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Stationary response of colored noise excited vibro-impact system*

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The generalized cell mapping (GCM) method is used to obtain the stationary response of a single-degree-of-freedom. Vibro-impact system under a colored noise excitation. In order to show the advantage of the GCM method, the stochastic averaging method is also presented. Both of the two methods are tested through concrete examples and verified by the direct numerical simulation. It is shown that the GCM method can well predict the stationary response of this noise-perturbed system no matter whether the noise is wide-band or narrow-band, while the stochastic averaging method is valid only for the wide-band noise.

Keywords: vibro-impact system, stationary probability density function, stochastic averaging method, generalized cell mapping method

PACS: 05.10.–a, 05.10.Gg, 05.10.Ln, 05.40.–a **DOI:** 10.1088/1674-1056/abf12a

1. Introduction

The vibro-impact system is an essential kind of non-linear dynamic system, which has attracted considerable attention in the past few decades.[1–6] Owing to its non-smooth characteristics, some interesting and distinguishing phenomena such as gazing bifurcation, tour bifurcation, chatter, and sticking motions are observed in the deterministic system.[7–12] In terms of stochastic dynamics, efforts are made to derive the stationary response of the vibro-impact system under the excitation of various noises. The most common ways are the stochastic averaging method and path integration method.[13–15] Huang[16] developed a stochastic averaging method to obtain the stationary response of a multi-degree-freedom vibro-impact system under the white Gaussian noise excitation. Recently, a solution procedure consisting of Zhuravlev non-smooth transformation and exponential polynomial closure method has been proposed to obtain the probability density function (PDF) of a lightly vibro-impact system excited by different white Gaussian noises.[16] In Ref. [17], the path integration was greatly simplified to accurately obtain the stationary response by introducing a suitable transformation of the state space to deal with the vibration systems' discontinuities with strong inelastic impact. More recently, an iterative weighted residual method was employed to obtain the approximate closed-form solution of the random response of a single-degree-freedom vibro-impact system under the white Gaussian noise.[18] More recent efforts dedicated to the vibro-impact system's stationary responses under various random excitations are available in Refs. [19–27].

For the stochastic averaging method, one important step is to transfer the physical process to the Ito diffusion process. Because the existing theory can only obtain the accurate drift coefficient and diffusion coefficient for the system under the physical white noise or the wide-band noise, the application of the stochastic averaging method is limited.[28] It can be seen that almost all the excitations in the above researches are in white or wide-band. However, in nature, the real noises always have a finite correlation time. When the noises' correlation time is considerably long, there are no effective methods to obtain the stationary response.

In recent years, the stochastic GCM method has been wildly used to obtain the response PDFs for systems subjected to white excitations.[29–36] In this paper, based on the Khasminskii limit theorem, the GCM method is improved to obtain the stationary response of a vibro-impact system under a colored noise excitation. Owing to no need to obtain the precise drift coefficient or diffusion coefficient of the approximated Markov process but the process's evolution law, the GCM method can be applied to both the wide-band noise excited system and the narrow-band noise excited system.

The rest of the paper is organized as follows. In Section 2, the model of the vibro-impact system is described. The procedure of the stochastic averaging method for the system is presented in Section 3. In Section 4, the GCM method for the white noise excited system is introduced, and its applicability to the colored noise excited system is investigated. In Section 5, both the methods are tested for wide-band noise and narrow-band noise, and the results are verified by the Monte Carlo simulation. Finally, the conclusions are summarized in Section 6.

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2. Vibro-impact system

Vibro-impact systems are encountered in many engineering applications such as structures and machines with rigid barriers, clearance, and gaps. Therefore, it is essential to study the dynamics of the vibro-impact system. Here we consider a vibro-impact system with a clearance type nonlinearity as shown in Fig. 1. The motion equation of the system is given as[37]

$$\ddot{x}+2\zeta\dot{x}+kx=F+f(t), \tag{1}$$

where x is the system displacement and overhead dot denotes differentiation with respect to time t; ζ is the linear damping coefficient; k is the linear stiffness; $b_{\rm l}$ is the distance between the equilibrium position of the mass and the lift barrier; $b_{\rm r}$ is for the right barrier; F is the contact force governed by Hertz contact model when the mass crashes to the barriers [38] and it is expressed as

$$F=\begin{cases}-\eta_{\rm r}(x-b_{\rm r})^{3/2}, & x>b_{\rm r},\\ 0, & -b_{\rm l}<x<b_{\rm r},\\ -\eta_{\rm l}(x+b_{\rm l})^{3/2}, & x<-b_{\rm l},\end{cases} \tag{2}$$

where $\eta_{\rm l}$ and $\eta_{\rm r}$ represent contact stiffness between the mass and the barriers on both sides respectively, and they are dependent on their geometries and material properties; the excitation f in Eq. (1) is assumed to represent a stationary and ergodic process, which is generated by a white noise passing through a second-order linear filter, and it satisfies the following equation:

$$\ddot{f}+2\zeta_1\omega_1\dot{f}+\omega_1^2 f=\sqrt{2DW}(t). \tag{3}$$

Let $S(\omega)$ be the rational spectral density and can be presented as

$$S(\omega)=\frac{D}{\pi}\frac{1}{(\omega^2-\omega_1^2)+4\zeta_1^2\omega_1^2\omega^2}, \tag{4}$$

where ζ_1, ω_1, and D are constants, and they determine the bandwidth of the noise; $W(t)$ is a standard white Gaussian noise, which can be either a wide-band noise or a narrow-band noise.

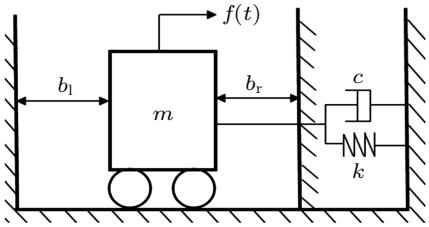


Fig. 1. Schematic model of vibro-impact oscillator.

Combining the contact force F with the spring force kx, equation (1) can be rewritten as

$$\begin{cases}\dot{x}=y,\\ \dot{y}=-2\zeta y-g(x)+f(t),\end{cases} \tag{5}$$

where $g(x)$ is a restoring force of the system and it is expressed as

$$g(x)=\begin{cases}kx+\eta_{\rm r}(x-b_{\rm r})^{3/2}, & x>b_{\rm r},\\ kx, & -b_{\rm l}<x<b_{\rm r},\\ kx-\eta_{\rm l}(-x-b_{\rm l})^{3/2}, & x<-b_{\rm l}.\end{cases} \tag{6}$$

The potential energy function and the total energy function of the system can be obtained as

$$\begin{aligned}V(x) &= \int_0^x g(X)\,{\rm d}X\\ &= \begin{cases}\dfrac{k}{2}x^2+\dfrac{2}{5}\eta_{\rm r}(x-b_{\rm r})^{5/2}, & x>b_{\rm r},\\ \dfrac{k}{2}x^2, & -b_{\rm l}<x<b_{\rm r},\\ \dfrac{k}{2}x^2+\dfrac{2}{5}\eta_{\rm l}(-x-b_{\rm l})^{5/2}, & x<-b_{\rm l},\end{cases}\end{aligned} \tag{7}$$

$$H=\frac{1}{2}y^2+V(x). \tag{8}$$

3. Stochastic averaging method

In this section, the stochastic averaging method is conducted to obtain the stationary responses of the system. Before moving on, it should be noted that equation (5) is a strongly nonlinear oscillator and does not meet the requirement of the stochastic averaging method. According to Ref. [39], the generalized harmonic transformation should be used to modify the system into a correct form.

Assuming that the damping coefficient is slight and the external random excitation is weak, the motion of Eq. (5) is nearly periodic and can be written as

$$\begin{cases}x=A\cos(\Phi(t))+B,\\ y=-Av(A,\Phi)\sin(\Phi(t)),\\ \Phi(t)=\psi(t)+\theta(t),\end{cases} \tag{9}$$

where

$$v(A,\Phi)=\frac{{\rm d}\psi}{{\rm d}t}=\sqrt{\frac{2[V(A+B)-V(A\cos\Phi+B)]}{A^2\sin^2\Phi}}, \tag{10}$$

with B being a deviation process, A an amplitude process, Φ, ψ, and θ the phase processes, and v the instantaneous frequency of the oscillator. A and B are related to H by

$$V(A+B)=V(-A+B)=H. \tag{11}$$

Transform the variable from x and y into A and θ respectively, then equation (5) will become

$$\begin{cases}\dfrac{{\rm d}A}{{\rm d}t}=m_1+\sigma_1 f(t),\\ \dfrac{{\rm d}\theta}{{\rm d}t}=m_2+\sigma_2 f(t),\end{cases} \tag{12}$$

where

$$\begin{cases} m_1 = -\dfrac{2\zeta A^2 \nu^2(A,\Phi)\sin^2\Phi}{g(A+B)(1+h)}, \\ \sigma_1 = -\dfrac{A\nu(A,\Phi)\sin\Phi}{g(A+B)(1+h)}, \\ m_2 = -\dfrac{2\zeta A\nu^2(A,\Phi)\sin\Phi(\cos\Phi+h)}{g(A+B)(1+h)}, \\ \sigma_2 = -\dfrac{\nu(A,\Phi)(\cos\Phi+h)}{g(A+B)(1+h)}, \\ h = \dfrac{\mathrm{d}B}{\mathrm{d}A} = \dfrac{g(-A+B)+g(A+B)}{g(-A+B)-g(A+B)}. \end{cases} \tag{13}$$

On the assumption that the damping coefficient is slight and the random excitation is weak, the terms on the right-hand side of transformed equation (12) are indeed small; hence, the transformed equation (12) is a standard form for the Stratonovich–Khasminskii limit theorem, A and θ will converge weakly to a two-dimensional diffusive Markov process as ζ, $D \to 0$. Furthermore, when the random excitation is in the wide-band, the Markov process can be represented by the following Ito stochastic differential function

$$\begin{cases} \mathrm{d}A = \tilde{m}_1 \mathrm{d}t + \tilde{\sigma}_{1r}\mathrm{d}B_r(t), \\ \mathrm{d}\theta = \tilde{m}_2 \mathrm{d}t + \tilde{\sigma}_{2r}\mathrm{d}B_r(t), \end{cases} \quad r = 1,2, \tag{14}$$

where $B_r(t)$ represents the standard Wiener process. $\tilde{m}_i$ and $\tilde{b}_{ij} = \tilde{\sigma}_{ir}\tilde{\sigma}_{jr}$ are called drift coefficient and diffusion coefficient, respectively. They can be obtained from the following formula:

$$\begin{cases} \tilde{m}_i = m_i + \int_{-\infty}^{0} \left(\dfrac{\partial \sigma_i}{\partial A}|_t \sigma_1|_{t+\tau} + \dfrac{\partial \sigma_i}{\partial \theta}|_t \sigma_2|_{t+\tau} \right) R_{\mathrm{f}}(\tau)\mathrm{d}\tau, \\ \tilde{b}_{ij} = \tilde{\sigma}_{ir}\tilde{\sigma}_{jr} = \int_{-\infty}^{\infty} (\sigma_i|_t \sigma_j|_{t+\tau}) R_{\mathrm{f}}(\tau)\mathrm{d}\tau, \\ i,j,r = 1,2, \end{cases} \tag{15}$$

where R_{f} is the correlation function of the random noise.

Considering the fact that the A and θ are the slowly varying variables, the drift and diffusion coefficient in Eq. (14) can be smoothened by time-averaging. Denoting the time-averaged drift and diffusion coefficient by

$$\begin{cases} \bar{m}_i = \dfrac{1}{2\pi}\int_0^{2\pi} \tilde{m}_i(A,\Phi)\mathrm{d}\Phi, \\ \bar{\sigma}_{ik}\bar{\sigma}_{jk} = \bar{b}_{ij} = \dfrac{1}{2\pi}\int_0^{2\pi} \tilde{\sigma}_{ik}(A,\Phi)\tilde{\sigma}_{jk}(A,\Phi)\mathrm{d}\Phi, \\ \bar{\sigma}_{ik}\bar{\sigma}_{jk} = \bar{b}_{ij} = 0, \quad (i \neq j)\ \ i,j = 1,2, \end{cases} \tag{16}$$

the Ito equation becomes

$$\begin{cases} \mathrm{d}A = \bar{m}_1 \mathrm{d}t + \bar{\sigma}_{1r}\mathrm{d}B_r(t), \\ \mathrm{d}\theta = \bar{m}_2 \mathrm{d}t + \bar{\sigma}_{2r}\mathrm{d}B_r(t), \end{cases} \quad r = 1,2. \tag{17}$$

It can be seen that the smoothed A process becomes independent of θ. Hence it can be presented by the following one-dimensional Ito equation:

$$\mathrm{d}A = \bar{m}_1 \mathrm{d}A + \bar{b}_{11}^{1/2}\mathrm{d}B(t). \tag{18}$$

The associated FPK equation has the form

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial A}\left[\bar{m}_1(A)\,p\right] + \frac{1}{2}\frac{\partial^2}{\partial A^2}\left[\bar{b}_{11}(A)\,p\right], \tag{19}$$

where $p = p(A,t|A_0)$ is the transition probability density of A with the initial condition

$$p(A,0|A_0) = \delta(A - A_0). \tag{20}$$

Assuming that the boundary condition p at $A = 0$ is finite, and

$$p, \mathrm{d}p/\mathrm{d}A \to 0, \quad \text{as} \quad A \to \infty, \tag{21}$$

the stationary solution of FPK equation can be derived analytically as

$$p(A) = \frac{C}{\bar{b}_{11}}\exp\left[\int_0^A \frac{2\bar{m}_1(x)}{\bar{b}_{11}(x)}\mathrm{d}x\right], \tag{22}$$

where C is the normalization constant.

As the stationary $p(A)$ is solved, the stationary PDF of the total energy $p(H)$ can be derived from $p(A)$ as

$$p(H) = \left.\frac{p(A)}{|\mathrm{d}H/\mathrm{d}A|}\right|_{A=V^{-1}(H)}, \tag{23}$$

where $A = V^{-1}(H)$ is the inverse function of $H = V(A + B(A))$.

The stationary joint PDF of displacement and velocity can be further obtained as

$$p(x,y) = \left.\frac{p(H)}{T(A(H))}\right|_{H=\frac{1}{2}y^2+V(x)}, \tag{24}$$

where T is the instantaneous period of the system and expressed as

$$T = \int_0^{2\pi} 1/v(A,\Phi)\mathrm{d}\Phi. \tag{25}$$

The stationary PDFs of displacement and velocity can be expressed as

$$p(x) = \int_{-\infty}^{\infty} p(x,y)\mathrm{d}y, \tag{26}$$

$$p(y) = \int_{-\infty}^{\infty} p(x,y)\mathrm{d}x. \tag{27}$$

Finally, it is mentioned that as the stochastic averaging method isoperated, v, m_i, and σ_i in Eqs. (10) and (13) are usually approximated by only the first 6 terms of the Fourier series to simplify the computation procedure with a relative error of less than 0.03%[39]

$$v^{-1}(A,\Phi) = C_0(A) + \sum_{n=1}^{\infty} C_n(A)\cos(n\Phi), \tag{28}$$

$$m_i(A,\Phi) = m_{i0}(A) + \sum_{n=1}^{\infty} m_{in}^c(A)\cos(n\Phi) + m_{in}^s(A)\sin(n\Phi), \quad (29)$$

$$\sigma_i(A,\Phi) = \sigma_{i0}(A) + \sum_{n=1}^{\infty} \sigma_{in}^c(A)\cos(n\Phi) + \sigma_{in}^s(A)\sin(n\Phi). \quad (30)$$

4. Generalized cell mapping method

In recent years, the stochastic GCM method is greatly developed to obtain the response PDFs for systems subjected to random excitations.[29–36] Compared with the traditional numerical methods, the GCM can greatly reduce the simulation time and achieve a very accurate result. However, the application of the GCM method requires the response process of the random excited system to be a Markov process. Hence its application is limited only to the white noise excited system. So far, no work has been done on the applicability of the GCM method to the colored noise excited system.

In this section, we introduce the GCM method for the systems subjected to the white noise, then the possibility of extending the GCM method to our colored noise excited vibro-impact system is investigated.

4.1. GCM for system with white noise

First, we give the theoretical mechanism for the GCM method. As is well known, the response $x=(x,y)$ in Eq. (5) will form a Markov process if the excitation f is a white noise. Therefore according to the Markov property, for any finite value $0 \le t_1 < t_2 < \cdots < t_n$, the following equations hold

$$p\left(x_{t_n}|x_{t_1},x_{t_2},\ldots x_{t_{n-1}}\right) = p\left(x_{t_n}|x_{t_{n-1}}\right), \quad (31)$$

$$p(x,t|x_0,t_0) = \int_{\Omega} p\left(x,t|x',t'\right) p\left(x',t'|x_0,t_0\right) \mathrm{d}x', \quad (32)$$

where $t_0 \le t' \le t$, Ω is the state space for x. Dividing the interval $[t,t_0]$ into n subinterval $t_0 < t_1 < \cdots < t_n = t$, equation (32) can be rewritten as

$$p(x,t|x_0,t_0) = \int_{\Omega} p(x,t|x_{n-1},t_{n-1})\mathrm{d}x_{n-1} \times \int_{\Omega} p(x_{n-1},t_{n-1}|x_{n-2},t_{n-2})\,\mathrm{d}x_{n-2}\cdots \times \int_{\Omega} p(x_2,t_2|x_1,t_1)\,p(x_1,t_1|x_0,t_0)\,\mathrm{d}x_1. \quad (33)$$

Note that equation (5) is the time-homogenous $p(x_n,t_n|x_{n-1},t_{n-1}) = p(x_n,t_n-t_{n-1}|x_{n-1},0)$ and $t_i - t_{i-1} = \Delta t$, $\forall i \in 1,\ldots,n$. Then the PDF $p(x,t)$ at any time t can be obtained as

$$p(x,t) = p(x,t_0+n\Delta t) = \int_{\Omega} p(x,\Delta t|x_{n-1})\,\mathrm{d}x_{n-1} \times \int_{\Omega} p(x_{n-1},\Delta t|x_{n-2})\,\mathrm{d}x_{n-2} \times \cdots \times \int_{\Omega} p(x_1,\Delta t|x_0)p(x_0,t_0)\,\mathrm{d}x_0. \quad (34)$$

Now, we give the detailed procedure of operating the GCM method. First, we discretize the continuous state space D into N small cells $C_i\,(i=1,2,\ldots,N)$ as shown in Fig. 2. Let $p_i(n)$ denote the probability of the cell C_i at n-th step in steps of mapping length Δt, and p_{ij}, the one-step transition probability from cell C_j to cell C_i is

$$p_i(n) = \int_{C_i} p(x,t_0+n\Delta t)\,\mathrm{d}x, \quad (35)$$

$$p_{ij} = \int_{C_i}\int_{C_j} p\left(x,\Delta t|x'\right)\mathrm{d}x'\,\mathrm{d}x. \quad (36)$$

Then the probability evolution of the system can be described by a homogeneous Markov chain

$$p(n) = P\cdot p(n-1) = P^n\cdot p(0), \quad (37)$$

where $p(0)$ denotes the initial probability distribution and P is the one-step transition probability matrix with elements p_{ij}. Thus, once the one-step transition probability matrix P and the initial probability distribution $p(0)$ are given, the transient and the stationary PDF can be computed.

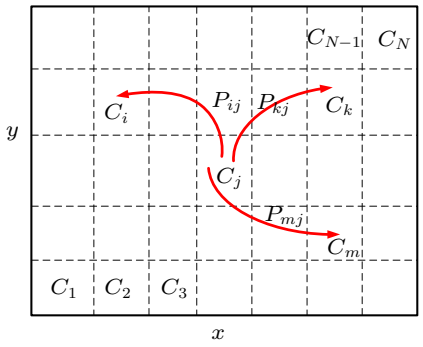


Fig. 2. Schematic diagram for GCM method.

Finally, we take p_{ij} for example to show how to obtain the matrix P by using the GCM method. First, Q_j inner points are uniformly selected in the cell C_j and s random sample trajectories are generated from each point. In this manner, there are total $S = sQ_j$ trajectories constructed in the cell C_j. It is assumed that there are S_i trajectories of time duration Δt falling into cell C_i, then the one-step transition probability from cell C_j to cell C_i can be calculated from $p_{ij} = S_i/S$.

4.2. GCM for colored noise excited system

The method mentioned above is based on the fact that the response process under white noise is a homogenous Markov process. Hence, equation (31) is always tenable whatever $\Delta t = t_n - t_{n-1}$ is chosen. However, when the excitation is a colored noise in our system, the response process of Eq. (5) is no longer a Markov process. Therefore, will the GCM method not be applicable to our system anymore? The following analysis will indicate that the answer is yes.

For the systems having the following form:

$$\frac{\mathrm{d}X_j}{\mathrm{d}t} = \varepsilon^2 f_j(X) + \varepsilon g_{j\mathrm{r}}(X)\xi_{\mathrm{r}}(t), \quad 0<\varepsilon\ll 1, \quad (38)$$

where the f_j and g_{jr} are deterministic functions and ξ_r are stationary random excitations with zero means, the Khasminskii limit theorem[40] shows that the process $X(t)$ will converge weakly to a Markov diffusion process in a semi-infinite time interval as $\varepsilon \to 0$. Furthermore, supposing that $X(t)$ is not observed continuously, but discretely, beginning from t in time intervals of $\Delta t'$. Then, if the observation time interval $\Delta t' \geq \tau_{rs}$, where τ_{rs} is the correlation time between ξ_r and ξ_s, the authors in Refs. [28,41] show that the observed increment $X(t+\Delta t') - X(t)$ between those time intervals will become independent. The stochastic averaging method is exactly based on this theory and uses the time-averaged first and second increment moments as the drift coefficient and diffusion coefficient as follows:[28]

$$\begin{aligned} a_j(X,t) &= \frac{1}{\Delta t'} E[\Delta X_j \,|X_t\,], \\ b_{jk}(X,t) &= \frac{1}{\Delta t'} E[\Delta X_j \Delta X_k \,|X_t\,]. \end{aligned} \tag{39}$$

Especially when the noise is wide-band ($\tau_{rs} \ll 1$), the observation time interval can be chosen as $0 < \tau_{rs} < \Delta t' \ll 1$; in this short interval, the variation of the f_j and g_{jr} are very slight, the drift coefficient and the diffusion coefficient of the diffusion process can be given accurately as[28]

$$\begin{cases} a_j(X,t) = \varepsilon^2 f_j(X_t) + \varepsilon^2 \dfrac{1}{\Delta t'} \displaystyle\int_t^{t+\Delta t'} \mathrm{d}u \int_t^u g_{ls}(X_t) \frac{\partial}{\partial x_l} g_{jr}(X_t)\, E\left[\xi_r(u)\,\xi_s(v)\right] \mathrm{d}v, \\ b_{jk}(X,t) = \varepsilon^2 \dfrac{1}{\Delta t'} \displaystyle\int_t^{t+\Delta t'} \mathrm{d}u \int_t^{t+\Delta t'} g_{jr}(X_t) g_{ks}(X_t)\, E\left[\xi_r(u)\,\xi_s(v)\right] \mathrm{d}v. \end{cases} \tag{40}$$

The physical process equation (12) approximated by the Ito stochastic differential function Eq. (14) in Section 3 is exactly based on Eq. (40). When the noises' correlation time is considerably long, the observation time interval $\Delta t'$ should also be chosen long. In this time interval, f_j and g_{jr} change significantly, and equation (40) will no longer hold. As a consequence, the stochastic averaging method will become invalid. While for the application of the GCM method, we do not need to calculate precisely the drift coefficient or the diffusion coefficient but only use the conclusion that the observed process $X(t)$ is a Markov-like process when $\Delta t' > \tau_{rs}$. Hence, no matter whether the noise is wide-band or narrow-band, if only the one-step mapping time in Eq. (12) is chosen to be $\Delta t > \tau_{rs}$, the transition probability $p\left(x_{t_n} | x_{t_{n-1}} \ldots, x_{t_2}, x_{t_1}\right)$ can be approximated by $p\left(x_{t_n} | x_{t_{n-1}}\right)$ and the GCM method will be applicable. Furthermore, according to the transformation in Eq. (9), the observed original process equation (5) should also be a Markov chain. Therefore, if the one-step mapping time Δt of the GCM method is chosen to be greater than the correlation time of noise f, the GCM method can be directly applied to the original process Eq. (5).

5. Examples

In this section, both the Stochastic averaging method and the generalized cell mapping method are tested by using concrete examples, and the MC simulation is conducted to validate their effectiveness.

The system parameters are given as $\zeta = 0.1$, $k = 1$, $\eta_l = \eta_r = 80$, $b_l = 0.2$, and $b_r = 0.1$. In the first case, the noise's parameters are chosen as $\zeta_l = 0.6$, $\omega_l = 6$, and $D = 0.4$. With such specified parameters, the excitation is in wide band. The excitation's correlation function is shown in Fig. 3. It can be seen that when the absolute value of the time difference τ is greater than 2 s, the two forces $f(t)$ and $f(t+\tau)$ become uncorrelated $E\left[f(t)\, f(t+\tau)\right] \approx 0$.

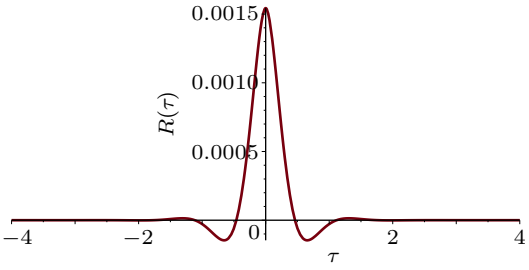


Fig. 3. Correlation function of excitation f.

The concrete steps of the GCM method are given as follows. First, we choose the domain of interest, $Z = \{-0.4 < x < 0.4,\ -0.4 < y < 0.4\}$, in the state space and divide the domain Z into 80×80 cells. After that, 5×5 interior points are uniformly selected within each cell, and 1000 sample trajectories with the time duration Δt are generated from each of these points. Hence, the one-step transition probability matrix P is constructed by the total 2.5×10^4 sample trajectories in each cell. The initial probability is assumed to be a uniform distribution, then the stationary PDF of the response process can be computed according to Eq. (37).

The stationary PDFs of amplitude, total energy, displacement, and velocity derived by the stochastic averaging method are shown in Fig. 4 with the solid lines; those of the GCM method with different one-step mapping times are shown with dot-dash lines. From Fig. 4, we can see that the results obtained by the averaging method agree well with those from the MC simulation method. Meanwhile, the results of the GCM method gradually converge to those of the averaging method as the one-step mapping time Δt increases. When Δt is chosen

to be greater than 5 s, it is sufficient to have a good result with a high accuracy. The joint probability densities of velocity and displacement shown in Fig. 5 are also in good agreement with the results from all the methods.

Now, we choose another set of noise parameters $\zeta_1 = 0.6$, $\omega_1 = 0.5$, $D = 0.0005$, and $D = 0.002$. The low frequency $\omega_1 = 0.5$ in Eq. (3) ensures that the noise is a narrow-band noise, while the small D ensures that the noise is weak. The excitation's correlation function is shown in Fig. 6, and when the time difference τ is greater than 20 s, $E[f(t)f(t+\tau)] \approx 0$. Hence, when the GCM method is applied, the one-step mapping time Δt must be chosen to be greater than 20 s, and here we chose $\Delta t = 30$ s.

The stationary PDFs of amplitude, total energy, displacement, and velocity derived by both the stochastic averaging method and the GCM method are shown in Fig. 7. In this case, the stochastic averaging method is not valid anymore. The reason is that as the correlation time of the noise increases, the observation time gap should also be increased, leading the drift coefficient and diffusion coefficient of the Markov diffusion process 1 not to be in the form of Eq. (40) anymore. In contrast, since the GCM method does not need to calculate those specific quantities but only to obtain the evolution law of the process, it can still predict the results well.

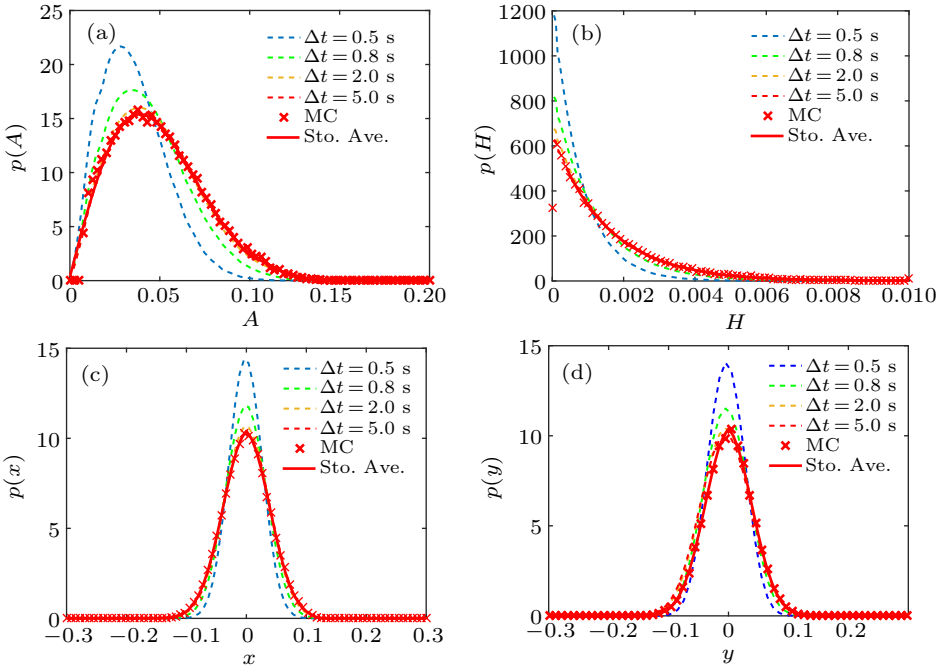


Fig. 4. Stationary probability densities: (a) amplitude, (b) total energy, (c) displacement, and (d) velocity. (dash line: stochastic averaging; solid line: generalized cell mapping; Cross ×: Monte Carlo simulation).

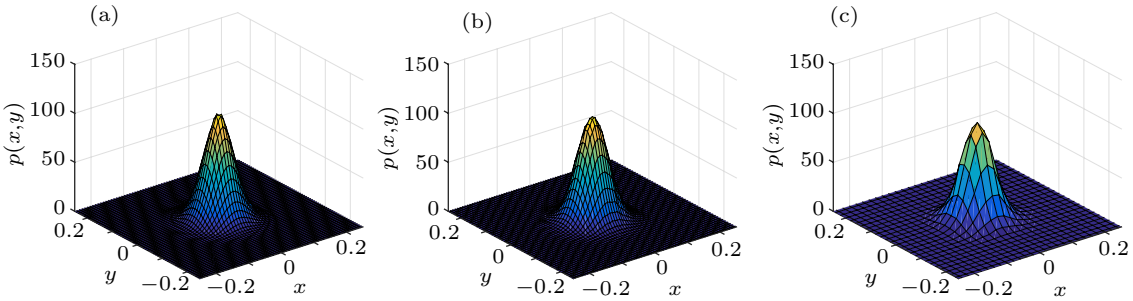


Fig. 5. Joint probability density of velocity and displacement (a) stochastic averaging, (b) generalized cell mapping with $\Delta t = 5$ s, and (c) Monte Carlo simulation.

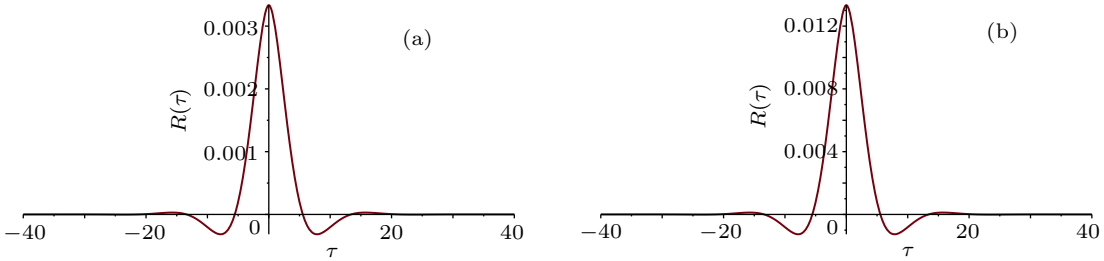


Fig. 6. Correlation function of the excitation f for (a) $D = 0.0005$ and (b) 0.002.

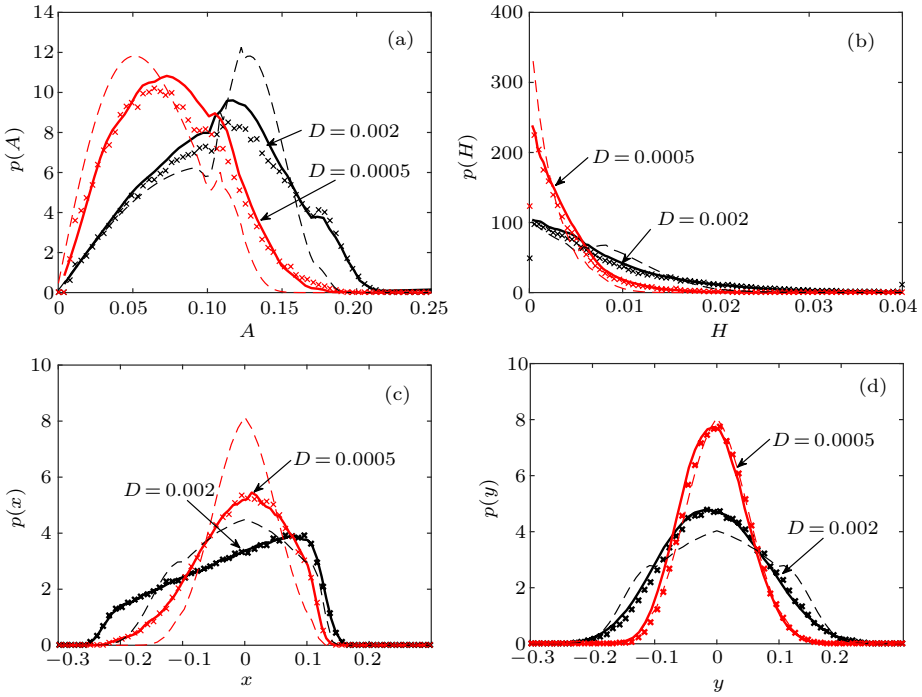


Fig. 7. Probability densities (red, $D = 0.0005$; black, $D = 0.002$): (a) amplitude, (b) total energy, (c) displacement, (d) velocity. The dash line is stochastic averaging; solid line is generalized cell mapping $\Delta t = 30$ s; cross $\times$: Monte Carlo simulation.

6. Conclusions

Based on the Hertz impact model, the response of a single-degree-of-freedom vibro-impact system under the random colored noise is investigated. First, the stochastic averaging method is carried out routinely. Then the GCM method, which is theoretically only applicable to the Markov process, is also tried for this colored noise excited system. The results show that if the one-step mapping time is chosen to be greater than the correlation time of the noise, the probability evolution of the system can be approximated by the one-step transition probability matrix. The application examples show that the GCM method can well predict the results no matter whether the noise is wide-band or narrow-band, while the stochastic averaging method is valid only for the wide-band noise. Hence, the GCM method's application is greatly extended. It is no longer limited to the white noise excited system, but still to the colored noise excited system. On the other hand, due to the successful prediction of the GCM method, if we accurately work out the drift coefficient and diffusion coefficient of the approximated Markov process, the application of the stochastic averaging method to a narrow-band noise excited system still seems possible.

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