1}}, \quad (13)$$

where

$$\begin{aligned}
&h=(k+1)^{\frac{\alpha}{2}}-k^{\frac{\alpha}{2}}, \\
&J_{aAB_i}=\frac{k\alpha}{2(k+1)^2}\mathcal{C}^2_{aA|B_{i+1}\cdots B_{N-1}}\left(k\mathcal{C}^{\alpha-2}_{aAB_i}-k^{\frac{\alpha}{2}}\mathcal{C}^{\alpha-2}_{aA|B_{i+1}\cdots B_{N-1}}\right).
\end{aligned}$$

Proof By partitioning the last qubit system $B_iB_{i+1}\cdots B_{N-1}$ $(i = 1,2,\ldots N-2)$ into two subsystems and using the above inequality (11) repeatedly, one gets the result.

The Propositions 2.3 and 2.4 show that the new polygamy inequalities (12) and (13) are tighter than (18) and (19) in Ref. [22]. Let us illustrate this by an example.

Example 2.2 Consider the three-qubit state $\rho_{ABC} = |\psi\rangle_{ABC}\langle\psi|$ (9). By direct calculation we have $\mathcal{C}_{aA|BC} = 2\lambda_0\sqrt{\lambda_2^2+\lambda_3^2+\lambda_4^2}$, $\mathcal{C}_{aAB} = 2\lambda_0\sqrt{\lambda_2^2+\lambda_4^2}$ and $\mathcal{C}_{aAC} = 2\lambda_0\sqrt{\lambda_3^2+\lambda_4^2}$. In view of $\mathcal{C}_{aAB} \ge \sqrt{k}\mathcal{C}_{aAC}$, we set $\lambda_0 = \frac{1}{2}$, $\lambda_2 = \sqrt{\frac{1}{2}}$, $\lambda_1 = \lambda_4 = \sqrt{\frac{5}{72}}$ and $\lambda_3 = \frac{1}{3}$, then $\mathcal{C}^\alpha_{aA|BC} = \left(\frac{49}{72}\right)^{\frac{\alpha}{2}}$, and

$$\begin{aligned}
&\mathcal{C}^\alpha_{aAB}+h\mathcal{C}^\alpha_{aAC}+\frac{k\alpha}{2(k+1)^2}\mathcal{C}^2_{aAC}(k\mathcal{C}^{\alpha-2}_{aAB}-k^{\frac{\alpha}{2}}\mathcal{C}^{\alpha-2}_{aAC}) \\
&=\left(\frac{41}{72}\right)^{\frac{\alpha}{2}}+h\left(\frac{13}{72}\right)^{\frac{\alpha}{2}} \\
&\quad +\frac{k\alpha}{18(k+1)^2}\left[k\left(\sqrt{\frac{41}{72}}\right)^{\alpha-2}-k^{\frac{\alpha}{2}}\left(\sqrt{\frac{13}{72}}\right)^{\alpha-2}\right].
\end{aligned}$$

For $k \ge 2$, our polygamy inequality of concurrence is tighter than the formula $\mathcal{C}^\alpha_{aAB}+\frac{(1+r)^{\alpha/2}-1}{r^{\alpha/2}}\mathcal{C}^\alpha_{aAC}$ $(r = 0.5)$ given in Ref. [22] for $\alpha \ge 2$ (see Fig. 2).

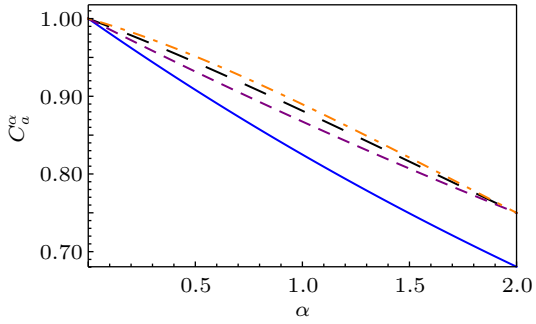


Fig. 2. The axis C_a^α stands for the αth power CoA of $|\psi\rangle$. The blue solid line represents the upper bound of concurrence of $|\psi\rangle$ in Example 2.2, the orange dotdashed line represents the upper bound of polygamy of CoA in Ref. [22], the the black dotted (purple dashed) line represents the upper bound from our result when $k=2$ ($k=3$).

3. Optimized relations for entanglement negativity

Entanglement negativity is a quantifier of bipartite entanglement. The negativity of a bipartite state ρ_{AB} in $\mathbb{H}_A\otimes\mathbb{H}_B$ is defined by Ref. [31], $N(\rho_{AB})=(||\rho_{AB}^{\mathrm{T}_A}||-1)/2$, where $\rho_{AB}^{\mathrm{T}_A}$ is the partial transposed matrix of ρ_{AB} with respect to the subsystem A, and $||x||=\mathrm{tr}\sqrt{xx^\dagger}$ is the trace norm of x. The negativity vanishes if and only if ρ_{AB} is separable for the $2\otimes 2$ and $2\otimes 3$ systems.[32] For the sake of convenience, we use the following definition of negativity, $N(\rho_{AB})=||\rho_{AB}^{\mathrm{T}_A}||-1$. For a bipartite pure state $|\psi\rangle_{AB}$, the negativity is given by $N(|\psi\rangle_{AB})=2\sum_{i<j}\sqrt{\lambda_i\lambda_j}=(\mathrm{tr}\sqrt{\rho_A})^2-1$, where $\lambda_i\geq 0$, $\sum_i\lambda_i=1$ and $\rho_A=\mathrm{tr}_B|\psi\rangle_{AB}\langle\psi|$.

To give a discrimination of positive partial transposed bound entangled states[32,33] and separable states in any bipartite quantum systems, one modification of negativity is convex-roof extended negativity (CREN).[34]

For a mixed state ρ_{AB}, the CREN is defined as

$$N_{\mathrm{c}}(\rho_{AB})=\min\sum_i p_i N(|\psi_i\rangle_{AB}), \tag{14}$$

where the minimum is taken over all possible pure state decompositions $\{p_i,|\psi_i\rangle_{AB}\}$ of ρ_{AB}.

The convex-roof extended negativity of assistance (CRENoA) of ρ_{AB} is defined by

$$N_{\mathrm{a}}(\rho_{AB})=\max\sum_i p_i N(|\psi_i\rangle_{AB}), \tag{15}$$

with the maximum taking over all possible pure state decompositions $\{p_i,|\psi_i\rangle_{AB}\}$ of ρ_{AB}.

It has been showed that the negativity is equivalent to concurrence for any pure states with Schmidt rank 2.[34] Consequently, it follows that for any two-qubit mixed state $\rho_{AB}=\sum p_i|\psi_i\rangle_{AB}\langle\psi_i|$,

$$\begin{aligned}N_{\mathrm{c}}(\rho_{AB}) &= \min\sum_i p_i N(|\psi_i\rangle_{AB})\\ &= \min\sum_i p_i \mathcal{C}(|\psi_i\rangle_{AB})=\mathcal{C}(\rho_{AB}),\end{aligned} \tag{16}$$

$$\begin{aligned}N_{\mathrm{a}}(\rho_{AB}) &= \max\sum_i p_i N(|\psi_i\rangle_{AB})\\ &= \max\sum_i p_i \mathcal{C}(|\psi_i\rangle_{AB})=\mathcal{C}_a(\rho_{AB}).\end{aligned} \tag{17}$$

Analogously, we obtain a new monogamy relation as follows.

Lemma 3.1 For any $2\otimes 2\otimes 2^{N-2}$ quantum state ρ_{ABC} such that $N_{cAB}\geq\sqrt{k}N_{cAC}$, we have

$$\begin{aligned}N_{c_{A|BC}}^{\alpha} \geq\ & N_{c_{AB}}^{\alpha}+\frac{\alpha}{2(k+1)}N_{cAC}^2\left(kN_{cAB}^{(\alpha-2)}-k^{\frac{\alpha}{2}}N_{cAC}^{(\alpha-2)}\right)\\ &+\left[(k+1)^{\frac{\alpha}{2}}-k^{\frac{\alpha}{2}}\right]N_{cAC}^{\alpha},\end{aligned} \tag{18}$$

with $\alpha\geq 2$ and $k\geq 1$.

Proof The prove is similar with Lemma 2.1.

Note $N_{cA|BC}$ stands for the negativity of ρ_{ABC}, which quantifies the entanglement between the subsystem A and BC, and $N_{cA|BC}=0$ implies ρ_{ABC} is separable. Lemma 3.1 shows that there exits limited shareability among the subsystem A, B and C. If $N_{cA|BC}\approx N_{cAB}\approx 1$, then ρ_{AB} is a maximally entangled state and $N_{cAC}\approx 0$.

According to Lemma 3.1, we obtain a new monogamy relation of negativity.

Proposition 3.1 For $\alpha\geq 2$, $k\geq 1$ and any N-qubit state $\rho_{AB_1B_2\cdots B_{N-1}}$, if $N_{cAB_i}\geq\sqrt{k}N_{cA|B_i\cdots B_{N-1}}$ for $i=1,2,\ldots,m$, and $\sqrt{k}N_{cAB_j}\leq N_{cA|B_j\cdots B_{N-1}}$ for $j=m+1,\ldots,N-2$, then the following monogamy relation holds:

$$\begin{aligned}N_{cA|B_1B_2\cdots B_{N-1}}^{\alpha}\geq & \sum_{i=1}^{m}h^{i-1}(N_{cAB_i}^{\alpha}+Q_{aAB_i})\\ &+h^m\sum_{j=m+1}^{N-2}(hN_{cAB_j}^{\alpha}+\bar{Q}_{aAB_j})+h^mN_{cAB_{N-1}}^{\alpha},\end{aligned} \tag{19}$$

where

$$\begin{aligned}&h=(k+1)^{\frac{\alpha}{2}}-k^{\frac{\alpha}{2}},\\ &Q_{aAB_i}=\frac{\alpha}{2(k+1)}N_{cA|B_{i+1}\cdots B_{N-1}}^2\left(kN_{cAB_i}^{(\alpha-2)}-k^{\frac{\alpha}{2}}N_{cA|B_{i+1}\cdots B_{N-1}}^{(\alpha-2)}\right),\\ &\bar{Q}_{aAB_j}=\frac{\alpha}{2(k+1)}N_{cAB_j}^2(kN_{cA|B_{j+1}\cdots B_{N-1}}^{(\alpha-2)}-k^{\frac{\alpha}{2}}N_{cAB_j}^{(\alpha-2)}).\end{aligned}$$

Proof For $i=1,2,\ldots,m$, according to the inequality (4), we have

$$\begin{aligned}&N_{cA|B_1B_2\cdots B_{N-1}}^{\alpha}\\ &\geq N_{cAB_1}^{\alpha}+hN_{cA|B_2\cdots B_{N-1}}^{\alpha}+\frac{\alpha}{2(k+1)}\\ &\quad\times N_{cA|B_2\cdots B_{N-1}}^2\left(kN_{cAB_1}^{(\alpha-2)}-k^{\frac{\alpha}{2}}N_{cA|B_2\cdots B_{N-1}}^{(\alpha-2)}\right)\\ &\geq N_{cAB_1}^{\alpha}+h\left[N_{cAB_2}^{\alpha}+hN_{cA|B_3\cdots B_{N-1}}^{\alpha}+\frac{\alpha}{2(k+1)}\right.\\ &\quad\left.\times N_{cA|B_3\cdots B_{N-1}}^2\left(kN_{cAB_2}^{(\alpha-2)}-k^{\frac{\alpha}{2}}N_{cA|B_3\cdots B_{N-1}}^{(\alpha-2)}\right)\right]\\ &\quad+\frac{\alpha}{2(k+1)}N_{cA|B_2\cdots B_{N-1}}^2\left(kN_{cAB_1}^{(\alpha-2)}-k^{\frac{\alpha}{2}}N_{cA|B_2\cdots B_{N-1}}^{(\alpha-2)}\right)\\ &\geq\cdots\\ &\geq\sum_{i=1}^{m}h^{i-1}(N_{cAB_i}^{\alpha}+Q_{aAB_i})+h^mN_{cA|B_{m+1}\cdots B_{N-1}}^{\alpha},\end{aligned} \tag{20}$$

for $j=m+1,\ldots,N-2$, and get

$$N^{\alpha}_{cA|B_{m+1}\cdots B_{N-1}} \geq N^{\alpha}_{cA|B_{m+2}\cdots B_{N-1}} + hN^{\alpha}_{cAB_{m+1}} + \frac{\alpha}{2(k+1)} \times N^{2}_{cAB_{m+1}} \left(kN^{(\alpha-2)}_{cA|B_{m+2}\cdots B_{N-1}} - k^{\frac{\alpha}{2}} N^{(\alpha-2)}_{cAB_{m+1}}\right) \geq \sum_{j=m+1}^{N-2} (hN^{\alpha}_{cAB_j} + \bar{Q}_{aAB_j}) + N^{\alpha}_{cAB_{N-1}}. \quad (21)$$

Combining (20) and (21), one gets the results.

In particular, if $i=1,2,\ldots,N-2$, then we get a new monogamy relation.

Proposition 3.2 For any N-qubit mixed state $\rho_{AB_1B_2\cdots B_{N-1}}$, if $N_{cAB_i} \geq \sqrt{k}N_{cA|B_{i+1}\cdots B_{N-1}}$, then

$$N^{\alpha}_{c\ A|B_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{N-2} h^{i-1}(N^{\alpha}_{c\ AB_i} + Q_{aAB_i}) + h^{N-2}N^{\alpha}_{c\ AB_{N-1}}, \quad (22)$$

where

$$h = (k+1)^{\frac{\alpha}{2}} - k^{\frac{\alpha}{2}},$$
$$Q_{aAB_i} = \frac{\alpha}{2(k+1)} N^{2}_{cA|B_{i+1}\cdots B_{N-1}} \left(kN^{\alpha-2}_{cAB_i} - k^{\frac{\alpha}{2}} N^{\alpha-2}_{cA|B_{i+1}\cdots B_{N-1}}\right).$$

Example 3.1 Let us consider again the three-qubit state $|\psi\rangle$ (9). From the definition of CREN, we have $(N_c)_{A|BC} = 2\lambda_0\sqrt{\lambda_2^2+\lambda_3^2+\lambda_4^2}$, $(N_c)_{AB} = 2\lambda_0\lambda_2$ and $(N_c)_{AC} = 2\lambda_0\lambda_3$. Accounting to $N_{cAB_i} \geq \sqrt{k}N_{cA|B_{i+1}\cdots B_{N-1}}$, we take $\lambda_0 = \frac{1}{2}$, $\lambda_2 = \sqrt{\frac{1}{2}}$, $\lambda_1 = \lambda_4 = \sqrt{\frac{5}{72}}$ and $\lambda_3 = \frac{1}{3}$, then $(N_c)_{A|BC} = \sqrt{\frac{49}{72}}$, $(N_c)_{AB} = \sqrt{\frac{1}{2}}$, $(N_c)_{AC} = \frac{1}{3}$, and $(N_c)^{\alpha}_{A|BC} = \left(\frac{49}{72}\right)^{\frac{\alpha}{2}} \geq (N_c)^{\alpha}_{AB} + h(N_c)^{\alpha}_{AC} + \frac{\alpha}{2(k+1)}(N_c)^{2}_{AC}(k(N_c)^{\alpha-2}_{AB} - k^{\frac{\alpha}{2}}(N_c)^{\alpha-2}_{AC}) = (\sqrt{\frac{1}{2}})^{\alpha} + h\cdot(\frac{1}{3})^{\alpha} + \frac{\alpha}{18(k+1)}[k(\sqrt{\frac{1}{2}})^{\alpha-2} - k^{\frac{\alpha}{2}}(\frac{1}{3})^{\alpha-2}]$. For $k = 1/r$, one has $(N_c)^{\alpha}_{A|BC} = \left(\frac{49}{72}\right)^{\frac{\alpha}{2}} \geq (N_c)^{\alpha}_{AB} + \frac{(1+r)^{\alpha/2}-1}{r^{\alpha/2}}(N_c)^{\alpha}_{AC}$ from Ref. [22]. Our monogamy inequality is tighter than the corresponding result ($r=0.5$) in Ref. [22] for $\alpha \geq 2$ and any $k \geq 2$ (see Fig. 3).

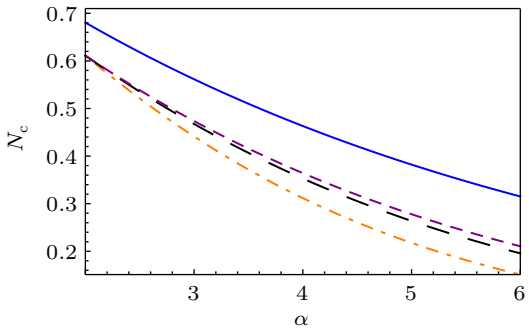


Fig. 3. The axis N_c is the CREN of $|\psi\rangle$. The blue solid line represents the CREN of $|\psi\rangle$ in Example 3.1. The black dotted (purple dashed) line represents the lower bound from our result for $k=2$ ($k=3$). The green dashed line represents the lower bound from the result ($k=0.5$) in Ref. [22].

Next, we investigate the polygamy relations of the negativity of assistant N_a. We first introduce a lemma.

Lemma 3.2 For $0 \leq \alpha \leq 2$, $k \geq 1$ and $2\otimes 2\otimes 2^{N-2}$ quantum state $|\psi\rangle_{ABC}$, if $N_{aAB} \geq \sqrt{k}N_{aAC}$, then we have

$$N^{\alpha}_{aA|BC} \leq (N_{aAB})^{\alpha} + \frac{k\alpha}{2(k+1)^2} N^{2}_{aAC}\left(kN^{(\alpha-2)}_{aAB} - k^{\frac{\alpha}{2}} N^{(\alpha-2)}_{aAC}\right) + \left[(k+1)^{\frac{\alpha}{2}} - k^{\frac{\alpha}{2}}\right] N^{\alpha}_{aAC}. \quad (23)$$

Using the Lemma 3.2, we can get a new polygamy relation.

Proposition 3.3 For a N-qubit quantum state $|\psi\rangle_{AB_1\cdots B_{N-1}}$, if $N_{aAB_i} \geq \sqrt{k}N_{aA|B_{i+1}\cdots B_{N-1}}$ for $i=1,2,\ldots,m$, and $\sqrt{k}N_{aAB_j} \leq N_{aA|B_{j+1}\cdots B_{N-1}}$ for $j=m+1,\ldots,N-2$, then the following polygamy relation holds with $0 \leq \alpha \leq 2$ and $k \geq 1$:

$$N^{\alpha}_{a\ A|B_1B_2\cdots B_{N-1}} \leq \sum_{i=1}^{m} h^{i-1}\left(N^{\alpha}_{a\ AB_i} + Q_{aAB_i}\right) + h^{m}\sum_{j=m+1}^{N-2}(hN^{\alpha}_{a\ AB_j} + \bar{Q}_{aAB_j}) + h^{m}N^{\alpha}_{a\ AB_{N-1}}, \quad (24)$$

where

$$h = (k+1)^{\frac{\alpha}{2}} - k^{\frac{\alpha}{2}},$$
$$Q_{aAB_i} = \frac{k\alpha}{2(k+1)^2} N^{2}_{aA|B_{i+1}\cdots B_{N-1}}\left(kN^{\alpha-2}_{aAB_i} - k^{\frac{\alpha}{2}} N^{\alpha-2}_{aA|B_{i+1}\cdots B_{N-1}}\right),$$
$$\bar{Q}_{aAB_j} = \frac{k\alpha}{2(k+1)^2} N^{2}_{aAB_j}\left(kN^{\alpha-2}_{aA|B_{j+1}\cdots B_{N-1}} - k^{\frac{\alpha}{2}} N^{\alpha-2}_{aAB_j}\right).$$

Proof The proof is similar to the proof of Proposition 2.3 by using inequality (23).

In particular, for $i=1,\ldots,N-2$, we get the following polygamy relation.

Proposition 3.4 For $0 \leq \alpha \leq 2$ and quantum state $|\psi\rangle_{AB_1\cdots B_{N-1}}$, if $N_{aAB_i} \geq \sqrt{k}N_{aA|B_{i+1}\cdots B_{N-1}}$ with $k \geq 1$, then

$$N^{\alpha}_{aA|B_1B_2\cdots B_{N-1}} \leq \sum_{i=1}^{N-2} h^{i-1}(N^{\alpha}_{aAB_i} + Q_{aAB_i}) + h^{N-2}N^{\alpha}_{aAB_{N-1}}, \quad (25)$$

where

$$h = (k+1)^{\frac{\alpha}{2}} - k^{\frac{\alpha}{2}},$$
$$Q_{aAB_i} = \frac{k\alpha}{2(k+1)^2} N^{2}_{aA|B_{i+1}\cdots B_{N-1}}\left(kN^{\alpha-2}_{aAB_i} - k^{\frac{\alpha}{2}} N^{\alpha-2}_{aA|B_{i+1}\cdots B_{N-1}}\right).$$

Proof The proof is similar to the proof of Proposition 2.4.

The inequalities (24) and (25) give the upper bounds of polygamy relations of $N_a(\rho_{A|B_1\cdots B_{N-1}})$ for $\rho_{AB_1\cdots B_{N-1}}$.

Example 3.2 Consider again the three-qubit state ρ_{ABC} (9). We have $N_{aA|BC} = 2\lambda_0\sqrt{\lambda_2^2+\lambda_3^2+\lambda_4^2}$, $N_{aAB} = 2\lambda_0\sqrt{\lambda_2^2+\lambda_4^2}$ and $N_{aAC} = 2\lambda_0\sqrt{\lambda_3^2+\lambda_4^2}$. In view of $N_{aAB} \geq \sqrt{k}N_{aAC}$, we set $\lambda_0 = \frac{1}{2}$, $\lambda_2 = \sqrt{\frac{1}{2}}$, $\lambda_1 = \lambda_4 = \sqrt{\frac{5}{72}}$ and $\lambda_3 = \frac{1}{3}$. Then $N^{\alpha}_{aA|BC} = (\frac{49}{72})^{\frac{\alpha}{2}}$, $N^{\alpha}_{aAB} + hN^{\alpha}_{aAC} + \frac{k\alpha}{2(k+1)^2}N^{2}_{aAC}(kN_{aAB})^{\frac{\alpha}{2}} + h(\frac{13}{72})^{\frac{\alpha}{2}} + \frac{k\alpha}{18(k+1)^2}[k(\sqrt{\frac{41}{72}})^{\alpha-2} - k^{\frac{\alpha}{2}}(\sqrt{\frac{13}{72}})^{\alpha-2}]$. Our polygamy inequality of negativity of assistant is tighter than the formula $N^{\alpha}_{aAB} + \frac{(1+r)^{\alpha/2}-1}{r^{\alpha/2}}N^{\alpha}_{aAC}$ ($r=0.5$) given in Ref. [22] for $\alpha \geq 2$ and $k \geq 2$ (see Fig. 4).

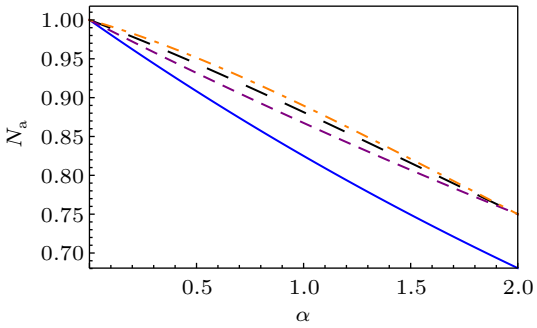


Fig. 4. The axis N_a is the negativity of assistant of $|\psi\rangle$, as a function of α. The blue solid line represents the negativity of $|\psi\rangle$ in Example 3.1. The black dotted (purple dashed) line represents the lower bound from our result for $k=2$ ($k=3$). The orange dotdashed line represents lower bound from the result ($k=0.5$) in Ref. [22].

4. Optimized relations for EoF and EoA

The EoF[35,36] is a well defined important measure of entanglement for bipartite systems. Let $\mathbb{H}_A$ and $\mathbb{H}_B$ be m and n dimensional ($m \le n$) vector spaces, respectively. The EoF of a pure state $|\psi\rangle_{AB} \in \mathbb{H}_A \otimes \mathbb{H}_B$ is defined by

$$E(|\psi\rangle_{AB}) = S(\rho_A), \tag{26}$$

where $\rho_A = \mathrm{tr}_B(|\psi\rangle_{AB}\langle\psi|)$ and $S(\rho) = -\mathrm{tr}(\rho \ln \rho)$. For a bipartite mixed state $\rho_{AB} \in \mathbb{H}_A \otimes \mathbb{H}_B$, the EoF is given by

$$E(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle_{AB}), \tag{27}$$

with the minimum taking over all possible pure state decompositions of ρ_{AB}. The entanglement of assistance (EoA) of a bipartite mixed state $\rho_{AB} \in \mathbb{H}_A \otimes \mathbb{H}_B$ is defined by

$$E_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle_{AB}), \tag{28}$$

with the maximum taking over all possible pure state decompositions of ρ_{AB}.

Denote $f(x) = H\left(\frac{1+\sqrt{1-x}}{2}\right)$, where $H(x) = -x\log_2(x) - (1-x)\log_2(1-x)$. From Eqs. (26) and (27), one has $E(|\psi\rangle) = f\left(\mathcal{C}^2(|\psi\rangle)\right)$ for $2 \otimes m$ ($m \ge 2$) pure state $|\psi\rangle$, and $E(\rho) = f\left(\mathcal{C}^2(\rho)\right)$ for two-qubit mixed state ρ.[37] It has been shown that $f(x)$ is a monotonically increasing function for $0 \le x \le 1$,[37] and satisfies the following relations:

$$f^{\sqrt{2}}(x^2+y^2) \ge f^{\sqrt{2}}(x^2) + f^{\sqrt{2}}(y^2), \tag{29}$$

$$f(x^2+y^2) \le f(x^2) + f(y^2). \tag{30}$$

For an arbitrary N-qubit state $\rho_{AB_1B_2\cdots B_{N-1}}$, it was shown that

$$\begin{aligned} &E^\alpha(\rho_{A|B_1B_2\cdots B_{N-1}}) \\ &\ge E^\alpha(\rho_{AB_1}) + E^\alpha(\rho_{AB_2}) + \cdots + E^\alpha(\rho_{AB_{N-1}}), \end{aligned} \tag{31}$$

with $\alpha \ge \sqrt{2}$, where $A|B_1B_2\cdots B_{N-1}$ is a bipartition system, and ρ_{AB_i} ($i = 1,2,\ldots,N-1$) is the reduced density matrices.[9]

Lemma 4.1 For $\alpha \ge \sqrt{2}$, and $k \ge 1$, if $x \ge \sqrt{k}y$, then

$$\begin{aligned} f^\alpha(x^2+y^2) \ge\ & f^\alpha(x^2) + \left[(k+1)^{\frac{\alpha}{\sqrt{2}}} - k^{\frac{\alpha}{\sqrt{2}}}\right] f^\alpha(y^2) \\ & + \frac{\alpha}{\sqrt{2}(k+1)} f^{\sqrt{2}}(y^2) \\ & \times \left[k f^{\alpha-\sqrt{2}}(x^2) - k^{\frac{\alpha}{\sqrt{2}}} f^{\alpha-\sqrt{2}}(y^2)\right]. \end{aligned} \tag{32}$$

Proof Due to inequalities (3) and (29), we can obtain

$$\begin{aligned} f^\alpha(x^2+y^2) &= \left(f^{\sqrt{2}}(x^2+y^2)\right)^{\frac{\alpha}{\sqrt{2}}} \\ &\ge \left(f^{\sqrt{2}}(x^2) + f^{\sqrt{2}}(y^2)\right)^{\frac{\alpha}{\sqrt{2}}} \\ &\ge f^\alpha(x^2) + \left[(k+1)^{\frac{\alpha}{\sqrt{2}}} - k^{\frac{\alpha}{\sqrt{2}}}\right] f^\alpha(y^2) \\ &\quad + \frac{\alpha}{\sqrt{2}(k+1)} f^{\sqrt{2}}(y^2) \\ &\quad \times \left[k f^{\alpha-\sqrt{2}}(x^2) - k^{\frac{\alpha}{\sqrt{2}}} f^{\alpha-\sqrt{2}}(y^2)\right]. \end{aligned}$$

Lemma 4.2 For any $2 \otimes 2 \otimes 2^{N-2}$ mixed state ρ_{ABC}, if $\mathcal{C}_{AB} \ge \sqrt{k}\mathcal{C}_{AC}$, then

$$\begin{aligned} E^\alpha_{A|BC} \ge\ & E^\alpha_{AB} + [(k+1)^{\frac{\alpha}{\sqrt{2}}} - k^{\frac{\alpha}{\sqrt{2}}}] E^\alpha_{AC} \\ & + \frac{\alpha}{\sqrt{2}(k+1)} E^{\sqrt{2}}_{AC} \left(k E^{\alpha-\sqrt{2}}_{AB} - k^{\frac{\alpha}{\sqrt{2}}} E^{\alpha-\sqrt{2}}_{AC}\right), \end{aligned} \tag{33}$$

with $\alpha \ge \sqrt{2}$ and $k \ge 1$.

Proof The proof is similar to the prove of Lemma 4.1.

Note $E_{A|BC}$ stands for the EoF of the state ρ_{ABC}, which quantifies the entanglement between the subsystems A, B and C. Lemma 4.1 implies that there exits restrictive condition among the subsystems A, B and C when they sharing information.

According to Lemma 4.1, we can obtain a new monogamy relation based on EoF.

Proposition 4.1 For $\alpha \ge \sqrt{2}$, $k \ge 1$ and any N-qubit state $\rho_{AB_1B_2\cdots B_{N-1}}$, if $\mathcal{C}_{AB_i} \ge \sqrt{k}\mathcal{C}_{A|B_{i+1}\cdots B_{N-1}}$, then

$$E^\alpha_{A|B_1B_2\cdots B_{N-1}} \ge \sum_{i=1}^{N-2} h^{i-1}(E^\alpha_{AB_i} + R_{AB_i}) + h^{N-2} E^\alpha_{AB_{N-1}}, \tag{34}$$

where

$$\begin{aligned} R_{AB_i} &= \frac{\alpha}{\sqrt{2}(k+1)} \left(E^{\sqrt{2}}_{AB_{i+1}} + \cdots + E^{\sqrt{2}}_{AB_{N-1}}\right) \\ &\quad \times \left(k E^{\alpha-\sqrt{2}}_{AB_i} - k^{\frac{\alpha}{\sqrt{2}}} E^{\alpha-\sqrt{2}}_{A|B_{i+1}\cdots B_{N-1}}\right). \\ h &= (k+1)^{\frac{\alpha}{\sqrt{2}}} - k^{\frac{\alpha}{\sqrt{2}}}. \end{aligned}$$

Proof Let $\rho_{A|B_1B_2\cdots B_{N-1}} = \sum_i p_i |\psi_i\rangle_{A|B_1B_2\cdots B_{N-1}}\langle\psi_i|$ be an optimal decomposition such that $E(\rho_{A|B_1B_2\cdots B_{N-1}}) = \sum_i p_i E(|\psi_i\rangle_{A|B_1B_2\cdots B_{N-1}})$. Because $|\psi_i\rangle_{A|B_1B_2\cdots B_{N-1}}$ is a N-qubit pure state, then

$$\begin{aligned} E(\rho_{A|B_1B_2\cdots B_{N-1}}) &= \sum_i p_i E(|\psi_i\rangle_{A|B_1B_2\cdots B_{N-1}}) \\ &= \sum_i p_i f\left(\mathcal{C}^2(|\psi_i\rangle_{A|B_1B_2\cdots B_{N-1}})\right) \\ &\ge f\left(\sum_i p_i \mathcal{C}^2(|\psi_i\rangle_{A|B_1B_2\cdots B_{N-1}})\right) \\ &\ge f\left(\sum_i (\sqrt{p_i})^2 \sum_i (\sqrt{p_i}\mathcal{C}(|\psi_i\rangle_{A|B_1B_2\cdots B_{N-1}}))^2\right) \end{aligned}$$

$$\geq f\left(\left[\sum_i p_i \mathcal{C}(|\psi_i\rangle_{A|B_1B_2\cdots B_{N-1}})\right]^2\right)$$
$$\geq f\left(\mathcal{C}^2(\rho_{A|B_1B_2\cdots B_{N-1}})\right), \quad (35)$$

where the first inequality is due to that $f(x)$ is a convex function. The second inequality is due to the Cauchy–Schwarz inequality. Due to the definition of concurrence and that $f(x)$ is a monotonically increasing function, we obtain the third inequality. Therefore, we have

$$\begin{aligned}
&E^\alpha(\rho_{A|B_1B_2\cdots B_{N-1}})\\
\geq\ & f^\alpha(\mathcal{C}^2_{AB_1}+\mathcal{C}^2_{AB_2}+\cdots+\mathcal{C}^2_{AB_{N-1}})\\
\geq\ & f^\alpha(\mathcal{C}^2_{AB_1})+hf^\alpha(\mathcal{C}^2_{AB_2}+\cdots+\mathcal{C}^2_{AB_{N-1}})\\
&+\frac{\alpha}{\sqrt{2}(k+1)}f^{\sqrt{2}}(\mathcal{C}^2_{AB_2}+\cdots+\mathcal{C}^2_{AB_{N-1}})\\
&\times\left[kf^{\alpha-\sqrt{2}}(\mathcal{C}^2_{AB_1})-k^{\frac{\alpha}{\sqrt{2}}}f^{\alpha-\sqrt{2}}(\mathcal{C}^2_{AB_2}+\cdots+\mathcal{C}^2_{AB_{N-1}})\right]\\
\geq\ & f^\alpha(\mathcal{C}^2_{AB_1})+hf^\alpha(\mathcal{C}^2_{AB_2}+\cdots+\mathcal{C}^2_{AB_{N-1}})\\
&+\frac{\alpha}{\sqrt{2}(k+1)}\left[f^{\sqrt{2}}(\mathcal{C}^2_{AB_2})+\cdots+f^{\sqrt{2}}(\mathcal{C}^2_{AB_{N-1}})\right]\\
&\times\left[kf^{\alpha-\sqrt{2}}(\mathcal{C}^2_{AB_1})-k^{\frac{\alpha}{\sqrt{2}}}f^{\alpha-\sqrt{2}}(\mathcal{C}^2_{A|B_2\cdots B_{N-1}})\right]\\
\geq\ & f^\alpha(\mathcal{C}^2_{AB_1})+hf^\alpha(\mathcal{C}^2_{AB_2})+\cdots+h^{N-2}f^\alpha(\mathcal{C}^2_{AB_{N-1}})\\
&+\frac{\alpha h^{N-3}}{\sqrt{2}(k+1)}f^{\sqrt{2}}(\mathcal{C}^2_{AB_{N-1}})\\
&\times\left[kf^{\alpha-\sqrt{2}}(\mathcal{C}^2_{AB_{N-2}})-k^{\frac{\alpha}{\sqrt{2}}}f^{\alpha-\sqrt{2}}(\mathcal{C}^2_{AB_{N-1}})\right]\\
&+\cdots+\frac{\alpha}{\sqrt{2}(k+1)}\left[f^{\sqrt{2}}(\mathcal{C}^2_{AB_2})+\cdots+f^{\sqrt{2}}(\mathcal{C}^2_{AB_{N-1}})\right]\\
&\times\left[kf^{\alpha-\sqrt{2}}(\mathcal{C}^2_{AB_1})-k^{\frac{\alpha}{\sqrt{2}}}f^{\alpha-\sqrt{2}}(\mathcal{C}^2_{A|B_2\cdots B_{N-1}})\right]\\
\geq\ & E^\alpha_{AB_1}+hE^\alpha_{AB_2}+\cdots+h^{N-2}E^\alpha_{AB_{N-1}}+\frac{\alpha h^{N-3}}{\sqrt{2}(k+1)}\left(E^{\sqrt{2}}_{AB_{N-1}}\right)\\
&\times\left[kE^{\alpha-\sqrt{2}}_{AB_{N-2}}-k^{\frac{\alpha}{\sqrt{2}}}E^{\alpha-\sqrt{2}}_{AB_{N-1}}\right]+\cdots\\
&+\frac{\alpha}{k+1}\left(E^{\sqrt{2}}_{AB_2}+\cdots+E^{\sqrt{2}}_{AB_{N-1}}\right)\left(kE^{\alpha-\sqrt{2}}_{AB_1}-k^{\frac{\alpha}{\sqrt{2}}}E^{\alpha-\sqrt{2}}_{A|B_2\cdots B_{N-1}}\right), \quad (36)
\end{aligned}$$

where the first inequality is obtained by using the monogamy inequality (31). Due to the monotonicity of the function $f^{\sqrt{2}}(x)$ and inequality(33), we get the third and the fourth inequalities. As for any $2\otimes 2$ state ρ_{AB_i}, $E(\rho_{AB_i})=f\left[\mathcal{C}^2(\rho_{AB_i})\right]$, one gets the last inequality accounting to inequality (35).

Our monogamy relation (34) is tighter than the formula (30) in Ref. [22], and we illustrate this with an example.

Example 4.1 Let us consider the three-qubit state ρ_{ABC} (9). Accounting to the conditions $\mathcal{C}_{AB_i}\geq\sqrt{k}\mathcal{C}_{A|B_{i+1}\cdots B_{N-1}}$ in Proposition 4.1, we take $\lambda_0=\frac{1}{2}$, $\lambda_2=\sqrt{\frac{1}{2}}$, $\lambda_1=\lambda_4=\sqrt{\frac{5}{72}}$ and $\lambda_3=\frac{1}{3}$. Then $E^\alpha_{A|BC}=(0.7554)^\alpha$, $E^\alpha_{AB}+hE^\alpha_{AC}+\frac{\alpha}{\sqrt{2}(k+1)}(E^{\sqrt{2}}_{AC})\left(kE^{\alpha-\sqrt{2}}_{AB}-k^{\frac{\alpha}{\sqrt{2}}}E^{\alpha-\sqrt{2}}_{AC}\right)=(0.6009)^\alpha+h(0.1873)^\alpha+\frac{\alpha(0.1873)^{\sqrt{2}}}{\sqrt{2}(k+1)}\left[k(0.6009)^{\alpha-\sqrt{2}}-k^{\frac{\alpha}{\sqrt{2}}}(0.1873)^{\alpha-\sqrt{2}}\right]$.

Assuming $k=1/r$, one has $E^\alpha_{A|BC}\geq E^\alpha_{AB}+\frac{(1+r)^{\alpha/2}-1}{r^{\alpha/2}}E^\alpha_{AC}$ given in Ref. [22]. When $k=2$, our monogamy inequality is tighter than the results ($r=0.5$) in Ref. [22] for $\alpha\geq\sqrt{2}$. The larger the k is, the tighter the monogamy inequality becomes (see Fig. 5).

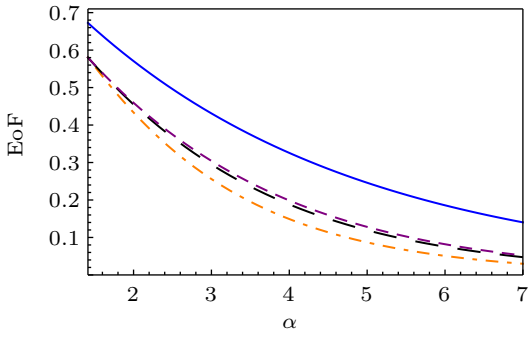


Fig. 5. EoF as a function of α. The blue solid line represents the EOF of the state $|\psi\rangle$ in Example 4.1. The black dotted (purple dashed) line represents the lower bound from our result for $k=2$ ($k=3$). The orange dotdashed line represents the lower bound from the corresponding result ($k=0.5$) in Ref. [22].

Before investigating the polygamy relations of EoA, we introduce a lemma.

Lemma 4.3 For $0\leq\alpha\leq 1$ and $2\otimes 2\otimes 2^{N-2}$ state ρ_{ABC}, if $E_{aAB}\geq kE_{aAC}$ ($k\geq 1$), then

$$E^\alpha_{aA|BC}\leq E^\alpha_{aAB}+\frac{k\alpha}{\sqrt{2}(k+1)^2}E^{\sqrt{2}}_{aAC}\left(kE^{(\alpha-\sqrt{2})}_{aAB}-k^{\frac{\alpha}{\sqrt{2}}}E^{(\alpha-\sqrt{2})}_{aAC}\right)+\left[(k+1)^{\frac{\alpha}{\sqrt{2}}}-k^{\frac{\alpha}{\sqrt{2}}}\right]E^\alpha_{aAC}. \quad (37)$$

Proof By using inequality (10) and the polygamy relation $E_{aA|BC}\leq E_{aAB}+E_{aAC}$, we achieve the result.

Note Lemma 4.3 implies that the information shared among the subsystems A, B and C restricted by the polygamy relation (37).

According to Lemma 4.3, we obtain a new polygamy relation of EoA as follows.

Proposition 4.2 For $0\leq\alpha\leq 1$ and any N-qubit quantum state $\rho_{AB_1\cdots B_{N-1}}$, if $E_{aAB_i}\geq kE_{aA|B_{i+1}\cdots B_{N-1}}$, then

$$E^\alpha_{aA|B_1B_2\cdots B_{N-1}}\leq\sum_{i=1}^{N-2}h^{i-1}(E^\alpha_{aAB_i}+R_{aAB_i})+h^{N-2}E^\alpha_{aAB_{N-1}}, \quad (38)$$

where

$$\begin{aligned}
h&=(k+1)^{\frac{\alpha}{\sqrt{2}}}-k^{\frac{\alpha}{\sqrt{2}}},\\
R_{aAB_i}&=\frac{k\alpha}{\sqrt{2}(k+1)^2}\left(E^{\sqrt{2}}_{aAB_{i+1}}+\cdots+E^{\sqrt{2}}_{aAB_{N-1}}\right)\\
&\quad\times\left(kE^{\alpha-\sqrt{2}}_{aAB_i}-k^{\frac{\alpha}{\sqrt{2}}}E^{\alpha-\sqrt{2}}_{aA|B_{i+1}\cdots B_{N-1}}\right).
\end{aligned}$$

Proof Firstly, let us consider a multiparty quantum state $\rho_{A|B_1B_2\cdots B_{N-1}}$ rather than just three parties, then

$$E^\alpha_{aA|B_1B_2\cdots B_{N-1}}\leq(E_{aAB_1}+E_{aA|B_2\cdots B_{N-1}})^\alpha, \quad (39)$$

by using the inequality (10) and iterating inequality (37) on the last term of inequality (39), we obtain the result.

Proposition 4.2 implies that polygamy inequality (38) tightens again the upper bounds of $E_a(\rho_{A|B_1\cdots B_{N-1}})$ given in Ref. [22].

5. Optimized relations for Tsallis entanglement

The Tsallis-q entropy[38] with respect to a non-negative number q can be used to characterize the classical statistical correlations inherent in quantum states.[39] For a bipartite pure state $|\psi\rangle_{AB}$, the Tsallis-q entanglement is defined by[11]

$$T_q(|\psi\rangle_{AB}) = S_q(\rho_A) = \frac{1}{q-1}\left(1-\mathrm{tr}(\rho_A^q)\right),$$

for any $q>0$ and $q\neq 1$. $T_q(\rho)$ converges to the von Neumann entropy if q tends to 1. For a bipartite mixed state ρ_{AB}, the Tsallis-q entanglement is defined via the convex-roof extension, $T_q(\rho_{AB}) = \min\sum_i p_i T_q(|\psi_i\rangle_{AB})$, with the minimum taking over all possible pure state decompositions of ρ_{AB}.

As a dual quantity to Tsallis-q entanglement, the Tsallis-q entanglement of assistant is defined as[11]

$$T_{qa}(\rho_{AB}) = \max\sum_i p_i T_q(|\psi_i\rangle_{AB}),$$

with the maximum taking over all possible pure state decompositions of ρ_{AB}.

In Ref. [40] the authors have proved an analytic relation between the Tsallis-q entanglement and the concurrence, $T_q(|\psi\rangle_{AB}) = g_q(\mathcal{C}^2(|\psi\rangle_{AB}))$. Generally, for $2\otimes m$ $(m\geq 2)$ pure state $|\psi\rangle$, $T_q(|\psi\rangle) = g_q\left(\mathcal{C}^2(|\psi\rangle)\right)$, and for two-qubit mixed state ρ, $T_q(\rho) = g_q\left(\mathcal{C}^2(\rho)\right)$,[11] where the function $g_q(x)$ is defined by $g_q(x) = \frac{1}{q-1}\left[1-\left(\frac{1+\sqrt{1-x}}{2}\right)^q-\left(\frac{1-\sqrt{1-x}}{2}\right)^q\right]$. It has been further proved that $T^2_{q\,A|B_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{N-1} T^2_{q\,A|B_i}$ for $\frac{5-\sqrt{13}}{2}\leq q\leq\frac{5+\sqrt{13}}{2}$.

It has been shown that $g_q(x)$ is monotonically increasing and convex with q, in particular, $g_q(x)$ satisfies the following relation for $2\leq q\leq 3$:

$$g_q(x^2+y^2)\geq g_q(x^2)+g_q(y^2). \tag{40}$$

For the case when $1\leq q\leq 2$ or $3\leq q\leq 4$, the function $g_q(x)$ satisfies

$$g_q(x^2+y^2)\leq g_q(x^2)+g_q(y^2). \tag{41}$$

Due to the relation (40), we can obtain the following inequality.

Lemma 5.1 For $\alpha\geq 1$ and $k\geq 1$, if $x\geq\sqrt{k}y$, then

$$\begin{aligned} g_q(x^2+y^2) \geq\ & {g_q}^{\alpha}(x^2)+((k+1)^{\alpha}-k^{\alpha})\,{g_q}^{\alpha}(y^2) \\ & +\frac{\alpha}{k+1}g_q(y^2)\left(k{g_q}^{\alpha-1}(x^2)-k^{\alpha}{g_q}^{\alpha-1}(y^2)\right). \end{aligned} \tag{42}$$

Proof Due to inequalities (3) and (40), we have

$$\begin{aligned} g_q^{\alpha}(x^2+y^2) &\geq \left(g_q(x^2)+g_q(y^2)\right)^{\alpha} \\ &= g_q^{\alpha}(x^2)\left(1+\frac{g_q(y^2)}{g_q(x^2)}\right)^{\alpha} \\ &\geq {g_q}^{\alpha}(x^2)+((k+1)^{\alpha}-k^{\alpha})\,{g_q}^{\alpha}(y^2) \\ &\quad +\frac{\alpha}{k+1}g_q(y^2)\left(k{g_q}^{\alpha-1}(x^2)-k^{\alpha}{g_q}^{\alpha-1}(y^2)\right). \end{aligned} \tag{43}$$

Lemma 5.2 For any $2\otimes 2\otimes 2^{N-2}$-dimensional mixed state ρ_{ABC}, if $\mathcal{C}_{AB}\geq\sqrt{k}\mathcal{C}_{AC}$, then $T_q(\rho)$ satisfies

$$\begin{aligned} T^{\alpha}_{q_{A|BC}} \geq\ & T^{\alpha}_{q_{AB}}+((k+1)^{\alpha}-k^{\alpha})\,T^{\alpha}_{q_{AC}} \\ & +\frac{\alpha}{k+1}T_{q_{AC}}\left(kT^{\alpha-1}_{q_{AB}}-k^{\alpha}T^{\alpha-1}_{q_{AC}}\right) \end{aligned} \tag{44}$$

with $\alpha\geq 1$ and $2\leq q\leq 3$.

Proof The proof is similar with the proof of Lemma 5.1.

Note $T_{q_{A|BC}}$ stands for the Tsallis-q entanglement between the subsystems A and BC, thus the Lemma 5.2 shows that there exits limitation among the subsystem when they sharing information.

Due to Lemma 5.2, we have the following new monogamy relation of Tsallis-q entanglement.

Proposition 5.1 For an arbitrary N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $\mathcal{C}_{AB_i}\geq\sqrt{k}\mathcal{C}_{A|B_{i+1}\cdots B_{N-1}}$ for $i=1,2,\ldots,N-2$, then Tsallis-q entanglement satisfies the following monogamy relations with $\alpha\geq 1$ and $k\geq 1$:

$$(T_q)^{\alpha}_{A|B_1B_2\cdots B_{N-1}}\geq\sum_{i=1}^{N-2}h^{i-1}((T_q)^{\alpha}_{AB_i}+G_{AB_i})+h^{N-2}T^{\alpha}_{qAB_{N-1}}, \tag{45}$$

where

$$\begin{aligned} G_{AB_i} &= \frac{\alpha}{k+1}\left((T_q)_{AB_{i+1}}+\cdots+(T_q)_{AB_{N-1}}\right) \\ &\quad\times\left(k(T_q)^{\alpha-1}_{AB_i}-k^{\alpha}(T_q)^{\alpha-1}_{A|B_{i+1}\cdots B_{N-1}}\right), \\ h &= (k+1)^{\alpha}-k^{\alpha}. \end{aligned}$$

Note that $(T_q)_{A|B_1B_2\cdots B_{N-1}}$ quantifies the Tsallis entanglement in the partition $A|B_1B_2\cdots B_{N-1}$ and $(T_q)_{AB_i}$ quantifies that in two-qubit subsystem AB_i with $i=1,2,\ldots,N-1$.

Proof Let $\rho_{A|B_1B_2\cdots B_{N-1}}=\sum_i p_i|\psi_i\rangle_{A|B_1B_2\cdots B_{N-1}}\langle\psi_i|$ be a optimal decomposition, then

$$\begin{aligned} (T_q)(\rho_{A|B_1B_2\cdots B_{N-1}}) &= \sum_i p_i T_q(|\psi_i\rangle_{A|B_1B_2\cdots B_{N-1}}) \\ &= \sum_i p_i g_q\left[\mathcal{C}^2_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right] \\ &\geq g_q\left[\sum_i p_i\mathcal{C}^2_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right] \\ &\geq g_q\left(\left[\sum_i p_i\mathcal{C}_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right]^2\right) \\ &= g_q\left[\mathcal{C}^2_{A|B_1B_2\cdots B_{N-1}}(\rho)\right], \end{aligned} \tag{46}$$

where the first inequality is due to that $g_q(x)$ is a convex function. The second inequality is due to the Cauchy–Schwarz inequality. In terms of the definition of Tsallis-q entanglement and that $g_q(x)$ is a monotonically increasing function, we obtain the third inequality. Therefore, we have

$$\begin{aligned} & T^{\alpha}_q(\rho_{A|B_1B_2\cdots B_{N-1}}) \\ &\geq g^{\alpha}_q\left[\sum_i\mathcal{C}^2(\rho_{AB_i})\right] \\ &\geq g^{\alpha}_q\left(\mathcal{C}^2_{AB_1}\right)+hg^{\alpha}_q(\mathcal{C}^2_{AB_2})+\cdots+h^{(N-3)}g^{\alpha}_q(\mathcal{C}^2_{AB_{N-2}}) \end{aligned}$$

$$
\begin{aligned}
&+h^{(N-2)}{g_q}^{\alpha}(\mathcal{C}^2_{AB_{N-1}})+h^{(N-3)}\frac{\alpha}{k+1}{g_q}^{\alpha}(\mathcal{C}^2_{AB_{N-1}})\\
&\times\left[k{g_q}^{\alpha-1}(\mathcal{C}^2_{AB_{N-2}})-k^{\alpha}{g_q}^{\alpha-1}(\mathcal{C}^2_{AB_{N-1}})\right]\\
&+\cdots+\frac{\alpha}{k+1}\left[{g_q}^{\alpha}(\mathcal{C}^2_{AB_2})+\cdots+{g_q}^{\alpha}(\mathcal{C}^2_{AB_{N-1}})\right]\\
&\times\left[k{g_q}^{\alpha-1}(\mathcal{C}^2_{AB_1})-k^{\alpha}{g_q}^{\alpha-1}(\mathcal{C}^2_{A|B_2\cdots B_{N-1}})\right]\\
\geq\ &(T_q)^{\alpha}_{AB_1}+h(T_q)^{\alpha}_{AB_2}+\cdots+h^{N-3}(T_q)^{\alpha}_{AB_{N-2}}+h^{N-2}(T_q)^{\alpha}_{AB_{N-1}}\\
&+h^{(N-3)}\frac{\alpha}{k+1}(T_q)^{\alpha}_{AB_{N-1}}\left[k(T_q)^{\alpha-1}_{AB_{N-2}}-k^{\alpha}(T_q)^{\alpha-1}_{AB_{N-1}}\right]+\cdots\\
&+\frac{\alpha}{k+1}\left((T_q)^{\alpha}_{AB_2}+\cdots(T_q)^{\alpha}_{AB_{N-1}}\right)\\
&\times\left[k(T_q)^{\alpha-1}_{AB_1}-k^{\alpha}(T_q)^{\alpha-1}_{A|B_2\cdots B_{N-1}}\right],
\end{aligned}
\tag{47}
$$

where we have used the monogamy inequality in (36) for N-qubit states to obtain the first inequality. Using (43) that $g_q(x)$ is a monotonically increasing function and the similar consideration in the proof of Proposition 2.1, we get the second inequality. Since for any $2\otimes 2$-dimensional quantum state ρ_{AB_i}, $T_q(\rho_{AB_i})=g_q\left[\mathcal{C}^2(\rho_{AB_i})\right]$, one gets the last inequality due to (46).

Example 5.1 Let us consider again the three-qubit state $|\psi\rangle$ (9). By using the definition of Tsallis-q entanglement, we have $T_{q_{A|BC}}=g_q\left[(2\lambda_0\sqrt{\lambda_2^2+\lambda_3^2+\lambda_4^2})^2\right]$, $T_{q_{AB}}=g_q(4\lambda_0^2\lambda_2^2)$ and $T_{q_{AC}}=g_q(4\lambda_0^2\lambda_3^2)$. Accounting to $\mathcal{C}_{AB_i}\geq\sqrt{k}\mathcal{C}_{A|B_{i+1}\cdots B_{N-1}}$, we take $\lambda_0=\frac{1}{2}$, $\lambda_2=\sqrt{\frac{1}{2}}$, $\lambda_1=\lambda_4=\sqrt{\frac{5}{72}}$ and $\lambda_3=\frac{1}{3}$. When $q=2$, one gets $(T_2)^{\alpha}_{A|BC}=(49/144)^{\alpha}\geq(T_2)^{\alpha}_{AB}+h(T_2)^{\alpha}_{AC}+\frac{\alpha}{k+1}(T_2)_{AC}\left(k(T_2)^{\alpha-1}_{AB}-k^{\alpha}(T_2)^{\alpha-1}_{AC}\right)=(1/4)^{\alpha}+h(1/18)^{\alpha}+\frac{\alpha}{18(k+1)}\left[k(1/4)^{(\alpha-1)}-k^{\alpha}(1/18)^{(\alpha-1)}\right]$. Assuming $k=\frac{1}{r}$, one has the relation $(T_q)^{\alpha}_{A|BC}=\left(\frac{49}{72}\right)^{\frac{\alpha}{2}}\geq(T_q)^{\alpha}_{AB}+\frac{(1+r)^{\alpha/2}-1}{r^{\alpha/2}}(T_q)^{\alpha}_{AC}$ given in Ref. [22]. Our monogamy inequality is tighter than the corresponding results ($r=0.5$) given in Ref. [22] for $\alpha\geq 1$ (see Fig. 6).

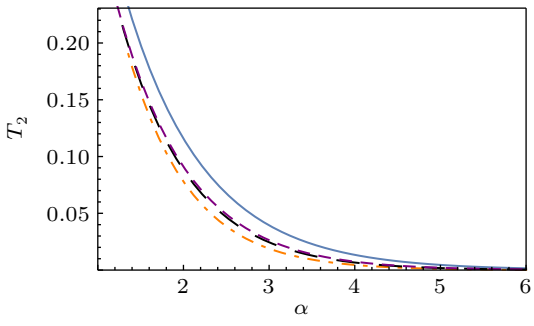


Fig. 6. The axis T_2 is the Tsallis-q ($q=2$) entanglement of $|\psi\rangle$, which is functions of α. The blue solid line represents the lower bound of the Tsallis-q of $|\psi\rangle$ ($q=2$) in Example 5.1. The black dotted (purple dashed) line represents the lower bound from our result for $k=2$ ($k=3$). The orange dotdashed line represents the lower bound from the result ($k=0.5$) in Ref. [22].

Next, we investigate the polygamy relations of the Tsallis-q entanglement of assistant (ToA). Firstly, we introduce a new polygamy relation of mixed state ρ_{ABC} as follows.

Lemma 5.3 For any $2\otimes 2\otimes 2^{N-2}$ quantum state ρ_{ABC}, if $(T_{qa})_{AB}\geq k(T_{qa})_{AC}$, then

$$
\begin{aligned}
(T_{qa})^{\alpha}_{A|BC}\ \leq\ &(T_{qa})^{\alpha}_{AB}+\frac{k\alpha}{(k+1)^2}(T_{qa})_{AC}\\
&\times\left(k(T_{qa})^{(\alpha-1)}_{AB}-k^{\alpha}(T_{qa})^{(\alpha-1)}_{AC}\right)\\
&+\left[(k+1)^{\alpha}-k^{\alpha}\right](T_{qa})^{\alpha}_{AC}
\end{aligned}
\tag{48}
$$

with $0\leq\alpha\leq 1$ and $k\geq 1$.

Analogously, due to Lemma 5.3, we can obtain a new polygamy relation for $T_{qa}(\rho_{AB_1\cdots B_{N-1}})$.

Proposition 5.2 For any N-qubit quantum state $\rho_{AB_1\cdots B_{N-1}}$, if $(T_{qa})_{AB_i}\geq k(T_{qa})_{A|B_{i+1}\cdots B_{N-1}}$ with $0\leq\alpha\leq 1$ and $k\geq 1$, then

$$
\begin{aligned}
(T_{qa})^{\alpha}_{A|B_1B_2\cdots B_{N-1}}\ \leq\ &\sum_{i=1}^{N-2}h^{i-1}\left((T_{qa})^{\alpha}_{AB_i}+G_{aAB_i}\right)\\
&+h^{N-2}(T_{qa})^{\alpha}_{AB_{N-1}},
\end{aligned}
\tag{49}
$$

where

$$
\begin{aligned}
G_{aAB_i}=\ &\frac{k\alpha}{(k+1)^2}\left((T_{qa})_{AB_{i+1}}+\cdots+(T_{qa})_{AB_{N-1}}\right)\\
&\times\left(k(T_{qa})^{\alpha-1}_{AB_i}-k^{\alpha}(T_{qa})^{\alpha-1}_{A|B_{i+1}\cdots B_{N-1}}\right),\\
h=\ &(k+1)^{\alpha}-k^{\alpha}.
\end{aligned}
$$

Note that, the new polygamy inequality (49) gives upper bounds of $T_{qa}(\rho_{A|B_1\cdots B_{N-1}})$ which is tighter than the ones given in Ref. [22].

6. Conclusion

Multipartite entanglement is of particular significance in quantum information processing. Monogamy and polygamy relations characterize the multipartite entanglement distributions among the subsystems. Using the α-th power of the bipartite entanglement, we have presented new monogamy and polygamy inequalities related to the αth power of concurrence, EoF, CREN and Tsallis-q entanglement, which are tighter than the existing entanglement monogamy and polygamy inequalities.[22–24,26] Our approach may be also applied to the investigation on monogamy and polygamy inequalities of other entanglement measures.

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