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Detection of multi-spin interaction of a quenched *XY* chain by the average work and the relative entropy*

Xiu-Xing Zhang(张修兴)[1,†], Fang-Jv Li(李芳菊)[1], Kai Wang(王凯)[1], Jing Xue(薛晶)[1], Guang-Wen Huo(霍广文)[1], Ai-Ping Fang(方爱平)[2], and Hong-Rong Li(李宏荣)[2,‡]

[1] *Department of Physics, Weinan Normal University, Wei'nan* 714099, *China*
[2] *School of Physics, Xi'an Jiaotong University, Xi'an* 710049, *China*

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We investigate the nonequilibrium thermodynamics of a quenched *XY* spin chain with multi-spin interaction in a transverse field. The analytical expressions of both the average work and the relative entropy are obtained under different quenching parameters. The influences of the system parameters on the nonequilibrium thermodynamics are investigated. We find that at finite temperature the critical phenomenon induced by the multi-spin interaction and the external field can be revealed by the properties of the system nonequilibrium thermodynamics. In addition, our results indicate that the average work and the relative entropy can be used to detect both the existence and strength of the multi-spin interaction in the nonequlibrium system.

Keywords: nonequilibrium thermodynamics, spin chain, multi-spin interaction

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1. Introduction

In recent years, quantum thermodynamics has become a thriving research field.[1–12] The theory of quantum thermodynamics aims at investigating how the concepts of heat, work and temperature can be extended to the quantum realm. It provides us the opportunity to understand the microscopic foundations of the laws of thermodynamics. In addition, it also spurs the designation of various quantum thermal devices, such as quantum heat engines,[13–19] quantum refrigerator,[20–22] and quantum batteries.[23,24]

At present, much focus has been put on the investigation of thermodynamics in the quenched quantum systems.[25–38] On the one hand, with help of ultra-cold atomic techniques, one can measure the probability distribution of work done on quenched gases of two-level systems.[39,40] These experimental results stimulate researchers' interest in the quenched systems. On the other hand, following a quench the quantum fluctuations will be generated, which has enriched the system's phase diagram.[41,42] Thus, thoroughly investigating the critical behavior of a quenched system may open up a deep underlying connection between the quantum phase transition (QPT) and the quantum thermodynamics theory. Along this line, many works have been carried out.[34–37] Dorner *et al.*[34] investigated thermodynamics in a quenched transverse Ising model and found that the irreversible entropy production exhibits sharp peaks near the critical point. In Ref. [35], the work statistics has been studied in the quenched anisotropic *XY* model. It shows that the work fluctuation exhibits non-analytical behavior when the quench is done across quantum critical point. Wang and coworkers[37] investigate the influence of the DM interaction on the nonequilibrium thermodynamics in the *XY* chain. Their results indicate that the critical behavior induced by the DM interaction can be revealed via the behavior of the work statistics and the irreversible entropy production. Moreover, it is also found that the work statistics in the vicinity of criticality is connected to the critical Casimir free energy.[43,44]

In the previous studies of nonequilibrium thermodynamics in spin chain systems, only the nearest-neighbor spin interaction is considered.[26–28] Besides the two-spin kind of interactions, however, multi-spin interactions may also exist in spin chains.[45–48] It has been proved that the phase transition points can be changed by the multi-spin interaction.[48] Consequently, some physical quantities, such as the entanglement, the quantum discord and the geometric phase, exhibit singular properties at the phase transition point.[48] A question then arises: how can we detect the multi-spin interaction in a spin system? Moreover, it has been shown that the average work changes greatly around the phase transition points of the spin system and can signal out the appearance of phase transitions. According to this point, the average work is expected as a detector of phase transitions in the nonequilibrium systems.[34–37] With the same reason, we expect that the average work and relative entropy may be used to detect the

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†Corresponding author. E-mail: zhangxiuxing216@163.com
‡Corresponding author. E-mail: hrli@mail.xjtu.edu.cn

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multi-spin interaction in the quenched systems. Starting from this point, we here investigate the nonequilibrium thermodynamics in the spin chain system, in which both the nearest-neighbor spin interaction and the three-spin interaction are included. We analytically evaluate the statistical properties of the average work and the relative entropy under different sudden quenching parameters. The results show that the average work and the relative entropy can be used as the detector of the multi-spin interaction in the nonequilibrium system. Furthermore, the QPT that occurs in the nonequilibrium process can also be revealed.

The rest of this paper is organized as follows. In Section 2, we briefly review some key concepts about the nonequilibrium thermodynamics. In Section 3, the XY spin chain with multi-spin interaction is diagonalized and the characteristic function is obtained. In Section 4, main results and related discussions are presented. A summary is given in Section 5.

2. Nonequilibrium thermodynamics

In this section, we will review the key concept about the nonequilibrium thermodynamics and the formalisms that will be used in the rest of this paper. We consider a quantum system, whose initial state can be described by the Gibbs state

$$\rho(\lambda_0)=\frac{\exp[-\beta H_0]}{Z(\lambda_0)}, \tag{1}$$

where λ_0 is the initial value of the system parameter λ; $Z(\lambda_0)=\mathrm{Tr}[\exp[-\beta H_0]]$ is the partition function; H_0 corresponds to the initial system Hamiltonian; and $\beta=1/T$ ($k_\mathrm{B}=1$) is the reciprocal temperature.

From time $t=0$, the system is quenched out of equilibrium through external work till $t=\tau$. As a result, the system parameter is varied from λ_0 to a final value λ_f. Consequently, the system's Hamiltonian becomes H_f. In the eigenenergy picture, the initial and final Hamiltonian can be decomposed as $H_0=\sum_n E_n(\lambda_0)|n\rangle\langle n|$ and $H_\mathrm{f}=\sum_m E_m(\lambda_\mathrm{f})|m\rangle\langle m|$, respectively. Here $E_n(\lambda_0)$ ($E_m(\lambda_\mathrm{f})$) is the n-th (m-th) eigenenergy of the initial (final) Hamiltonian, with $|n\rangle$ ($|m\rangle$) being the corresponding eigenstates.

The work distribution of this non-equilibrium process is[9]

$$P(W)=\sum_{n,m}p_n^0 p_{m|n}^\tau\delta\{W-(E_m(\lambda_\mathrm{f})-E_n(\lambda_0))\}, \tag{2}$$

where $p_n^0=\mathrm{e}^{-\beta E_n(\lambda_0)}/Z(\lambda_0)$ indicates the probability to find the system in the n-th eigenstate of H_0 when making the initial energy measurement; $p_{m|n}^\tau=|\langle m|\hat{U}(\tau)|n\rangle|^2$ is the transition probability between $|n\rangle$ and $|m\rangle$, with $\hat{U}(\tau)=\widehat{T}\exp\{-\frac{\mathrm{i}}{\hbar}\int_0^\tau H_\mathrm{f}\}$ and $\widehat{T}$ the time ordering operator.

The Fourier transformation of the work distribution (2) is[10]

$$\begin{aligned}\chi(u)&=\int P(W)\,\mathrm{e}^{\mathrm{i}uW}\,\mathrm{d}W\\&=\mathrm{Tr}\left\{\left[\hat{U}(\tau)\,\mathrm{e}^{-\mathrm{i}uH_0}\right]\rho(\lambda_0)\left[\mathrm{e}^{-\mathrm{i}uH_\mathrm{f}}\hat{U}(\tau)\right]^\dagger\right\}.\end{aligned} \tag{3}$$

Here $\chi(u)$ is called the characteristic function. In the case of a sudden quench $\hat{U}(\tau)=1$,[10] so $\chi(u)$ can be simplified to

$$\chi(u)=\frac{1}{Z(\lambda_0)}\mathrm{Tr}\left[\mathrm{e}^{\mathrm{i}uH_\mathrm{f}}\mathrm{e}^{-\mathrm{i}uH_0}\mathrm{e}^{-\beta H_0}\right]. \tag{4}$$

From Eq. (3), one can calculate the n-th moment of work $\langle W^n\rangle=(-\mathrm{i})^n\partial_u^n\chi(u)|_{u=0}$ and the cumulation of the work distribution $K_n=(-\mathrm{i})^n\partial_u^n\ln[\chi(u)]|_{u=0}$.[38] Therefore, the averaged work can be expressed as

$$\begin{aligned}\langle W\rangle&=-\mathrm{i}\partial_u\chi(u)|_{u=0}\\&=\mathrm{Tr}[H_\mathrm{f}\rho(\lambda_0)-H_0\rho(\lambda_0)].\end{aligned} \tag{5}$$

Consequently, the variance of the work distribution can also be obtained as

$$\Sigma^2=(-\mathrm{i})^2\partial_u^2\chi(u)|_{u=0}=\langle W^2\rangle-\langle W\rangle^2. \tag{6}$$

In order to know more about the nonequilibrium process, we also introduce the relative entropy to describe the distinction between the pre-quenched and post-quenched states[23–25]

$$\Delta S=S(\rho(\lambda_0)||\rho(\lambda_\mathrm{f})), \tag{7}$$

where the relative entropy between density matrices ρ and κ is defined as

$$S(\rho||\kappa)=\mathrm{Tr}(\rho\log\rho-\rho\log\kappa). \tag{8}$$

From Eq. (8), one knows that the relative entropy $S(\rho||\kappa)\geq 0$. The magnitude of $S(\rho||\kappa)$ increases with the difference between ρ and κ. It has been proved that in the sudden quenching process, the relative entropy can be obtained as[23,24,26]

$$\Delta S=\beta(\langle W\rangle-\Delta F), \tag{9}$$

where $\Delta F=\beta^{-1}\ln[Z(\lambda_\mathrm{f})/Z(\lambda_0)]$ is the free energy difference between the post-quenched and the pre-quenched states.

3. The XY spin chain with multi-spin interaction

The model under consideration is a spin-1/2 chain, in which both the nearest-neighbor spin interaction and the three-spin interaction are included. Meanwhile, a transverse magnetic field is homogeneously applied to each spin of the chain. At time $t=0$, the Hamiltonian of this system is

$$\begin{aligned}H_0=&-\sum_{l=1}^{N}\left[\frac{(1+\gamma_0)}{2}\sigma_l^x\sigma_{l+1}^x+\frac{(1-\gamma_0)}{2}\sigma_l^y\sigma_{l+1}^y+h_0\sigma_l^z\right]\\&-\alpha_0\sum_{l=1}^{N}\left(\sigma_{l-1}^x\sigma_l^z\sigma_{l+1}^x+\sigma_{l-1}^y\sigma_l^z\sigma_{l+1}^y\right).\end{aligned} \tag{10}$$

Here γ_0 and α_0 are, respectively, the anisotropic parameter and the strength of the three-spin interaction; h_0 is the external magnetic field; σ_l^m $(m=x,y,z)$ are the Pauli matrices. It has been proved that the system undergoes QPT along the critical lines $h_0 \pm 1 - 2\alpha_0 = 0$ for any values of γ_0.[48]

The Hamiltonian (10) can be fully diagonalized after carrying out the Jordan–Wigner transformation, the Fourier transformation and the Bogoliubov transformation.[49] We firstly introduce the Jordan–Wigner transformation

$$\sigma_l^x = \prod_{i<l}\left(1-2c_i^\dagger c_i\right)\left(c_l + c_l^\dagger\right), \tag{11}$$

$$\sigma_l^y = -\mathrm{i}\prod_{i<l}\left(1-2c_i^\dagger c_i\right)\left(c_l - c_l^\dagger\right), \tag{12}$$

$$\sigma_l^z = 1-2c_l^\dagger c_l, \tag{13}$$

where $c_l^\dagger$ (c_l) is the fermionic creation (annihilation) operator. Substituting Eqs. (11)–(13) into Eq. (10), one obtains

$$\begin{aligned} H_0 = & -\sum_{l=1}^{N}\left\{\left[c_l^\dagger c_{l+1} + c_{l+1}^\dagger c_l + \gamma_0\left(c_l^\dagger c_{l+1}^\dagger - c_l c_{l+1}\right)\right]\right. \\ & +2\alpha_0\left(c_{l-1}^\dagger c_{l+1} + c_{l+1}^\dagger c_{l-1}\right) \\ & \left.+h_0\left(1-2c_l^\dagger c_l\right)\right\}. \end{aligned} \tag{14}$$

Based on Eqs. (14), we apply the Fourier transformation[49]

$$c_l = -\frac{1}{\sqrt{N}}\sum_{k=-M}^{M} \mathrm{e}^{\mathrm{i}2\pi kl/N} d_k, \tag{15}$$

$$c_l^\dagger = -\frac{1}{\sqrt{N}}\sum_{k=-M}^{M} \mathrm{e}^{-\mathrm{i}2\pi kl/N} d_k^\dagger, \tag{16}$$

with $M = N/2$ for N even and $M = (N-1)/2$ for N odd.

From Eqs. (14)–(16), one gets

$$\begin{aligned} H_0 = & \sum_k 2\left(h_0 - \cos ka - 2\alpha_0\cos(2ka)\right) d_k^\dagger d_k \\ & +\mathrm{i}\gamma_0 \sin ka\left(d_k^\dagger d_{-k}^\dagger - d_{-k} d_k\right), \end{aligned} \tag{17}$$

where $d_k^\dagger$ (d_k) is the fermionic creation (annihilation) operator in the momentum space.

In order to fully diagonalize the Hamiltonian (17), we introduce the Bogoliubov transformation[49]

$$\eta_k = \cos\frac{\theta_k^0}{2} d_k - \mathrm{i}\sin\frac{\theta_k^0}{2} d_{-k}^\dagger, \tag{18}$$

By substituting Eq. (18) into Eq. (17), the Hamiltonian (10) can be fully diagonalized as

$$H_0 = \sum_k \Omega_k^0\left(\eta_k^\dagger \eta_k - \frac{1}{2}\right) = \sum_k H_k^0. \tag{19}$$

where the dispersion relation in terms of Ω_k^0 is

$$\Omega_k^0 = 2\sqrt{\left(h_0 - \cos ka - 2\alpha_0\cos 2ka\right)^2 + \gamma_0^2 \sin^2 ka}, \tag{20}$$

and the Bogoliubov angle θ_k^0 is

$$\theta_k^0 = \arctan\left(\frac{\gamma_0 \sin ka}{h_0 - \cos ka - 2\alpha_0\cos(2ka)}\right) \tag{21}$$

with $a = 2\pi/N$.

After carrying out the sudden quenching operation, the system Hamiltonian changes to H_f. Using a similar diagonalization procedure outlined above, the quenched Hamiltonian can also be fully diagonalized as

$$H_\mathrm{f} = \sum_k \Omega_k^\mathrm{f}\left(\widetilde{\eta}_k^\dagger \widetilde{\eta}_k - \frac{1}{2}\right) = \sum_k H_k^\mathrm{f}, \tag{22}$$

where Ω_k^f is the energy spectrum of the quenched Hamiltionian. We here note that Ω_k^f can be obtained from Ω_k^0 by replacing the initial system parameter λ_0 with the quenched one λ_f.

It can be proved that the operators η_k for the initial spin chain and $\widetilde{\eta}_k$ for the quenched one are related by

$$\widetilde{\eta}_k = \eta_k \cos\phi_k - \mathrm{i}\eta_{-k}^\dagger \sin\phi_k, \tag{23}$$

where $\phi_k = \left(\theta_k^\mathrm{f} - \theta_k^0\right)/2$, and θ_k^f is given by Eq. (21) with the initial system parameter replaced by the quenched one.

From Eqs. (18) and (23), the relationship between the eigenstates of H_k^0 and H_k^f are as follows:

$$|0_k, 0_{-k}\rangle = \cos\phi_k \left|\widetilde{0}_k, \widetilde{0}_{-k}\right\rangle + \mathrm{i}\sin\phi_k \left|\widetilde{1}_k, \widetilde{1}_{-k}\right\rangle, \tag{24}$$

$$|1_k, 1_{-k}\rangle = \cos\phi_k \left|\widetilde{1}_k, \widetilde{1}_{-k}\right\rangle - \mathrm{i}\sin\phi_k \left|\widetilde{0}_k, \widetilde{0}_{-k}\right\rangle, \tag{25}$$

$$|0_k, 1_{-k}\rangle = \left|\widetilde{0}_k, \widetilde{1}_{-k}\right\rangle, \tag{26}$$

$$|1_k, 0_{-k}\rangle = \left|\widetilde{1}_k, \widetilde{0}_{-k}\right\rangle, \tag{27}$$

where $|n_k, n_{-k}\rangle$ $(|\widetilde{n}_k, \widetilde{n}_{-k}\rangle)$ are the eigenstates of H_k^0 (H_k^f).

According to Eqs. (3) and (26), the characteristic function can be obtained as follows:

$$\chi(u) = \frac{1}{Z(\lambda_0)}\prod_{k>0}\left\{A\,\mathrm{e}^{\beta\Omega_k^0} + A^*\,\mathrm{e}^{-\beta\Omega_k^0} + 2\right\}, \tag{28}$$

with

$$A = \mathrm{e}^{\mathrm{i}u\Omega_k^0}\left(\cos^2\phi_k\,\mathrm{e}^{-\mathrm{i}u\Omega_k^\mathrm{f}} + \sin^2\phi_k\,\mathrm{e}^{\mathrm{i}u\Omega_k^\mathrm{f}}\right),$$

where A^* is the conjugate of A. From Eq. (28), both the Tasaki–Crooks and the Jarzynski relations can be verified.[5–8] Combining Eqs. (5) and (28), we obtain the analytical expression of the averaged work as follows:

$$\langle W\rangle = \sum_{k>0}\left(\Omega_k^0 - \Omega_k^\mathrm{f}\cos 2\phi_k\right)\tanh\frac{\beta\Omega_k^0}{2}. \tag{29}$$

Furthermore, the fluctuation of the work distribution can also be obtained as follows:

$$\Sigma^2 = \sum_{k>0}\left[\left(\Omega_k^0 - \Omega_k^\mathrm{f}\right)^2 + 4\Omega_k^0\Omega_k^\mathrm{f}\sin^2\phi_k\right]$$

$$\times \frac{\cosh \beta \Omega_k^0}{2\cosh^2\left(\beta\Omega_k^0/2\right)} - \langle W\rangle^2 . \tag{30}$$

In addition, from Eqs. (9) and (29), the expression of the relative entropy can also be obtained as

$$\Delta S = \beta \sum_{k>0}\left(\Omega_k^0 - \Omega_k^{\rm f}\cos 2\phi_k\right)\tanh\frac{\beta\Omega_k^0}{2} + \sum_{k>0}\ln\frac{\cosh^2\frac{\beta\Omega_k^{\rm f}}{2}}{\cosh^2\frac{\beta\Omega_k^0}{2}}. \tag{31}$$

We here note that the expressions for the average work (29) and the relative entropy (31) are identical in form to the corresponding equations obtained in the Ising[2] and the XY models.[35] The difference comes from the dispersion relation and the Bogoliubov angle under different quenching conditions. Based on Eqs. (29) and (31), we will investigate the behavior of the average work and the relative entropy in the following section.

4. Results and discussion

In this section, we investigate the behavior of average work and relative entropy under different quenching conditions. There are three parameters in this system, i.e., the anisotropic parameter γ_0, the external field h_0 and the multi-spin interaction α_0. Since γ_0 can not change the critical behavior of the XY model, we will concentrate on the behavior of the average work in the following two cases: (a) γ_0 and α_0 are fixed, h_0 is the quenching parameter, (b) γ_0 and h_0 are fixed, α_0 is the quenching parameter.

From Eq. (29), the averaged work under external field h_0 quenching is

$$\langle W\rangle = \sum_{k>0}\langle W\rangle_k = \sum_{k>0} 2\Delta h\cos\theta_k^0\tanh\frac{\beta\Omega_k^0}{2}, \tag{32}$$

with $\Delta h = h_0 - h_{\rm f}$.

Similarly, the averaged work under multi-spin interaction α_0 quenching can be expressed as

$$\langle W\rangle = \sum_{k>0} 2\Delta\alpha\cos(2ka)\cos\theta_k^0\tanh\frac{\beta\Omega_k^0}{2}, \tag{33}$$

with $\Delta\alpha = \alpha_{\rm f} - \alpha_0$.

From Eqs. (32) and (33), we can see that $\langle W\rangle$ is proportional to $\Delta\lambda = |\lambda_{\rm f} - \lambda_0|$, with $\lambda = h, \alpha$. Thus the $\langle W\rangle$ will be mainly decided by the system initial Hamiltonian. Based on Eqs. (32) and (33), the behavior of $\langle W\rangle$ under sudden quenching can be investigated.

Figure 1 shows the average work $\langle W\rangle$ as a function of h_0 under various α_0 for $\Delta h = 0.01$ with $\gamma_0 = 0$, 0.5, 0.8 and 1. Other parameters are chosen as $N = 500$, $\beta = 100$. It is seen from Fig. 1 that with the increasing h_0, the average work $\langle W\rangle$ increases and approaches to an asymptotic value. In fact, the asymptotic value comes from the competition among the external field h_0 and other system parameters. If the external field h_0 dominates, the effects of other parameters will be hid. From Eqs. (20) and (21), we know that $\Omega_k^0|_{h_0\to\infty} = 2h_0$, $\cos\theta_k^0|_{h_0\to\infty} = 1$. Thus, in the $h_0\to\infty$ limit, from Eq. (32) one gets

$$\langle W\rangle_k|_{h_0\to\infty} = 2\Delta h\tanh\beta h_0. \tag{34}$$

In combination with Eq. (32), we can know that $\langle W\rangle$ approaches $N\Delta h$ in the $h_0\to\infty$ limit. This result is consistent with the asymptotic behavior that appears in the figures. In addition, $\langle W\rangle$ exhibits sudden change behavior during the evolution. The sudden change appears in the vicinity of the system's QPT points which will be further proved in Fig. 2. Moreover, Fig. 1 also shows that the change of γ_0 has no influence on the position of sudden change point, which further proves that γ_0 does not change the QPT point of the system.[49,50]

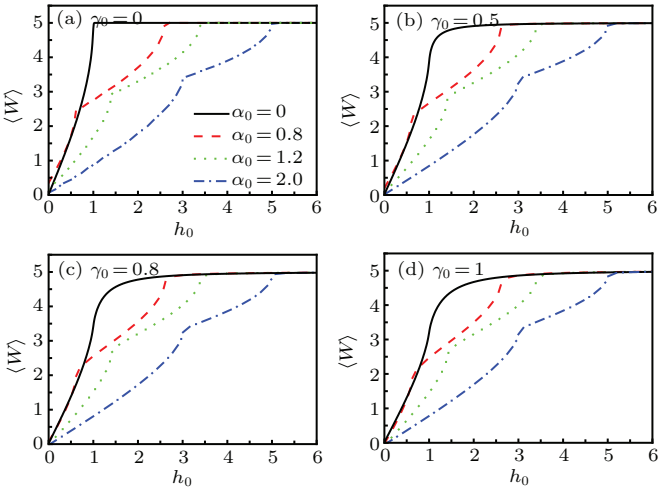


Fig. 1. The average work as a function of h_0 under various α_0 for $\Delta h = 0.01$, with (a) $\gamma_0 = 0$, (b) $\gamma_0 = 0.5$, (c) $\gamma_0 = 0.8$, and (d) $\gamma_0 = 1$. The other parameters are chosen as $N = 500$, $\beta = 100$. The symbols shown in the subset of Fig. 1(a) are applicable to all the curves.

In order to see more clear about the relations between $\langle W\rangle$ and the QPT, we investigate the evolution behavior of $\partial\langle W\rangle/\partial h_0$. From Eqs. (29) and (31), we obtain the first derivation of $\langle W\rangle$ to h_0,

$$\frac{\partial\langle W\rangle}{\partial h_0} = \frac{1}{\beta}\frac{\partial\Delta S}{\partial h_0} + \sum_{k>0} 2\left(\cos\theta_k^0\tanh\frac{\beta\Omega_k^0}{2} - \cos\theta_k^{\rm f}\tanh\frac{\beta\Omega_k^{\rm f}}{2}\right). \tag{35}$$

From Eq. (35), we infer that the evolution of $\partial\langle W\rangle/\partial h_0$ and ΔS may have some similarities.

Based on Eq. (35), one can investigate the evolution of $\partial\langle W\rangle/\partial h_0$ with h_0, as shown in Fig. 2. The plotting parameters are the same as those in Fig. 1. It is seen from Fig. 2 that when $\alpha_0 = 0$, $\partial\langle W\rangle/\partial h_0$ displays peaks at $h_0 = 1$ for arbitrary γ_0. When the three-spin interaction is switched

on ($\alpha_0 \neq 0$), the peaks have $2\alpha_0$ displacement from $h_0 = 1$. This result corresponds exactly to the critical line determined by $h_0 \pm 1 - 2\alpha_0 = 0$.[48] Moreover, one also knows that the peak positions in Fig. 2 are identical with those of the sudden change points in Fig. 1. Combining Figs. 1 and 2, we infer that $\langle W \rangle$ can be used as a detector of QPTs and multi-spin interaction at finite temperature in the non-equilibrium system.

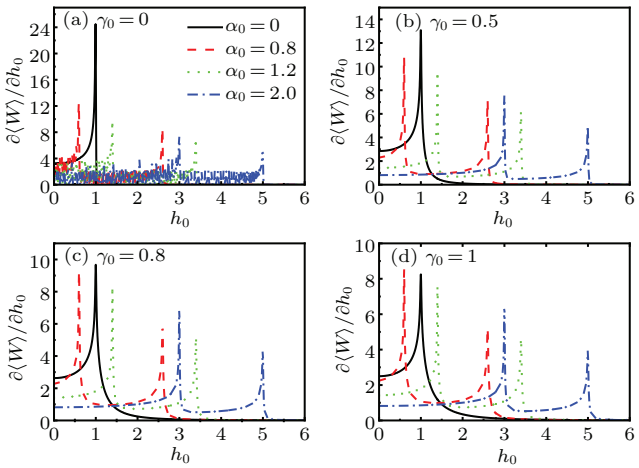


Fig. 2. The first derivation of average work as a function of h_0 under various α_0 for $\Delta h = 0.01$. The plotting parameters are the same as those of Fig. 1. The symbols shown in the subset of Fig. 2(a) are applicable to all the curves.

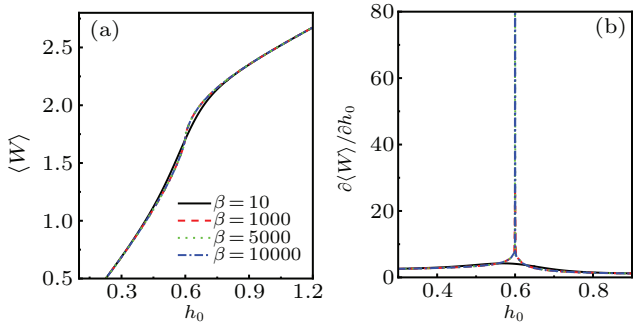


Fig. 3. The average work $\langle W \rangle$ and its first derivation $\partial \langle W \rangle / \partial h_0$ as a function of h_0 under various β. The other parameters are chosen as (a) $\gamma_0 = 1$, $\alpha_0 = 0.8$, and $N = 500$. The symbols shown in the subset of Fig. 3(a) are applicable to all the curves.

In Fig. 3, the influences of temperature β on the evolution of $\langle W \rangle$ and $\partial \langle W \rangle / \partial h_0$ are investigated. The other parameters are chosen as $\gamma_0 = 1$, $\alpha_0 = 0.8$, and $N = 500$. It shows that β hardly affects the evolution of $\langle W \rangle$ and $\partial \langle W \rangle / \partial h_0$ except in the vicinity of critical points. At the critical points, the peaks of $\partial \langle W \rangle / \partial h_0$ are sensitive to the decreasing β. This proves that the QPTs phenomenon is sensitive to the thermal fluctuation. In fact, the critical behavior of $\partial \langle W \rangle / \partial h_0$ can be predicted by Eq. (32) in the large N limit. From Eq. (32) we can obtain

$$\left.\frac{\partial \langle W \rangle_k}{\partial h_0}\right|_{N\to\infty} = 2\Delta h\beta \left(1 - \tanh^2 \beta \left(h_0 - 1 - 2\alpha_0\right)\right). \quad (36)$$

Equation (36) indicates that $\partial \langle W \rangle_k / \partial h_0$ will exhibit a peak if $h_0 - 1 - 2\alpha_0 = 0$. This result is consistent with the peaks appearing in Figs. 2 and 3(b).

In Fig. 4, $\langle W \rangle$ as a function of α_0 under various γ_0 for $\Delta\alpha = 0.01$, with $h_0 = 0$, 0.5, 1 and 3 are given. The other parameters are chosen as $N = 500$, $\beta = 100$. It shows that $\langle W \rangle$ decreases with the increasing α_0 for fixed γ_0 and h_0. There exists sudden change phenomenon during the evolution. The sudden change point has $h_0/2$ displacement from $\alpha_0 = 0.5$. The γ_0 just influences the magnitude of $\langle W \rangle$ rather than the sudden change point position. From Eq. (32) we know that the influences of γ_0 on $\langle W \rangle$ will be suppressed for large h_0 and α_0, which leads to the near superposition of the curves in Fig. 4(d).

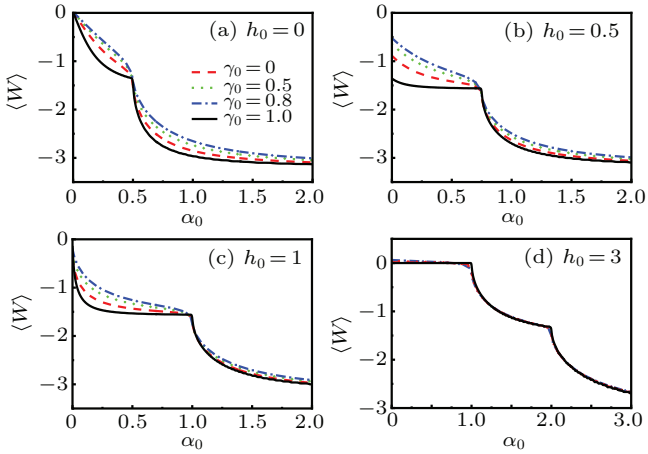


Fig. 4. The average work as a function of α_0 under various γ_0 for $\Delta\alpha = 0.01$, with (a) $h_0 = 0$, (b) $h_0 = 0.5$, (c) $h_0 = 1$, and (d) $h_0 = 3$. The other parameters are chosen as $N = 500$, $\beta = 100$. The symbols shown in (a) are applicable to all the curves.

The relations between the pre- and post-quenched states can be revealed through the investigation of the relative entropy. From Eqs. (9) and (28), the expressions of relative entropy under different quenching parameters can be obtained. From Eq. (9), the relative entropy under h_0 quenching can be expressed as

$$\Delta S = 2\beta \sum_{k>0} \Delta h \cos\theta_k^0 \tanh\frac{\beta\Omega_k^0}{2} + \sum_{k>0} \ln\frac{\cosh^2\frac{\beta\Omega_k^{\rm f}}{2}}{\cosh^2\frac{\beta\Omega_k^0}{2}}, \quad (37)$$

where $\Omega_k^{\rm f}$ is obtained from Eq. (20), with h_0 being replaced by $h_{\rm f}$ while keeping γ_0 and α_0 unchanged.

The relative entropy under external field α_0 quenching is

$$\Delta S = 2\beta \sum_{k>0} \Delta\alpha \cos(2ka) \cos\theta_k^0 \tanh\frac{\beta\Omega_k^0}{2} + \sum_{k>0} \ln\frac{\cosh^2\frac{\beta\Omega_k^{\rm f}}{2}}{\cosh^2\frac{\beta\Omega_k^0}{2}}, \quad (38)$$

where $\Omega_k^{\rm f}$ is given by Eq. (20), with α_0 being replaced by $\alpha_{\rm f}$ while keeping γ_0 and h_0 unchanged. From Eqs. (37) and (38), we can see that ΔS is related with both the initial and the final energy spectra, which differs from that of the $\langle W \rangle$. Based on Eqs. (37) and (38), we can investigate the behavior of ΔS under different quenching parameters.

Figure 5 plots the relative entropy ΔS as a function of h_0 under various α_0 for $\Delta h = 0.01$, with $\gamma_0 = 0$, 0.5, 0.8 and 1. Other parameters are chosen as $N = 500$ and $\beta = 100$. It shows that ΔS displays a peak at $h_0 = 1$ when $\alpha_0 = 0$. The peak has a $2\alpha_0$ displacement from $h_0 = 1$ when $\alpha_0 \neq 0$. This result is similar to that of $\partial \langle W \rangle / \partial h_0$ in Fig. 2 and is consistent with the critical line $h_0 \pm 1 - 2\alpha_0 = 0$ in Ref. [48]. Since ΔS describes the entropy distance between the initial equilibrium state and the final quenched one. The sharp peaks indicate that a small change of external field will lead to a dramatic change in the equilibrium state near the critical points.[34,37] Further, the multi-spin interaction suppresses ΔS and leads to the decreasing of the peaks. Based on Fig. 5, we say that ΔS can also be used as the detector of QPT and the three-spin interaction.

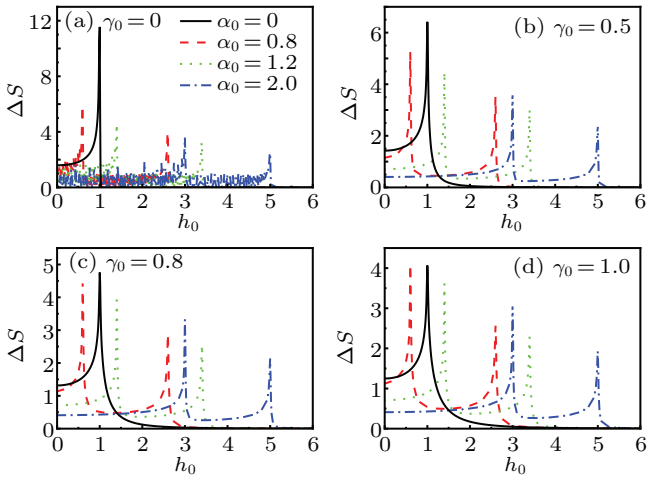


Fig. 5. The relative entropy as a function of h_0 under various α_0 for $\Delta h = 0.01$, with (a) $\gamma_0 = 0$, (b) $\gamma_0 = 0.5$, (c) $\gamma_0 = 0.8$, and (d) $\gamma_0 = 1$. The other parameters are chosen as $N = 500$, $\beta = 100$. The symbols shown (a) are applicable to all the curves.

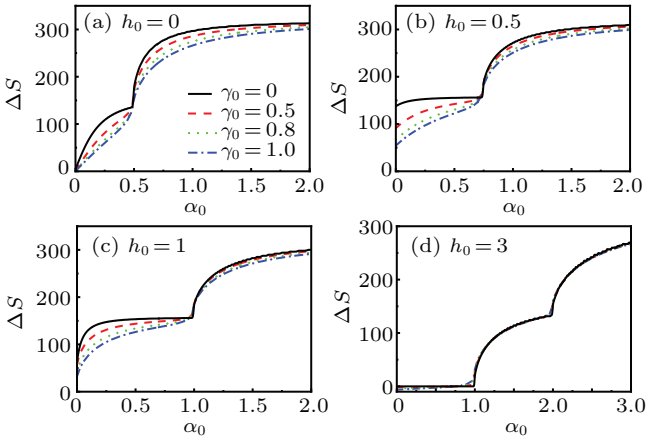


Fig. 6. The relative entropy as a function of α_0 under various γ_0 for $\Delta\alpha = 0.01$, with (a) $h_0 = 0$, (b) $h_0 = 0.5$, (c) $h_0 = 1$, and (d) $h_0 = 3$. The other parameters are chosen as $N = 500$, $\beta = 100$. The symbols shown in (a) are applicable to all the curves.

In Fig. 6, ΔS as a function of α_0 under various γ_0 for $\Delta\alpha = 0.01$ with $h_0 = 0$, 0.5, 1 and 3 are given. Other parameters are chosen as $N = 500$, $\beta = 100$. We can see that ΔS increases with the increasing α_0 and saturates to constant values. There also exists sudden change behavior during the evolution. For $h_0 = 0$, the sudden change point locates at $\alpha_0 = 0.5$ as shown in Fig. 6(a). For $h_0 \neq 0$, h_0 and α_0 jointly influence the phase transition process and lead to the displacement of sudden change point.

In Fig. 7, the influences of β on the evolution of ΔS with h_0 and α_0 are investigated. The other parameters are chosen as $N = 500$, (a) $\gamma_0 = 1$, $\alpha_0 = 0.8$ and (b) $\gamma_0 = 0.8$, $h_0 = 1$. It shows that ΔS exhibits peaks (sudden change behavior) during the evolution with h_0 (α_0). The peak values decrease with the decreasing β. This indicates that at high temperature the thermal fluctuation dominates the quantum one, which leads to the elimination of the QPT phenomenon.

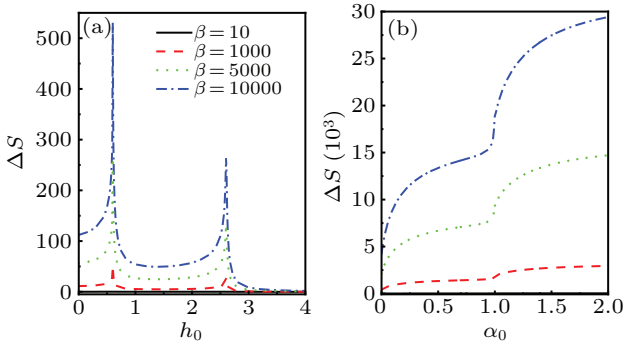


Fig. 7. The relative entropy as a function of (a) h_0 and (b) α_0 under various β. The other parameters are chosen as (a) $\gamma_0 = 1$, $\alpha_0 = 0.8$; (b) $\gamma_0 = 0.8$, $h_0 = 1$, and $N = 500$. The symbols shown in (a) are applicable to all the curves.

5. Summary

In summary, we have studied the non-equilibrium thermodynamics of a quenched XY spin chain with multi-spin interaction in a transverse field. The spin chain is driven out of equilibrium through sudden quenching. The analytical expression of the average work $\langle W \rangle$ and the relative entropy ΔS are obtained under different quenching conditions. The influence of the external field h_0, the multi-spin interaction α_0 and the temperature β on the properties of the $\langle W \rangle$, the first derivation of $\langle W \rangle$ and the ΔS are investigated. Our results show that at finite temperature they exhibit sudden change or sharp peaks during the evolution with the quenching parameters. The locations of sudden change and the sharp peaks correspond to those of the quantum phase transition (QPT) points. In addition, our results show that the positions of sudden change and peaks are obviously shifted away from the original ones ($h_0 = 1$). The magnitude of multi-spin interaction dominates the displacement of the sudden change and the peaks. Based on these results we infer that the average work and the relative entropy can be used as a detector of QPT points and the multi-spin interaction in the non-equilibrium system.

As a comment, by comparing the sudden change (peak) locations of the average work (the relative entropy) with the

critical points which are identified by pairwise quantum discord and classical correlation in Ref. [48], we find that the sudden change (peak) locations of the average work (the relative entropy) are around the critical points. In previous investigations,[48,50–53] it has been shown that quantum discord, entanglement and fidelity or their derivatives with respect to parameters in the Hamiltonian display the singular behavior at critical points and can be used to detect QPTs. Therefore, we believe that these parameters can also be used to detect three-spin interaction in nonequilibrium systems.

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