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Edge states enhanced by long-range hopping: An analytical study*

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We analyze the behavior of edge states in long-range (LR) interacting systems. In terms of lattice model Hamiltonian with the LR coupling, we determine analytically the condition of existence of edge states within the transfer matrix method (TMM). The expressions we obtain are general and hold for any choice of the LR hopping. The reason why edge states can appear is the transfer matrix in the bulk different from that in the boundary layers. Our predictions are in good agreement with numerical results by exact diagonalization. Our result is helpful in solving novel edge states in one- and two-dimensional (2D) superconductors with LR hopping and pairing.

Keywords: edge states, transfer matrix method, long-range hopping

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1. Introduction

There is ongoing interest in understanding the effects of electronic correlations on topological and nontopological quantum states due to the promise of uncovering new phenomena in condensed matter physics. Among such topological states of matter, topological superconductors[1,2] offer promising perspectives for realizing robust quantum devices[3–5] due to the presence of topologically protected edge states. In addition, much attention has been paid to nontopological states in ferroelectric ABO_3 crystals[6] and some semiconductors (i.e., Si, Ge)[7–9] owing to their potential applications. Quantum states are mainly governed by Hamiltonians with short-range terms. However, the addition of LR hopping amplitudes is important, because they are ubiquitous in real systems and their role can be crucial.[10–12] In recent years, LR classical and quantum systems have become a central focus for physical investigation. The existing theoretical works have reported that LR systems can present many interesting and unusual phenomena, essentially owing to the violation of locality.[13–19] In particular, some LR models can present new phases, manifesting striking properties which are absent in the short-range limit.[20–32]

Motivated by these ideas, we study the behavior of edge states in the presence of the LR hopping in LR interacting systems. For most of the conventional crystals with low-index surfaces, the short-range terms will be dominant and the LR ones are relatively weak, which can be ignored. Previous works[33–37] have investigated the existence/absence of edge states based on some models with many (a few) electron modes per unit cell and with the short-range terms. However, for some strongly correlated materials and some possible naturally stable crystals with high-index surfaces and/or the strong spin-orbit coupling, the LR coupling cannot be ignored. It is natural to ask how LR terms affect edge states. With this motivation, the question we want to answer in this manuscript is what is the condition of the presence of edge states in the systems with LR hopping. As we know, an infinite n-dimensional crystal is made up of an infinite number of parallel $(n-1)$-dimensional crystals which are periodically arrayed by coupling. For the sake of discussion, here we take a 2D crystal as an example. The $(n-1)$-dimensional crystal represents a crystal line (CL) when n is equal to 2. Here we mainly focus on such model crystals with three different type structures (type I: $\cdots$–P–P–P–P–$\cdots$, type Ⅱ: $\cdots$–P–Q–P–Q–$\cdots$, and type Ⅲ: $\cdots$–P=P–P=P–$\cdots$ where P and Q denote CLs, and the signs – and = mark the distance between the nearest neighbor (NN) CLs).

In this work, we perform an analytical study of the behavior of edge states in LR interacting systems when LR hopping terms are considered. Our analysis includes LR coupling among CLs and all possible neighbor hoppings within each CL. For such model crystals with three-type structures, we determine analytically the condition of the existence of edge states in the presence of LR terms. Our findings are supported by numerical calculations for specific parameter choices. The conclusion that edge states can survive in such model systems with LR hopping terms derives from the transfer matrix in the bulk different from that in the boundary layers.

Before presenting the following demonstration, we give a definition of "edge state" firstly. Edge states can be described such that they can propagate along the direction of the cut boundary and their amplitudes decay exponentially in distance

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normal to the cut boundary. With the aid of the transfer matrix language, an edge state has the following decay relation:

$$|\psi_{i+1}/\psi_i| = \delta = \mathrm{e}^{-\gamma a} < 1,$$
$$a > 0,\ \gamma > 0,\ i \geq 1,$$

where γ is a decay rate and a is the distance between the 1NN CLs. When $\delta = 1$ ($\gamma = 0$), it corresponds to the extended state and must be associated with the bulk state.

2. Edge states in type I

In this section, we focus on such a semi-infinite crystal with type I P–P–P–P–$\cdots$. For each CL, it is a lower dimensional crystal than the original one. The Fourier transformation can be used for each CL because the wave vector $\boldsymbol{k}_{||}$ along the boundary is a good quantum number. When we consider LR coupling and utilize the diagonal representation of each CL Hamiltonian, our effective model Hamiltonian can be written as

$$\begin{aligned} H_0 = & \sum_{n=1}^{\infty} \{\varepsilon_{m\times m}(\boldsymbol{k}_{||})\Phi_n^{\dagger}(\boldsymbol{k}_{||})\Phi_n(\boldsymbol{k}_{||}) \\ & + [\Phi_n^{\dagger}(\boldsymbol{k}_{||})F_{m\times m}^{(1)}(\boldsymbol{k}_{||})\Phi_{n+1}(\boldsymbol{k}_{||}) \\ & + \Phi_n^{\dagger}(\boldsymbol{k}_{||})F_{m\times m}^{(2)}(\boldsymbol{k}_{||})\Phi_{n+2}(\boldsymbol{k}_{||}) + \cdots + \text{h.c.}]\}, \end{aligned} \quad (1)$$

where

$$\begin{aligned} & \Phi_n^{\mathrm{T}}(\boldsymbol{k}_{||}) = \{\phi_n^{(1)}(\boldsymbol{k}_{||}), \phi_n^{(2)}(\boldsymbol{k}_{||}), \ldots, \phi_n^{(m)}(\boldsymbol{k}_{||})\}, \\ & \varepsilon_{m\times m}(\boldsymbol{k}_{||}) = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m), \\ & \Phi_n(\boldsymbol{k}_{||}) = 0 : n \leq 0, \end{aligned}$$

$\{\phi_n^{(\alpha)}(\boldsymbol{k}_{||}) : \alpha = 1,2,\ldots,m\}$ is the second quantization Fermi wave function of the α^{th} electron mode in the n^{th} CL, $\{\varepsilon_i : i = 1,2,\ldots,m\}$ are the energies of renormalized eigen-modes in each CL, and $\{F_{m\times m}^{(\tau)} : \tau = 1,2,3,\ldots\}$ denote the $m\times m$ hopping matrix from the τNN CLs, where m is any finite positive integer. The elements of $\{F_{m\times m}^{(\tau)} : \tau = 1,2,3,\ldots\}$ and $\{\varepsilon_i : i = 1,2,\ldots,m\}$ are functions of the hopping parameters and $\boldsymbol{k}_{||}$. The above Hamiltonian is general, where each unit cell in each CL can contain multiple atoms (and maybe different) and each atom can also contribute multiple nonidentical atomic orbits. Thus the model Hamiltonian is reasonably reliable for some actual materials. For later convenience, let us omit $\boldsymbol{k}_{||}$. Since the original infinite crystal has geometric reflection symmetry for each CL, the quantum dynamical equations (QDEs) from Eq. (1) in the bulk and boundary layers read

$$E_{m\times m}\Phi_n = \sum_{i=1}^{\infty} F_{m\times m}^{(i)}(\Phi_{n-i} + \Phi_{n+i}), \quad (2)$$

where $E_{m\times m} = E - \varepsilon_{m\times m}$ and E is the eigen energy of electron waves propagating in cut crystals. In Ref. [38], it has also been shown that edge states are absent in the presence of LR hopping when only one electron mode near the Fermi surface is important in the cut crystal with type I, resulting from the transfer matrix in the bulk identical to that in the boundary layers. Consequently, an interesting question is as follows: Are there edge states when $m \geq 2$ and do LR terms exist? If yes, what could the condition for the existence of edge states be?

In our effort to answer these questions, let us first formulate a specific case, that is, $m = 2$ and $\{F_{m\times m}^{(1)} \neq 0_{2\times 2}, F_{m\times m}^{(2)} \neq 0_{2\times 2}, F_{m\times m}^{(\tau)} = 0_{2\times 2} : \tau = 3,4,\ldots,\infty\}$, then we can adapt Eq. (2) to the matrix form by some calculations

$$\begin{cases} \Theta_{n+8}^{(\alpha)} = \Theta_{n+7}^{(\alpha)} T_{8\times 8}, \ \Theta_8^{(\alpha)} = \Omega_8^{(\alpha)} S_{8\times 8}^{(\alpha)}, \\ T_{8\times 8} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \ n \geq 1, \end{cases} \quad (3)$$

where $T_{11}^{\mathrm{T}} = \{b_1, b_2, b_3, b_4, b_3, b_2, b_1\}$, $T_{21} = -1$, T_{12} is a 7×7 identity matrix, T_{22} is a 1×7 zero matrix, $\{\Theta_n^{(\alpha)} = \{\phi_n^{(\alpha)}, \phi_{n-1}^{(\alpha)}, \ldots, \phi_{n-6}^{(\alpha)}, \phi_{n-7}^{(\alpha)}\} : \alpha = 1,2\}$, $\Omega_8^{(\alpha)} = \{\phi_2^{(\alpha)}, \phi_1^{(\alpha)}, \phi_2^{(\beta)}, \phi_1^{(\beta)}, 0, 0, 0, 0\}$. Here $\{\alpha = 1(2), \beta = 2(1)\}$ and the specific expressions $\{b_r : i = 1,2,3,4\}$ and the matrix $S_{8\times 8}^{(\alpha)}$ are summarized in Appendix A. It is straightforward to notice that $b_1 = \mathrm{Tr}(T_{8\times 8}) = \sum_{i=1}^{8}\lambda_i$ is the trace of $T_{8\times 8}$, and $1 = \mathrm{Det}(T_{8\times 8}) = \prod_{i=1}^{8}\lambda_i$, the determinant, where λ_i are the eigenvalues of $T_{8\times 8}$. One can easily prove that the roots of characteristic polynomial $f(\lambda) = \lambda^8 - \sum_{i=1}^{4} b_i\lambda^{8-i} - \sum_{i=1}^{3} b_i\lambda^i + 1$ are the eigenvalues of the matrix $T_{8\times 8}$. We have noticed that when the two sides of $f(\lambda) = 0$ are divided by λ^4 and setting $z = \lambda + \frac{1}{\lambda}$, $f(\lambda) = 0$ can be reduced to a quartic equation. One can read out the roots by solving the quartic equation, and the corresponding value λ can be obtained by calculating $\lambda^2 - z\lambda + 1 = 0$, as shown in more detail in Appendix B. Thus the eigenvalues of $T_{8\times 8}$ can be found. To make the conclusion describing the presence of edge states more reliable, all possible forms of eigenvalues for the matrix $T_{8\times 8}$ are taken into account. With the help of Appendix A in Ref. [38] and TMM,[39,40] Eq. (3) can be taken the form

$$\begin{aligned} \Theta_{n+8}^{(\alpha)} &= \Theta_{n+7}^{(\alpha)} U_{8\times 8} J_{8\times 8} U_{8\times 8}^{-1} \\ &= \Omega_8^{(\alpha)} S_{8\times 8}^{(\alpha)} U_{8\times 8} J_{8\times 8}^{n} U_{8\times 8}^{-1} \\ &= \Omega_8^{(\alpha)} S_{8\times 8}^{(\alpha)} U_{8\times 8} \sum_{i=1}^{2l} J_{n_i}^{n}(\lambda_i) U_{8\times 8}^{-1} \\ &= \begin{pmatrix} N_1^{(\alpha)} & N_2^{(\alpha)} & \ldots & N_8^{(\alpha)} \end{pmatrix} W_{8\times 8}, \end{aligned} \quad (4)$$

with

$$R_{8\times 8}^{(\alpha)} = S_{8\times 8}^{(\alpha)} U_{8\times 8} = \begin{pmatrix} R_{4\times 4}^{(1)} & R_{4\times 4}^{(2)} \\ R_{4\times 4}^{(3)} & R_{4\times 4}^{(4)} \end{pmatrix},$$

$W_{8\times 8} = \sum_{i=1}^{2l} J_{n_i}^{n}(\lambda_i) U_{8\times 8}^{-1}$, $\{N_i^{(\alpha)} = R_{1i}^{(\alpha)}\phi_2^{(\alpha)} + R_{2i}^{(\alpha)}\phi_1^{(\alpha)} + R_{3i}^{(\alpha)}\phi_2^{(\beta)} + R_{4i}^{(\alpha)}\phi_2^{(\beta)} : i = 1,2,\ldots,8\}$, $n_1 + n_2 +, \cdots, +n_{2l-1} +$

$n_{2l}=8$, where $l\leq 4$ and we have arranged such that $\{|\lambda_i|\leq 1,|\lambda_{i+l}|\geq 1,\lambda_i\lambda_{i+l}=1:i=1,2,\ldots,l\}$ without loss of generality. If one of $\{|\lambda_i|=|\lambda_{i+l}|:i=1,2,\ldots,l\}$ is equal to 1, it will result in the bulk states that we do not care about. Now we need to concentrate on discussing this situation $\{|\lambda_i|<1,|\lambda_{i+l}|>1,\lambda_i\lambda_{i+l}=1:i=1,2,\ldots,l\}$. In order to reach the physically meaningful edge states, one has to ensure that all wave functions must be finite when n goes to infinity. Therefore, first of all, $\{N_j^{(\alpha)}=0:j=5,6,7,8\}$ must be satisfied for the existence of edge states, namely, $\mathrm{Det}(R^{(2)}_{4\times 4})=0$ ($\mathrm{Rank}(R^{(2)}_{4\times 4})<4$); secondly, in order to arrive at non-zero edge states, one has to require that at least one of them $\{N_k^{(\alpha)}:k=1,2,3,4\}$ is not equal to zero, namely, $\mathrm{Det}(R^{(1)}_{4\times 4})\neq 0$ ($\mathrm{Rank}(R^{(1)}_{4\times 4})=4$). To verify our analytical conclusion to be reasonable and reliable, a 2D example is provided in Fig. 1(a). Here, due to its infinity in y-direction, we can take the Fourier transformation on wave functions for $A(B)$ sublattices. When the hopping from 1NN to 2NN CLs is considered, let us write the QDEs in the $\{n,k_y\}$ representation and in terms of a set of Fermion operators $\{\phi_{n,}^{A(B)}(k_y):n=1;2,\ldots,\infty\}$

$$\begin{aligned}E_A\phi_n^{\mathrm{A}}&=t_0^{\mathrm{AB}}\phi_n^{\mathrm{B}}+t_i^{\mathrm{A}}\Delta_i\phi_n^{\mathrm{A}}+t_i^{\mathrm{AB}}\Delta_i\phi_n^{\mathrm{B}},\\ E_{\mathrm{B}}\phi_n^{\mathrm{B}}&=t_0^{\mathrm{AB}}\phi_n^{\mathrm{A}}+t_i^{\mathrm{AB}}\Delta_i\phi_n^{\mathrm{A}}+t_i^{\mathrm{B}}\Delta_i\phi_n^{\mathrm{B}},\end{aligned}\tag{5}$$

with

$$\begin{aligned}&E_\alpha=E-\varepsilon_\alpha;\phi_n^\alpha=0,\quad n\leq 0,\\ &\Delta_i\phi_n^\alpha=\phi_{n+i}^\alpha+\phi_{n-i}^\alpha,\quad i=1,2\\ &t_0^{\mathrm{AB}}=2t_1\cos(k_ya);\quad t_1^{\mathrm{A}}=t_1''+2t_4'\cos(2k_ya),\\ &t_1^{\mathrm{B}}=t_1'+2t_4''\cos(2k_ya);\quad t_1^{\mathrm{AB}}=2t_2\cos(k_ya),\\ &t_2^{\mathrm{A}}=t_3+2t_5\cos(2k_ya),\\ &t_2^{\mathrm{B}}=t_3'+2t_5'\cos(2k_ya),\\ &t_2^{\mathrm{AB}}=2t_4\cos(k_ya);\quad \alpha=\mathrm{A},\mathrm{B},\end{aligned}$$

t_0^{AB} is the hopping within each CL, $\{t_i^{\mathrm{A}},t_i^{\mathrm{B}},t_i^{\mathrm{AB}}:i=1,2\}$ denote the hopping from iNN CLs and are dependent on the wave vector $\boldsymbol{k}_{||}$ within each CL, ε_α is the on-site energy, $\{t_i:i=1,2,\ldots,5;t_1',t_1'',t_3',t_4',t_4'',t_5'\}$ are the hopping parameters and $\{\varepsilon_\alpha,t_i^{\mathrm{A}},t_i^{\mathrm{B}},t_i^{\mathrm{AB}}\}$ are real. By the exact diagonalization method, its energy spectrum of edge states is given, compatible with our above demonstration (see Fig. 1(b)).

In this work we are interested in the general case and now consider $m\geq 2$ and $\{F^{(\beta)}_{m\times m}\neq 0_{m\times m},:\beta=1,2,\ldots,r;F^{(\gamma)}_{m\times m}=0_{m\times m}:\gamma=r+1,r+2,\ldots,\infty\}$, one will check that Eq. (2) can be written in the form of the following matrix:

$$\begin{aligned}&\Theta^{(\alpha)}_{n+2m'}=\Theta^{(\alpha)}_{n+2m'-1}T_{2m'\times 2m'},\\ &\Theta^{(\alpha)}_{2m'}=\Omega_{2m'}S^{(\alpha)}_{2m'\times 2m'},\\ &\alpha=1,2,\ldots,m,\end{aligned}\tag{6}$$

where

$$T_{2m'\times 2m'}=\begin{pmatrix}T_{11}&T_{12}\\ T_{21}&T_{22}\end{pmatrix},$$

$m'=mr$, $T_{11}^{\mathrm{T}}=\{b_1,b_2,\ldots,b_{m'-1},b_{m'},b_{m'-1},\ldots,b_2,b_1\}$, T_{12} is a $(2m'-1)\times(2m'-1)$ identity matrix, T_{22} is a $1\times(2m'-1)$ zero matrix, $T_{21}=-1$, $\{b_i:i=1,2,\ldots,m'-1,m'\}$ and the matrix elements in $S^{(\alpha)}_{2m'\times 2m'}$ are arbitrary complex functions of the elements of $\{F^{(\tau)}_{m\times m}:\tau=1,2,3,\ldots,r\}$ and $E_{m\times m}$ and $\Theta_n^{(\alpha)}=\{\phi_n^{(\alpha)},\phi_{n-1}^{(\alpha)},\ldots,\phi_{n-2m'+2}^{(\alpha)},\phi_{n-2m'+1}^{(\alpha)}\}$ as well as $\Omega_{2m'}=\{\psi^{(\alpha)},\psi^{(m)},\psi^{(m-1)},\ldots,\psi^{(\alpha+1)},\psi^{(\alpha-1)},\ldots,\psi^{(1)},\psi^{(0)}\}=\{\varphi_1,\varphi_2,\ldots,\varphi_{m'},\psi^{(0)}\}$, where $\{\psi^{(\delta)}=\{\phi_r^{(\delta)},\phi_{r-1}^{(\delta)},\ldots,\phi_1^{(\delta)}\}:\delta=1,2,\ldots,m\}$ and $\psi^{(0)}$ is a $1\times m'$ zero matrix. From the discussion of the special case $m=2$, we know that the behavior of edge states is closely related to the eigenvalues and eigenstates of $T_{2m'\times 2m'}$. The problem is reduced, therefore, to find the eigenvalues and eigenstates of $T_{2m'\times 2m'}$. In order to arrive at its eigenvalues, one can write the characteristic polynomial $f_{2m'}(\lambda)=\lambda^{2m'}-\sum_{i=1}^{m'}b_i\lambda^{2m'-i}-\sum_{i=1}^{m'-1}b_i\lambda^i+1$ of $T_{2m'\times 2m'}$ in Eq. (6), and find the solutions of $f_{2m'}(\lambda)=0$, which will also be the eigenvalues of $T_{2m'\times 2m'}$. Generally speaking, it is difficult to find exact roots of $f_{2m'}(\lambda)=0$, but one can solve the approximate value of the root, which is enough for scientific research. The methods for reaching the approximate value of the root are Newton' method and quadratic factor algorithm, and so on. For quadratic factor algorithm, more detail is provided in Appendix B. To determine the existence of edge states, one has to talk about every possible scenario of eigenvalues of $T_{2m'\times 2m'}$ in detail. To take full advantage of Appendix A in Ref. [38] and TMM,[39,40] we can rephrase Eq. (6) as

$$\begin{aligned}\Theta^{(\alpha)}_{n+2m'}&=\Theta^{(\alpha)}_{n+2m'-1}U_{2m'\times 2m'}J_{2m'\times 2m'}U^{-1}_{2m'\times 2m'}\\ &=\Omega_{2m'}S^{(\alpha)}_{2m'\times 2m'}U_{2m'\times 2m'}J^n_{2m'\times 2m'}U^{-1}_{2m'\times 2m'}\\ &=\Omega_{2m'}R^{(\alpha)}_{2m'\times 2m'}\sum_{i=1}^{2l}J^n_{n_i}(\lambda_i)U^{-1}_{2m'\times 2m'}\\ &=\begin{pmatrix}N_1^{(\alpha)}&N_2^{(\alpha)}&\ldots&N_{2m'}^{(\alpha)}\end{pmatrix}W_{2m'\times 2m'},\end{aligned}\tag{7}$$

where

$$R^{(\alpha)}_{2m'\times 2m'}=S^{(\alpha)}_{2m'\times 2m'}U_{2m'\times 2m'}=\begin{pmatrix}R^{(1)}_{m'\times m'}&R^{(2)}_{m'\times m'}\\ R^{(3)}_{m'\times m'}&R^{(4)}_{m'\times m'}\end{pmatrix},$$

$W_{2m'\times 2m'}=\sum_{i=1}^{2l}J^n_{n_i}(\lambda_i)U^{-1}_{2m'\times 2m'}$ with λ_i being the eigenvalues of $T_{2m'\times 2m'}$, $N_i^{(\alpha)}=\sum_{j=1}^{m'}R^{(\alpha)}_{ji}\varphi_j$, $n_1+n_2+,\cdots,+n_{2l-1}+n_{2l}=2m'$, $l\leq m'$, and we've already performed this arrangement $\{|\lambda_i|\leq 1,|\lambda_{i+l}|\geq 1,\lambda_i\lambda_{i+l}=1:i=1,2,\ldots,l-1,l\}$ without losing of generality. As long as one of $\{|\lambda_i|=|\lambda_{i+l}|:i=1,2,\ldots,l-1,l\}$ is equal to one, it will lead to the extended states we do not interest in. Here we only need to pay attention to $\{|\lambda_i|<1,|\lambda_{i+l}|>1,\lambda_i\lambda_{i+l}=1:i=1,2,\ldots,l-$

$1,l\}$. The presence of edge states requires $\mathrm{Det}(R^{(2)}_{m'\times m'}) = 0$ ($\mathrm{Rank}(R^{(2)}_{m'\times m'}) < m'$) at first. Secondly, this condition $\mathrm{Det}(R^{(1)}_{m'\times m'}) \neq 0$ ($\mathrm{Rank}(R^{(1)}_{m'\times m'}) = m'$) must be satisfied in order to obtain non-zero edge states.

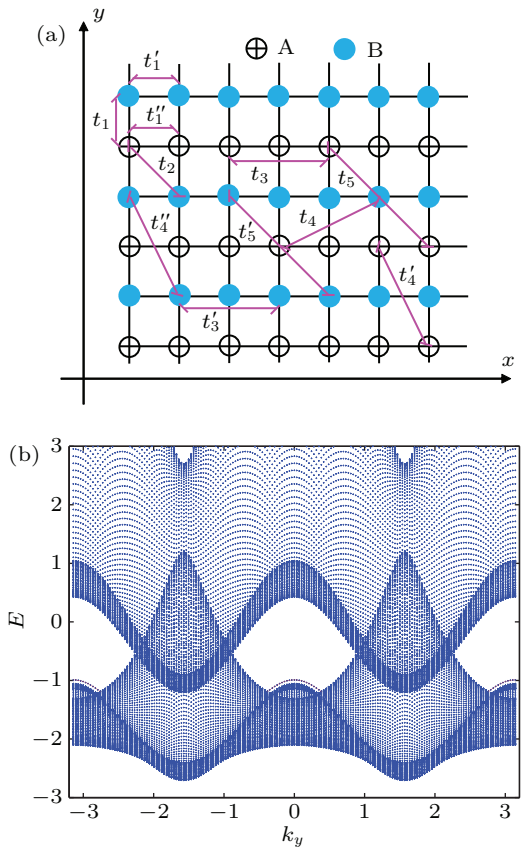


Fig. 1. The schematic illustration of the lattice structure of a semi-infinite 2D crystal in (a). The dispersion relation of edge states reached based on the exact diagonalization method (blue dashed line) and TMM (red dashed line) in (b) with $t_1' = t_1'' = 0.9t_1$, $t_2 = 0.7t_1$, $t_3 = t_3' = 0.4t_1$, $t_4 = t_4' = t_4'' = 0$, $t_5 = t_5' = 0$, $\varepsilon_{\mathrm{A}} = -0.6t_1$ and $\varepsilon_{\mathrm{B}} = 0.9t_1$.

3. Edge states in type II

In this section, we analyze such model crystals with type II, consisting of two different CLs: P and Q. Here we are only looking to the Q cut crystal Q–P–Q–P–$\cdots$, since the discussion for the P cut crystal will be similar. When the LR coupling is taken into account, QDEs for the Q cut crystal are

$$\begin{aligned}
&E^{(\mathrm{Q})}_{m\times m}\Phi_n = \sum_{i=1}^{\infty}\{F^{(2i-1)}_{m\times l}\blacktriangle_{2i-1}\Psi_n + A^{(2i)}_{m\times m}\Delta_{2i}\Phi_n\},\\
&E^{(\mathrm{P})}_{l\times l}\Psi_{n+1} = \sum_{i=1}^{\infty}\{(F^{(2i-1)}_{m\times l})^{\dagger}\blacktriangle_{2i-1}\Phi_{n+1} + B^{(2i)}_{l\times l}\Delta_{2i}\Psi_{n+1}\},\\
&n = 1,3,5,\ldots,\infty,
\end{aligned} \tag{8}$$

with $\Phi_n^{\mathrm{T}} = \{\phi_n^{(1)},\phi_n^{(2)},\ldots,\phi_n^{(m)}\}$, $\Psi_{n+1}^{\mathrm{T}} = \{\psi_{n+1}^{(1)},\psi_{n+1}^{(2)},\ldots,\psi_{n+1}^{(l)}\}$, $\blacktriangle_{2i-1}\Psi_n = \Psi_{n+2i-1} + \Psi_{n-(2i-1)}$, $\Delta_{2i}\Phi_n = \Phi_{n-2i} + \Phi_{n+2i}$, $\blacktriangle_{2i-1}\Phi_{n+1} = \Phi_{n+1-(2i-1)} + \Phi_{n+1+2i-1}$, $\Delta_{2i}\Psi_{n+1} = \Psi_{n+1-2i} + \Psi_{n+1+2i}$, the CL P has l modes and Q has m modes with l and m equal or unequal, $E^{(\alpha)}_{l_\alpha\times l_\alpha} = diag\{E^{\alpha}_1,E^{\alpha}_2,\ldots,E^{\alpha}_{l_\alpha}\}$ with $\{E^{\alpha}_{l_\alpha} = E - \varepsilon^{\alpha}_i : i = 1,2,\ldots,l_\alpha\}$, $l_{P(Q)} = l(m)$ when $\alpha = P(Q)$, $\{\Phi_n = 0, \Psi_{n+1} = 0 : n < 0\}$ and the matrix elements of $\{F^{(2i-1)}_{m\times l}, A^{(2i)}_{m\times m}, B^{(2i)}_{l\times l}\}$ are composed of the hopping parameters and $\boldsymbol{k}_{||}$. According to the conclusion in Ref. [34], edge states would not emerge when only the hopping between the 1NN CLs and all possible neighbor couplings within each CL exist. In the following, we will focus on the behavior of edge states in the presence of the LR hopping (the coupling from the rNN ($2 \leq r < \infty$) CLs). In order to make our general conclusion more acceptable, let us first discuss a special case, that is, $\{m = l = 1, F^{(1)}_{1\times1} \neq 0, F^{(3)}_{1\times1} \neq 0, A^{(2)}_{1\times1} \neq 0, B^{(2)}_{1\times1} \neq 0\}$ and $\{F^{(2i+1)}_{1\times1} = A^{(2i)}_{1\times1} = B^{(2i)}_{1\times1} = 0 : i = 2,3,\ldots,\infty\}$. Here we only give the transfer matrix of the electron mode in the CL Q, since the demonstration for the CL P will be similar. After some calculations, Eq. (8) can be rewritten as the following matrix form for the CL Q:

$$\begin{aligned}
&\Theta^{(\mathrm{Q})}_{n+12} = \Theta^{(\mathrm{Q})}_{n+10}T_{6\times6},\\
&\Theta^{(\mathrm{Q})}_{11} = \Omega_6 S^{(\mathrm{Q})}_{6\times6},\\
&n = 1,3,5,\ldots,\infty,
\end{aligned} \tag{9}$$

where

$$T_{6\times6} = \begin{pmatrix} T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix},$$

and $\Theta^{(\mathrm{Q})}_n = \{\phi^{(1)}_n,\phi^{(1)}_{n-2},\phi^{(1)}_{n-4},\phi^{(1)}_{n-6},\phi^{(1)}_{n-8},\phi^{(1)}_{n-10}\}$, $T_{11}^{\mathrm{T}} = \{b_1,b_2,b_3,b_2,b_1\}$, T_{12} is a 5×5 identity matrix, T_{22} is a 1×5 zero matrix, $T_{21} = -1$, $\Omega_6 = \{\phi^{(1)}_3,\phi^{(1)}_1,\psi^{(1)}_2,0,0,0\}$ and the specific expressions $\{b_r : i = 1,2,3\}$ and the matrix $S^{(\mathrm{Q})}_{6\times6}$ are given in Appendix C. In terms of Appendix A in Ref. [38] and TMM,[39,40] we can rewrite Eq. (9) as

$$\begin{aligned}
\Theta^{(\mathrm{Q})}_{n+12} &= \Theta^{(Q)}_{n+10}U_{6\times6}J_{6\times6}U^{-1}_{6\times6}\\
&= \Omega_6 S^{(\mathrm{Q})}_{6\times6}U_{6\times6}J^{(n+1)/2}_{6\times6}U^{-1}_{6\times6}\\
&= \Omega_6 S^{(\mathrm{Q})}_{6\times6}U_{6\times6}\sum_{i=1}^{2l}J^{(n+1)/2}_{n_i}(\lambda_i)U^{-1}_{6\times6}\\
&= \begin{pmatrix} N^{(\mathrm{Q})}_1 & N^{(\mathrm{Q})}_2 & \ldots & N^{(Q)}_6\end{pmatrix}W_{6\times6},
\end{aligned} \tag{10}$$

with

$$R^{(\mathrm{Q})}_{6\times6} = S^{(\mathrm{Q})}_{6\times6}U_{6\times6} = \begin{pmatrix} R^{(1)}_{3\times3} & R^{(2)}_{3\times3}\\ R^{(3)}_{3\times3} & R^{(4)}_{3\times3}\end{pmatrix},$$

where $n_1 + n_2 +,\cdots,+n_{2l-1} + n_{2l} = 6$ where $l \leq 3$ and $\{\lambda_i : i = 1,2,\ldots,2l\}$ are obtained from Appendix C with λ_i being the eigenvalues of $T_{6\times6}$. To facilitate the discussion, we have made such arrangement $\{|\lambda_i| \leq 1, |\lambda_{i+l}| \geq 1, \lambda_i\lambda_{i+l} = 1 : i = 1,2,\ldots,l-1,l\}$. Once one of $\{|\lambda_i| = |\lambda_{i+l}| : i = 1,2,\ldots,l-1,l\}$ is equal to 1, it will correspond to extended states we do not consider. Thus one needs to discuss $\{|\lambda_i| < 1, |\lambda_{i+l}| > 1, \lambda_i\lambda_{i+l} = 1 : i = 1,2,\ldots,l-1,l\}$. The existence of non-zero edge states confines $\{\mathrm{Det}(R^{(2)}_{3\times3}) = 0, \mathrm{Det}(R^{(1)}_{3\times3}) \neq 0\}$. In order to verify the correctness of our analytical result, let us take a 2D crystal as an example, as shown in Fig. 2(a).

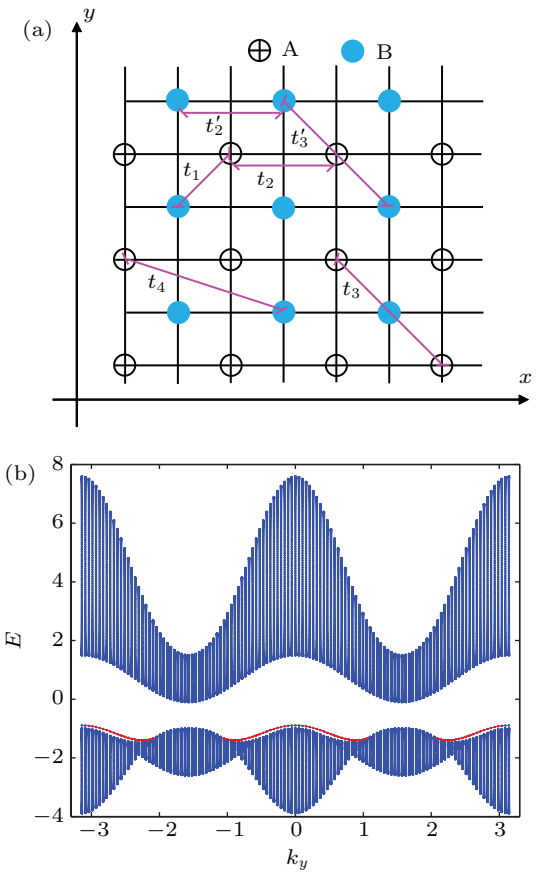


Fig. 2. The schematic illustration of the lattice structure of a semi-infinite 2D crystal in (a). The energy spectrum of edge states obtained in terms of the exact diagonalization method (blue dashed line) and TMM (red dashed line) in (b). The parameters are $t_2 = t_2' = 0.4t_1$, $t_3 = t_3' = 0$, $t_4 = 0.2t_1$, $\varepsilon_A = -1.0t_1$ and $\varepsilon_B = 1.5t_1$.

Here, owing to its infinity in y-direction, the Fourier transformation can be applied to wave functions for $A(B)$ sublattices. When we investigate the hopping from 1NN to 3NN CLs, one can adopt the $\{n, k_y\}$ representation and introduce QDEs based on a set of Fermion operators $\{\psi_{n,A}(k_y), \psi_{n+1,B}(k_y) : n = 1,3,5,\ldots,\infty\}$

$$
\begin{aligned}
E_A\psi_n^A &= \sum_{i=1}^{2} t_{2i-1}^{AB}\blacktriangle_{2i-1}\psi_n^B + t_2^{AA}\Delta_2\psi_n^A,\\
E_B\psi_{n+1}^B &= \sum_{i=1}^{2} t_{2i-1}^{AB}\blacktriangle_{2i-1}\psi_{n+1}^A + t_2^{BB}\Delta_2\psi_{n+1}^B,
\end{aligned}
\tag{11}
$$

with

$$
\begin{aligned}
&E_\alpha = E - \varepsilon_\alpha,\ \ \alpha = A,B,\\
&t_1^{AB} = 2[t_1\cos(k_y a) + t_4\cos(3k_y a)],\\
&t_3^{AB} = 2t_4\cos(k_y a),\\
&t_2^{AA} = t_2 + 2t_3\cos(2k_y a),\\
&t_2^{BB} = t_2' + 2t_3'\cos(2k_y a),
\end{aligned}
$$

$\blacktriangle_{2i-1}\psi_n^B = \psi_{n+2i-1}^B + \psi_{n-(2i-1)}^B$, $\Delta_2\psi_n^A = \psi_{n-2}^A + \psi_{n+2}^A$, $\blacktriangle_{2i-1}\psi_{n+1}^A = \psi_{n+1-(2i-1)}^A + \psi_{n+1+(2i-1)}^A$, $\Delta_2\psi_{n+1}^B = \psi_{n-1}^B + \psi_{n+3}^B$, $\{\psi_n^A = \psi_{n+1}^B = 0,\ n \le 0\}$, $\{t_2^{AA}, t_2^{BB}, t_1^{AB}, t_3^{AB}\}$ represent the coupling from 1NN to 3NN CLs, ε_α is the on-site energy, $\{t_1, t_2, t_3, t_4, t_2', t_3'\}$ are the hopping parameters and $\{\varepsilon_\alpha, t_2^{AA}, t_2^{BB}, t_1^{AB}, t_3^{AB}\}$ are real. By the use of the exact diagonalization method, it is shown that edge states can appear in the cut crystals of type Ⅱ (see Fig. 2(b)), completely consistent with our analytical calculation.

Now we focus on the general case in type Ⅱ where m and l are finite positive integers and LR couplings are introduced. Here we are still talking about electron modes within the CL Q because the similar result for the CL P can be obtained. Equation (8) can be rewritten as the following matrix form after some derivation:

$$
\begin{aligned}
&\Theta_{n+4l'}^{\prime(\alpha)} = \Theta_{n+4l'-2}^{\prime(\alpha)} T_{2l'\times 2l'}^{\prime}\ ,\\
&\Theta_{4l'-1}^{\prime(\alpha)} = \Omega_{2l'}^{\prime} S_{2l'\times 2l'}^{\prime(\alpha)},\\
&\alpha = 1,2,\ldots,m,\\
&n = 1,3,5,\ldots,\infty,
\end{aligned}
\tag{12}
$$

where

$$
T_{2l'\times 2l'}^{\prime} = \begin{pmatrix} T_{11}' & T_{12}' \\ T_{21}' & T_{22}' \end{pmatrix},
$$

$l' = \frac{(l+m)r}{2}(\frac{(l+m)(r-1)+2m}{2})$ when r is even (odd), $\Theta_n^{\prime(\alpha)} = \{\phi_n^{(\alpha)}, \phi_{n-2}^{(\alpha)}, \ldots, \phi_{n-4l'+4}^{(\alpha)}, \phi_{n-4l'+2}^{(\alpha)}\}$, $T_{2l'\times 2l'}^{\prime}$ has the same form as $T_{2m'\times 2m'}$ in Eq. (6), the matrix elements in T_{11}' and $S_{2l'\times 2l'}^{\prime(\alpha)}$ are arbitrary complex functions of the elements of $\{E_{m\times m}^{(P)}, E_{l\times l}^{(Q)}\}$ and the hopping parameters as well as $\boldsymbol{k}_{||}$. $\Omega_{2l'}' = \{\chi^{\prime(\alpha)}, \chi^{\prime(m)}, \chi^{\prime(m-1)}, \ldots, \chi^{\prime(\alpha+1)}, \chi^{\prime(\alpha-1)}, \ldots, \chi^{\prime(1)}, \varkappa^{\prime(l)}, \varkappa^{\prime(l-1)}, \ldots, \varkappa^{\prime(1)}, \Omega^{(0)}\} = \{\varphi_1, \varphi_2, \ldots, \varphi_{l'}, \Omega^{(0)}\}$, where $\Omega^{(0)}$ is a $1\times l'$ zero matrix; $\{\chi^{\prime(\delta)} = \{\phi_r^{(\delta)}, \phi_{r-2}^{(\delta)}, \ldots, \phi_3^{(\delta)}, \phi_1^{(\delta)}\}$, $\varkappa^{\prime(\delta')} = \{\psi_{r-1}^{(\delta')}, \psi_{r-3}^{(\delta')}, \ldots, \psi_4^{(\delta')}, \psi_2^{(\delta')}\}\}$, where r is odd and $\{\delta = 1,2,\ldots,m;\ \delta' = 1,2,\ldots,l\}$. $\{\chi^{\prime(\delta)} = \{\phi_{r-1}^{(\delta)}, \phi_{r-3}^{(\delta)}, \ldots, \phi_3^{(\delta)}, \phi_1^{(\delta)}\}, \varkappa^{\prime(\delta')} = \{\psi_r^{(\delta')}, \psi_{r-2}^{(\delta')}, \ldots, \psi_4^{(\delta')}, \psi_2^{(\delta')}\}\}$, where r is even and $\{\delta = 1,2,\ldots,m; \delta' = 1,2,\ldots,l\}$. To make full use of Appendix A in Ref. [38] and TMM,[39,40] let us adapt Eq. (12) to the following form:

$$
\begin{aligned}
\Theta_{n+4l'}^{\prime(\alpha)} &= \Theta_{n+4l'-1}^{\prime(\alpha)} U_{2l'\times 2l'} J_{2l'\times 2l'} U_{2l'\times 2l'}^{-1}\\
&= \Omega_{2l'}' S_{2l'\times 2l'}^{\prime(\alpha)} U_{2l'\times 2l'} J_{2l'\times 2l'}^{(n+1)/2} U_{2l'\times 2l'}^{-1}\\
&= \Omega_{2l'}' R_{2l'\times 2l'}^{\prime(\alpha)} \sum_{i=1}^{2l} J_{n_i}^{(n+1)/2}(\lambda_i) U_{2l'\times 2l'}^{-1}\\
&= \begin{pmatrix} N_1^{(\alpha)} & N_2^{(\alpha)} & \ldots & N_{2l'}^{(\alpha)} \end{pmatrix} W_{2l'\times 2l'},
\end{aligned}
\tag{13}
$$

with

$$
R_{2l'\times 2l'}^{\prime(\alpha)} = S_{2l'\times 2l'}^{\prime(\alpha)} U_{2l'\times 2l'} = \begin{pmatrix} R_{l'\times l'}^{\prime(1)} & R_{l'\times l'}^{\prime(2)} \\ R_{l'\times l'}^{\prime(3)} & R_{l'\times l'}^{\prime(4)} \end{pmatrix},
$$

$W_{2l'\times 2l'} = \sum_{i=1}^{2l} J_{n_i}^{(n+1)/2}(\lambda_i) U_{2l'\times 2l'}^{-1}$, where λ_i are the eigenvalues of $T_{2l'\times 2l'}'$, $N_i^{(\alpha)} = \sum_{j=1}^{l'} R_{ji}^{\prime(\alpha)}\varphi_j$, $n_1 + n_2 + \ldots, + n_{2l-1} + n_{2l} = 2l'$, $l \le l'$ and we have arranged such that $\{|\lambda_i| \le 1,\ |\lambda_{i+l}| \ge 1,\ \lambda_i\lambda_{i+l} = 1 : i = 1,2,\ldots,l-1,l\}$ for the sake of argument. If one of $\{|\lambda_i| = |\lambda_{i+l}| = 1 : i = 1,2,\ldots,l-1,l\}$ exists, it will result in bulk states we do not interest in. Here

we only need to discuss $\{|\lambda_i|<1,|\lambda_{i+l}|>1,\lambda_i\lambda_{i+l}=1:i=1,2,\ldots,l-1,l\}$. The presence of non-zero edge states requires $\{\mathrm{Det}(R^{(1)}_{l'\times l'})\neq 0,\mathrm{Det}(R^{(2)}_{l'\times l'})=0\}$. So far, we have completed the demonstration of the condition for the existence of edge states for type I and Ⅱ. From the above discussion, we clearly know that the emergence of edge states originates from the transfer matrix in the bulk different from that in the boundary layers, fully compatible with the conjecture in Ref. [38]. Meanwhile, the edge of actual cutting materials is imperfect and the edge relaxation is always unavoidable near the edge, even edge reconstruction. Maybe in practical materials, this requirement that the transfer matrix in the bulk is different from that in the boundary layers is more likely to occur. Thus, actual finite-size materials may be more prone to induce edge states.

4. Edge states in type Ⅲ

In the section, we investigate such cut crystals with type Ⅲ in detail. Since the original infinite crystals with type Ⅲ have no reflection symmetry, the forward transfer matrix from nNN ($n=2i-1:i=1,2,\ldots$) CLs is not equal to the backward one, and QDEs read

$$\begin{aligned}
E_{m\times m}\Phi^{(\mathrm{o})}_n &= \textstyle\sum_{i=1}^{\infty}\{F^{(2i)}_{m\times m}(\Phi^{(\mathrm{o})}_{n-2i}+\Phi^{(\mathrm{o})}_{n+2i})\\
&\quad +F^{(2i-1)}_{m\times m}\Phi^{(\mathrm{e})}_{n+(2i-1)}+B^{(2i-1)}_{m\times m}\Phi^{(\mathrm{e})}_{n-(2i-1)}\},\\
E_{m\times m}\Phi^{(\mathrm{e})}_{n+1} &= \textstyle\sum_{i=1}^{\infty}\{F^{(2i)}_{m\times m}(\Phi^{(\mathrm{e})}_{n+1-2i}+\Phi^{(\mathrm{e})}_{n+1+2i})\\
&\quad +B^{(2i-1)}_{m\times m}\Phi^{(\mathrm{o})}_{n+1+2i-1}+F^{(2i-1)}_{m\times m}\Phi^{(\mathrm{o})}_{n+1-(2i-1)}\}\\
&\quad n=1,3,5,\ldots,\infty,
\end{aligned}\tag{14}$$

where $\{\Phi^{(\alpha)T}_n=\{\phi^{(1)}_n,\phi^{(2)}_n,\ldots,\phi^{(m)}_n\}:\alpha=\mathrm{o(e)}\}$ with the superscript $\alpha=\mathrm{o(e)}$ for odd (even) CL, $E_{m\times m}=diag\{E_1,E_2,\ldots,E_m\}$ with $\{E_i=E-\varepsilon_i:i=1,2,\ldots,m\}$, $\{\Phi^{(o)T}_n=\Phi^{(e)T}_{n+1}=0:n<0\}$ and the matrix elements of $\{F^{(2i)}_{m\times m},F^{(2i-1)}_{m\times m},B^{(2i-1)}_{m\times m}\}$ are made up of the hopping parameters and $\boldsymbol{k}_{||}$ within each CL. it has been clearly shown that the form of Eq. (14) is not the same as that in Eq. (8). Now we search for edge states in cut crystals with type Ⅲ. What we notice is that when the odd and even CLs are treated as different CLs, one can check that the form of the transfer matrix in the bulk for the modes within odd and even CLs is identical to that in Eq. (12), but different from that in the boundary layers. When this is true, the condition for the existence of edge states is just as we have performed in type Ⅱ. To determine the validity of this idea, we will proceed from the special case to the general one. Here we only give the transfer matrix of the electron mode in the odd CL, since the discussion for the even CL will be similar. First of all, we discuss the special case where each CL contains one mode ϕ and the couplings from 1NN to 3NN CLs are added, then Eq. (14) simplifies to the following matrix form after some derivation:

$$\begin{cases}\Theta^{(\mathrm{o})}_{n+12}=\Theta^{(o)}_{n+10}T'_{6\times 6}\,,\\ \Theta^{(\mathrm{o})}_{11}=\Omega'_6S^{(o)}_{6\times 6},\end{cases}\tag{15}$$

where

$$T'_{6\times 6}=\begin{pmatrix}T'_{11} & T'_{12}\\ T'_{21} & T'_{22}\end{pmatrix},$$

and the form of $T'_{6\times 6}$ is akin to that of $T_{6\times 6}$ in Eq. (9), where $T'^{\mathrm{T}}_{11}=\{b'_1,b'_2,b'_3,b'_2,b'_1\}$, $\Omega'_6=\{\phi^{(\mathrm{o})}_3,\phi^{(\mathrm{o})}_1,\phi^{(\mathrm{e})}_2,0,0,0\}$ and the specific expressions $\{b'_r:i=1,2,3\}$ and the matrix $S^{(\mathrm{o})}_{6\times 6}$ are summarized in Appendix D. Now we can do such demonstrations similar to that in the special case with type Ⅱ and further get the similar conclusion of the existence of edge states. To check the validity of our conclusion, a 2D example with one mode within each CL is discussed in Fig. 3(a). Here, because of its infinity in y-direction, one can utilize the Fourier transformation to wave functions. When the coupling from 1NN to 3NN CLs is taken into account, one can obtain QDEs in the $\{n,k_y\}$ representation and by terms of a set of Fermion operators $\{\phi^{(\mathrm{o})}_n(k_y),\phi^{(e)}_{n+1}(k_y):n=1,3,5,\ldots,\infty\}$, where the superscript $\alpha=\mathrm{o(e)}$ means odd (even) CL

$$\begin{aligned}
E_\phi\phi^{(\mathrm{o})}_n &= t^F_1\phi^{(\mathrm{e})}_{n+1}+t^{\mathrm{B}}_1\phi^{(\mathrm{e})}_{n-1}+t^F_3\phi^{(\mathrm{e})}_{n+3}\\
&\quad +t^{\mathrm{B}}_3\phi^{(\mathrm{e})}_{n-3}+t^N_2(\phi^{(\mathrm{o})}_{n+2}+\phi^{(\mathrm{o})}_{n-2}),\\
E_\phi\phi^{(\mathrm{e})}_{n+1} &= t^{\mathrm{B}}_1\phi^{(\mathrm{o})}_{n+2}+t^F_1\phi^{(\mathrm{o})}_n+t^{\mathrm{B}}_3\phi^{(\mathrm{o})}_{n+4}\\
&\quad +t^F_3\phi^{(\mathrm{o})}_{n-2}+t^N_2(\phi^{(\mathrm{e})}_{n+3}+\phi^{(\mathrm{e})}_{n-1}),
\end{aligned}$$

with

$$\begin{aligned}
&E_\phi=E-2t_2\cos(2k_ya)-\varepsilon_\phi,\\
&t^F_1=2t_1\cos(k_ya);t^{\mathrm{B}}_1=2t'_1\cos(k_ya),\\
&t^N_2=t_2+2t_3\cos(2k_ya),\\
&t^F_3=2t_4\cos(k_ya);t^{\mathrm{B}}_3=2t'_4\cos(k_ya),
\end{aligned}$$

$\{\phi^{(\mathrm{o})}_n=\phi^{(\mathrm{e})}_{n+1}=0,n\leq 0\}$, $\{t^F_1,t^F_3,t^{\mathrm{B}}_1,t^{\mathrm{B}}_3,t^N_2\}$ represent the coupling from 1NN to 3NN CLs, ε_ϕ is the on-site energy, $\{t_1,t_2,t_3,t_4,t'_1,t'_4\}$ are the hopping parameters and $\{\varepsilon_\phi,t^F_1,t^F_3,t^{\mathrm{B}}_1,t^{\mathrm{B}}_3,t^N_2\}$ are real. We have fitted the numerical results using specific hopping parameters, in good agreement with our analytical calculation (see Fig. 3(b)). Now one will concentrate on discussing the general situation in type Ⅲ. We find that we can follow the demonstration of the general case in type Ⅱ to discuss the general one in type Ⅲ, and as a result we can get a matrix equation similar to Eq. (12) and further reach the condition of the existence of edge states similar to that in type Ⅱ. It is straightforward to notice that the above discussion can generalize to such model crystals with type: ⋯–P=P≡P–P=P≡P–⋯ and ⋯–Q=P–Q=P–⋯, and so on. Meanwhile, we also note that the prior work[25] takes no consideration of the degeneracy of eigenvalues of the matrix T (Eq. (38)[25]). According to the matrix theory, when the eigenvalues degenerate, the matrix S cannot be simply written as such a form.[25]

The exact expressions for the matrix S should be given such a form.[38] Our work shows a way to solve zero energy states in extended Kitaev chain with LR hopping and pairing. Our conclusion also covers the three- and one-dimensional case, in which the "line" represents the atomic surface and point, respectively.

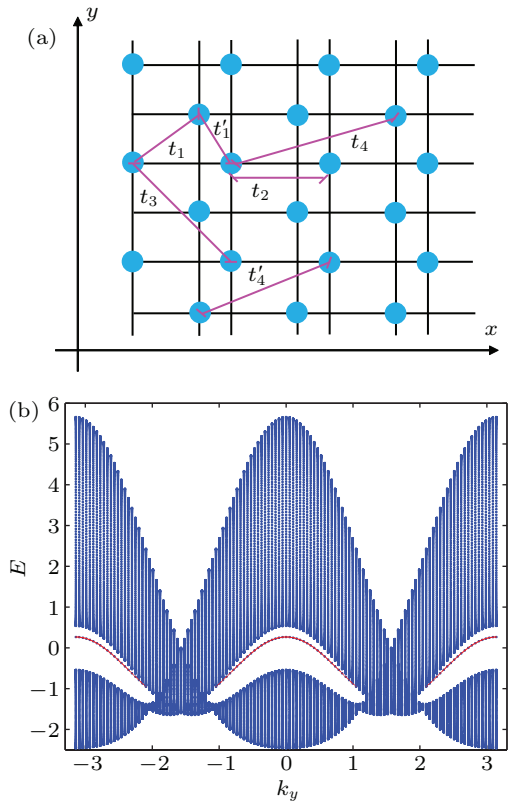


Fig. 3. The schematic illustration of the lattice structure of a semi-infinite 2D crystal in (a) and the dispersion relation of edge states shown via the exact diagonalization method (blue dashed line) and TMM (red dashed line) in (b) with $t_1 = 0.7t_1'$, $t_2 = 0.4t_1'$, $t_3 = 0.0$, $t_4 = 0.15t_1'$, $t_4{}' = 0.18t_1'$ and $\varepsilon_\phi = 0$.

5. Conclusion and perspectives

In conclusion, based on the lattice model Hamiltonian with the LR coupling among CLs and all possible neighbor hoppings within each CL, we have analytically determined the condition of the appearance of edge states in three model crystals. The expressions we obtain are general and hold for any choice of the LR hopping. In fact, the key in our demonstration is that the transfer matrix in the bulk is different from that in the boundary layers. It is this reason so that edge states can arise in such cut crystals. The study of edge states has become a central focus for physical investigation in recent years, such as extended Kitaev chain and Su–Schrieffer–Heeger chain. Our approach could be generalized to one- and three-dimensional systems. Meanwhile our demonstration can also apply to other model crystals, such as ⋯–P=P≡P–P=P≡P–⋯ and ⋯–Q=P–Q=P–⋯. We believe that this work can provide new proposals to solve novel edge states in extended Kitaev chain and Su–Schrieffer–Heeger chain. From a more general perspective, our study highlights the effects of the LR coupling in strongly correlated systems.

Appendix A: Type I with two electron modes

When the hopping from 1NN to 2NN CLs and two electron modes within each CL are discussed, QDEs of the bulk and boundary layers from Eq. (2) in the text read

$$\begin{aligned} E_1\phi_n^{(1)} &= \sum_{\alpha=1}^{2}\{F_{1\alpha}^{(1)}\Delta_1\phi_n^{(\alpha)} + F_{1\alpha}^{(2)}\Delta_2\phi_n^{(\alpha)}\}, \\ E_2\phi_n^{(2)} &= \sum_{\alpha=1}^{2}\{F_{2\alpha}^{(1)}\Delta_1\phi_n^{(\alpha)} + F_{2\alpha}^{(2)}\Delta_2\phi_n^{(\alpha)}\}, \end{aligned} \tag{A1}$$

where $\{F_{12}^{(\alpha)} = F_{21}^{(\alpha)} : \alpha = 1,2\}$ due to the original infinite crystal having geometric reflection symmetry for each CL. We can adapt Eq. (A1) to the matrix form by some calculations:

$$\begin{aligned} &\Theta_{n+8}^{(\alpha)} = \Theta_{n+7}^{(\alpha)}T_{8\times8}, \quad \Theta_8^{(\alpha)} = \Omega_8^{(\alpha)}S_{8\times8}^{(\alpha)}, \\ &T_{8\times8} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad n \geq 1, \end{aligned} \tag{A2}$$

where $T_{11}^{\mathrm{T}} = \{b_1, b_2, b_3, b_4, b_3, b_2, b_1\}$ with

$$\begin{aligned} &b_1 = G_1/G_0; \quad b_2 = G_2/G_0, \\ &b_3 = G_3/G_0; \quad b_4 = G_4/G_0, \\ &G_0 = F_{12}^{(2)}F_{12}^{(2)} - F_{11}^{(2)}F_{22}^{(2)}, \\ &G_1 = F_{22}^{(1)}F_{11}^{(2)} + F_{22}^{(2)}F_{11}^{(1)} - 2F_{12}^{(2)}F_{12}^{(1)}, \\ &G_2 = F_{11}^{(1)}F_{22}^{(1)} - F_{22}^{(2)}E_1 - F_{11}^{(2)}E_2 - F_{12}^{(1)}F_{12}^{(1)}, \\ &G_3 = G_1 - F_{22}^{(1)}E_1 - F_{11}^{(1)}E_2, \\ &G_4 = E_2E_1 - 2F_{12}^{(1)}F_{12}^{(1)} + 2F_{11}^{(1)}F_{22}^{(1)} - 2G_0. \end{aligned} \tag{A3}$$

For (1) electron mode, $S^{(1)}$ has the following form:

$$\begin{aligned} &S^{(1)} = \Pi_{\alpha=1}^{6}T^{(\alpha)}, \\ &T^{(\alpha)} = \begin{pmatrix} T_{11}^{(\alpha)} & T_{12}^{(\alpha)} \\ 0 & T_{22}^{(\alpha)} \end{pmatrix}, \\ &T_{12}^{(\alpha)} = T_{12}, T_{22}^{(\alpha)} = T_{22}, \end{aligned} \tag{A4}$$

with

$$\begin{aligned} T_{11}^{(6)T} &= T_{11}^{\mathrm{T}}, \quad T_{11}^{(5)T} = \{b_1, b_2, b_3, b_4, b_3, b_2, 0\}, \\ T_{11}^{(4)T} &= \{b_1, b_2, b_3, b_4+1, b_3 + (F_{12}^{(2)}F_{12}^{(1)} - F_{22}^{(2)}F_{11}^{(1)})/G_0, 0, (F_{12}^{(2)}F_{22}^{(1)} - F_{12}^{(1)}F_{22}^{(2)})/G_0\}, \\ T_{11}^{(3)T} &= \{b_1, b_2, b_3 + (F_{12}^{(1)}F_{12}^{(2)} - F_{22}^{(1)}F_{11}^{(2)})/G_0, b_4 + 1 + (F_{12}^{(1)}F_{12}^{(1)} - F_{22}^{(1)}F_{11}^{(1)})/G_0, (F_{12}^{(1)}F_{22}^{(2)} - F_{12}^{(2)}F_{22}^{(1)})/G_0, 0, 0\}, \end{aligned}$$

$$T_{11}^{(2)T} = \{b_1, b_2 + E_2F_{11}^{(2)}/G_0, b_3 + (F_{12}^{(1)}F_{12}^{(2)} + E_2F_{11}^{(1)} - F_{22}^{(1)}F_{11}^{(2)})/G_0, F_{12}^{(2)}E_2/G_0, (F_{12}^{(1)}F_{22}^{(2)} + F_{12}^{(1)}E_2 - F_{22}^{(1)}F_{12}^{(2)})/G_0, 0, 0\},$$
$$T_{11}^{(1)T} = \{(F_{22}^{(2)}F_{11}^{(1)} - F_{12}^{(2)}F_{12}^{(1)})/G_0, -F_{22}^{(2)}E_1/G_0, (F_{22}^{(2)}F_{12}^{(1)} - F_{12}^{(2)}F_{22}^{(1)})/G_0, F_{12}^{(2)}E_2/G_0, 0, 0, 0\}.$$

For (2) electron mode, $S^{(2)}$ has the similar matrix form to $S^{(1)}$, not given in detail.

Appendix B: Solutions of a general quartic equation and quadratic factor algorithm

For a general quartic equation, one can have the following form

$$az^4 + bz^3 + cz^2 + dz + e = 0, \quad a \neq 0, \tag{B1}$$

where we omit the detailed calculation process and can obtain four solutions in the complex number field as follows:

$$\begin{aligned} z_1 &= y_- - \frac{1}{2}\sqrt{y_1 - y_2}, \\ z_2 &= y_- + \frac{1}{2}\sqrt{y_1 - y_2}, \\ z_3 &= y_+ - \frac{1}{2}\sqrt{y_1 - y_2}, \\ z_4 &= y_+ + \frac{1}{2}\sqrt{y_1 - y_2}, \end{aligned} \tag{B2}$$

with

$$\begin{aligned} &y_\pm = \frac{-b}{4a} \pm \frac{1}{2}\sqrt{x + \Delta}; y_1 = 2x + \Delta, \\ &y_2 = (\frac{4bc}{a^2} - \frac{b^3}{a^3} - \frac{8d}{a})/(4\sqrt{x} + \Delta), \\ &x = \frac{b^2}{4a^2} - \frac{2c}{3a}, \\ &\Delta = \frac{\sqrt[3]{2}\Delta_1}{3a^3\sqrt[3]{\Delta_2 + \sqrt{\Delta_2^2 - 4\Delta_1^3}}} + \frac{\sqrt[3]{\Delta_2 + \sqrt{\Delta_2^2 - 4\Delta_1^3}}}{3\sqrt[3]{2}a}, \\ &\Delta_1 = c^2 - 3bd + 12ae, \\ &\Delta_2 = 2c^3 - 9c(bd + 8ae) + 27(ad^2 + b^2e). \end{aligned}$$

The definition of z amounts to an equation for λ:

$$\lambda^2 - z\lambda + 1 = 0. \tag{B3}$$

Thus one can insert the solutions $\{z_i : i = 1, 2, 3, 4\}$ and solve for λ. We find

$$\lambda_\pm^{(i)} = \frac{z_i \pm \sqrt{z_i^2 - 4}}{2}, \tag{B4}$$

yielding directly $\lambda_+^{(i)} \cdot \lambda_-^{(i)} = 1$. In our calculation, we have $\{a = 1, b_1 = -b,\ b_2 = -c - 4a,\ b_3 = -d - 3b,\ b_4 = -e - 6a - 2c\}$.

The quadratic factor algorithm is a good one for complex roots. As is well known, one has $x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/(2a)$ for $\{ax^2 + bx + c = 0; a \neq 0\}$. Quadratic factor algorithm is to find the quadratic factor $u(\lambda) = \lambda^2 + p\lambda + q$ of $f(\lambda)$. This method establishes a set of iterative methods for finding p and q and avoids complex number operation. Once p and q are determined, one can get the two roots of $f(\lambda)$. If $p^2 - 4q < 0$, a pair of conjugate complex roots can be obtained. Then one can use $f(\lambda) = (\lambda^2 + p\lambda + q) f_1(\lambda)$, where $f_1(\lambda)$ is a $(n-2)$ polynomial and arrive at the roots of $f_1(\lambda)$ by means of the same way. By iteration step by step, we can get all the roots of $f(\lambda) = 0$.

Appendix C: Type Ⅱ with one electron mode and solutions of a general cubic equation

For type Ⅱ, when we consider the hopping from 1NN to 3NN CLs and each CL with one electron mode, QDEs of the bulk and boundary layers from Eq. (8) in the text read

$$\begin{aligned} E_{m\times m}^{(\mathrm{Q})}\Phi_n &= \sum_{i=1}^{2} F_{m\times l}^{(2i-1)} \blacktriangle_{2i-1}\Psi_n + A_{m\times m}^{(2)}\Delta_2\Phi_n, \\ E_{l\times l}^{(\mathrm{P})}\Psi_{n+1} &= \sum_{i=1}^{2} (F_{m\times l}^{(2i-1)})^\dagger \blacktriangle_{2i-1}\Phi_{n+1} + B_{l\times l}^{(2)}\Delta_2\Psi_{n+1}. \end{aligned} \tag{C1}$$

Equation (C1) can be rewritten as the following matrix form after some calculations:

$$\begin{aligned} \Theta_{n+12}^{(\mathrm{Q})} &= \Theta_{n+10}^{(Q)} T_{6\times 6}\ , \\ \Theta_{11}^{(\mathrm{Q})} &= \Omega_6 S_{6\times 6}^{(\mathrm{Q})}, \end{aligned} \tag{C2}$$

with

$$T_{6\times 6} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

and $T_{11}^{\mathrm{T}} = \{b_1, b_2, b_3, b_2, b_1\}$, where

$$\begin{aligned} &b_1 = G_1/G_0; \quad b_2 = G_2/G_0; \quad b_3 = G_3/G_0; \\ &G_0 = F_{1\times1}^{(3)}F_{1\times1}^{(3)}; \quad G_1 = -2F_{1\times1}^{(1)}F_{1\times1}^{(3)} + A_{1\times1}^{(2)}B_{1\times1}^{(2)} \\ &G_2 = -E_{1\times1}^{(\mathrm{P})}A_{1\times1}^{(2)} - F_{1\times1}^{(1)}F_{1\times1}^{(1)} - 2F_{1\times1}^{(1)}F_{1\times1}^{(3)} - E_{1\times1}^{(\mathrm{Q})}B_{1\times1}^{(2)}, \\ &G_3 = E_{1\times1}^{(\mathrm{P})}E_{1\times1}^{(\mathrm{Q})} - 2F_{1\times1}^{(1)}F_{1\times1}^{(1)} + 2A_{1\times1}^{(2)}B_{1\times1}^{(2)} - 2F_{1\times1}^{(3)}F_{1\times1}^{(3)}. \end{aligned} \tag{C3}$$

For the CL Q, $S^{(\mathrm{Q})}$ has the following form:

$$\begin{aligned} &S^{(\mathrm{Q})} = \Pi_{\alpha=1}^{4} T^{(\alpha)}, \quad T_{11}^{(4)T} = T_{11}^{\mathrm{T}}, \\ &T^{(\alpha)} = \begin{pmatrix} T_{11}^{(\alpha)} & T_{12}^{(\alpha)} \\ 0 & T_{22}^{(\alpha)} \end{pmatrix}, \\ &T_{12}^{(\alpha)} = T_{12}, \quad T_{22}^{(\alpha)} = T_{22}, \end{aligned} \tag{C4}$$

with

$$\begin{aligned}
T_{11}^{(1)T} &= \{(A_{1\times1}^{(2)}B_{1\times1}^{(2)} - F_{1\times1}^{(1)}F_{1\times1}^{(3)})/G_0,\\
&\quad (-F_{1\times1}^{(1)}F_{1\times1}^{(3)} - E_{1\times1}^{(\mathrm{Q})}B_{1\times1}^{(2)})/G_0,\\
&\quad (B_{1\times1}^{(2)}F_{1\times1}^{(1)} + F_{1\times1}^{(3)}E_{1\times1}^{(\mathrm{P})})/G_0, 0, 0\},\\
T_{11}^{(2)T} &= \{b_1, b_2 + F_{1\times1}^{(1)}F_{1\times1}^{(3)}/G_0,\\
&\quad b_3 + (F_{1\times1}^{(1)}F_{1\times1}^{(1)} - A_{1\times1}^{(2)}B_{1\times1}^{(2)} + F_{1\times1}^{(3)}F_{1\times1}^{(3)})/G_0,\\
&\quad B_{1\times1}^{(2)}(F_{1\times1}^{(1)} - F_{1\times1}^{(3)})/G_0, 0\},\\
T_{11}^{(3)T} &= \{b_1, b_2, b_3 + 1, b_2 + F_{1\times1}^{(1)}F_{1\times1}^{(3)}/G_0, F_{1\times1}^{(3)}B_{1\times1}^{(2)}/G_0\}.
\end{aligned}$$

One can easily check that the roots of characteristic polynomial $f(\lambda) = \lambda^6 - \sum_{i=1}^{3} b_i\lambda^{6-i} - \sum_{i=1}^{2} b_i\lambda^i + 1$ are the eigenvalues of $T_{6\times6}$. When the two sides of $f(\lambda) = 0$ are divided by λ^3 and setting $z = \lambda + \frac{1}{\lambda}$, one can simplify $f(\lambda) = 0$ to a cubic equation. One can obtain the roots by calculating the cubic equation and further get the corresponding value λ by solving $\lambda^2 - z\lambda + 1 = 0$. To reach all possible forms of the eigenvalues of $T_{6\times6}$, now we discuss the general roots of a general cubic equation whose form is as follows:

$$\begin{aligned}
&az^3 + bz^2 + cz + d = 0,\\
&a \neq 0,
\end{aligned} \tag{C5}$$

here we skip the detailed calculation process and can arrive at three roots in the complex number field as follows:

$$\begin{aligned}
z_1 &= \sqrt[3]{\Delta - \frac{q}{2}} + \sqrt[3]{-\frac{q}{2} - \Delta},\\
z_2 &= \sqrt[3]{\Delta - \frac{q}{2}}\omega + \sqrt[3]{-\frac{q}{2} - \Delta}\omega^2,\\
z_3 &= \sqrt[3]{\Delta - \frac{q}{2}}\omega^2 + \sqrt[3]{-\frac{q}{2} - \Delta}\omega,
\end{aligned} \tag{C6}$$

with

$$\begin{aligned}
&p = \frac{3ac - b^2}{3a^2}; \quad q = \frac{2b^3 - 9abc + 27da^2}{27a^3},\\
&\Delta = \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}; \quad \omega = \frac{-1 + \mathrm{i}\sqrt{3}}{2}.
\end{aligned}$$

Based on the relationship $\lambda^2 - z\lambda + 1 = 0$ and inserting $\{z_i : i = 1, 2, 3\}$, one can have the same form as Eq. (B4). In our calculation, we find $\{a = 1;\ b_1 = -b;\ b_2 = -c - 3a;\ b_3 = -d - 2b\}$.

Appendix D: Type Ⅲ with one electron mode

For type Ⅲ, when the hopping from 1NN to 3NN CLs is added and each CL has one electron mode, QDEs of the bulk and boundary layers from Eq. (14) in the text simplify to the following form:

$$\begin{aligned}
E_{1\times1}\phi_n^{(\mathrm{o})} &= \sum_{i=1}^{2}\{F_{1\times1}^{(2i-1)}\phi_{n+(2i-1)}^{(\mathrm{e})} + B_{1\times1}^{(2i-1)}\phi_{n-(2i-1)}^{(\mathrm{e})}\}\\
&\quad + F_{1\times1}^{(2)}(\phi_{n-2}^{(\mathrm{o})} + \phi_{n+2}^{(\mathrm{o})}),\\
E_{1\times1}\phi_{n+1}^{(\mathrm{e})} &= \sum_{i=1}^{2}\{B_{1\times1}^{(2i-1)}\phi_{n+1+2i-1}^{(\mathrm{o})} + F_{1\times1}^{(2i-1)}\phi_{n+1-(2i-1)}^{(\mathrm{o})}\}\\
&\quad + F_{1\times1}^{(2)}(\phi_{n-1}^{(\mathrm{e})} + \phi_{n+3}^{(\mathrm{e})}).
\end{aligned} \tag{D1}$$

After some derivation, Eq. (D1) can be rewritten as

$$\begin{aligned}
\Theta_{n+12}^{(\mathrm{o})} &= \Theta_{n+10}^{(\mathrm{o})}T'_{6\times6},\\
\Theta_{11}^{(\mathrm{o})} &= \Omega'_6 S_{6\times6}^{(\mathrm{o})},
\end{aligned} \tag{D2}$$

where

$$T'_{6\times6} = \begin{pmatrix} T'_{11} & T'_{12}\\ T'_{21} & T'_{22}\end{pmatrix},$$

with $T_{11}^{\prime\mathrm{T}} = \{b'_1, b'_2, b'_3, b'_2, b'_1\}$ and

$$\begin{aligned}
&b'_1 = G_1/G_0; \quad b'_2 = G_2/G_0; \quad b'_3 = G_3/G_0;\\
&G_0 = F_{1\times1}^{(3)}B_{1\times1}^{(3)}; \quad G_1 = F_{1\times1}^{(2)}F_{1\times1}^{(2)} - F_{1\times1}^{(3)}B_{1\times1}^{(1)} - F_{1\times1}^{(1)}B_{1\times1}^{(3)},\\
&G_2 = -2E_{1\times1}F_{1\times1}^{(2)} - F_{1\times1}^{(1)}B_{1\times1}^{(1)} - F_{1\times1}^{(1)}F_{1\times1}^{(3)} - B_{1\times1}^{(1)}B_{1\times1}^{(3)},\\
&G_3 = E_{1\times1}E_{1\times1} - F_{1\times1}^{(1)}F_{1\times1}^{(1)} - B_{1\times1}^{(1)}B_{1\times1}^{(1)} + 2F_{1\times1}^{(2)}F_{1\times1}^{(2)}\\
&\quad - F_{1\times1}^{(3)}F_{1\times1}^{(3)} - B_{1\times1}^{(3)}B_{1\times1}^{(3)}.
\end{aligned} \tag{D3}$$

For the odd CL, $S^{(\mathrm{o})}$ has the following form

$$\begin{aligned}
S^{(\mathrm{o})} &= \Pi_{\alpha=1}^{4}T^{(\alpha)}, T_{11}^{(4)\mathrm{T}} = T_{11}^{\mathrm{T}},\\
T^{(\alpha)} &= \begin{pmatrix} T_{11}^{(\alpha)} & T_{12}^{(\alpha)}\\ 0 & T_{22}^{(\alpha)}\end{pmatrix},\\
T_{12}^{(\alpha)} &= B_{1\times1}^{(1)}, T_{22}^{(\alpha)} = F_{1\times1}^{(2)},
\end{aligned} \tag{D4}$$

with

$$\begin{aligned}
T_{11}^{(3)\mathrm{T}} &= \Big\{b_1, b_2, b_3 + B_{1\times1}^{(3)}B_{1\times1}^{(3)}/G_0, b_2\\
&\quad + B_{1\times1}^{(1)}B_{1\times1}^{(3)}/G_0, B_{1\times1}^{(3)}F_{1\times1}^{(2)}/G_0\Big\},\\
T_{11}^{(2)\mathrm{T}} &= \Big\{b_1, b_2 + B_{1\times1}^{(1)}B_{1\times1}^{(3)}/G_0, b_3\\
&\quad + \left(B_{1\times1}^{(1)}B_{1\times1}^{(1)} - F_{1\times1}^{(2)}F_{1\times1}^{(2)} + B_{1\times1}^{(3)}B_{1\times1}^{(3)}\right)/G_0,\\
&\quad - F_{1\times1}^{(2)}(F_{1\times1}^{(3)} - B_{1\times1}^{(1)})/G_0, 0\Big\},\\
T_{11}^{(1)\mathrm{T}} &= \Big\{\left(F_{1\times1}^{(2)}F_{1\times1}^{(2)} - B_{1\times1}^{(1)}F_{1\times1}^{(3)}\right)/G_0,\\
&\quad \left(-F_{1\times1}^{(1)}F_{1\times1}^{(3)} - E_{1\times1}F_{1\times1}^{(2)}\right)/G_0,\\
&\quad \left(F_{1\times1}^{(2)}F_{1\times1}^{(1)} + F_{1\times1}^{(3)}E_{1\times1}\right)/G_0, 0, 0\Big\}.
\end{aligned}$$

References

[1] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045
[2] Bernevig A B and Hughes T L 2013 *Topological Insulators and Topological Superconductors* (Princeton, NJ: Princeton University Press)
[3] Kitaev A Y 2003 *Ann. Phys.* **303** 2
[4] Nayak C, Simon S H, Stern A, Freedman M and Das Sarma S 2008 *Rev. Mod. Phys.* **80** 1083
[5] Stern A 2010 *Nature* **464** 187
[6] Meier D, Seidel J, Cano A, Delaney K, Kumagai Y, Mostovoy M, Spaldin N A, Ramesh R and Fiebig M 2012 *Nat. Mater.* **11** 284
[7] Miao M S, Yan Q, Van de Walle C G, Lou W K, Li L L and Chang K 2012 *Phys. Rev. Lett.* **109** 186803

[8] Zhang D, Lou W, Miao M, Zhang S C and Chang K 2013 *Phys. Rev. Lett.* **111** 156402
[9] Chang K and Lou W K 2011 *Phys. Rev. Lett.* **106** 206802
[10] Viyuela O, Fu L and Martin-Delgado M A 2018 *Phys. Rev. Lett.* **120** 017001
[11] Di Liberto M, Malpetti D, Japaridze G I and Morais Smith C 2014 *Phys. Rev. A* **90** 023634
[12] DeGottardi W, Thakurathi M, Vishveshwara S and Sen D 2013 *Phys. Rev. B* **88** 165111
[13] Hastings M B and Koma T 2006 *Commun. Math. Phys.* **265** 781
[14] Santos L F, Borgonovi F and Celardo G L 2016 *Phys. Rev. Lett.* **116** 250402
[15] Maghrebi M F, Gong Z X and Gorshkov A V 2017 *Phys. Rev. Lett.* **119** 023001
[16] Vodola D, Lepori L, Ercolessi E, Gorshkov A V and Pupillo G 2014 *Phys. Rev. Lett.* **113** 156402
[17] Hauke P and Tagliacozzo L 2013 *Phys. Rev. Lett.* **111** 207202
[18] Lepori L, Giuliano D and Paganelli S 2018 *Phys. Rev. B* **97** 041109
[19] Vodola D, Lepori L, Ercolessi E and Pupillo G 2016 *New J. Phys.* **18** 015001
[20] Lepori L and DellAnna L 2017 *New J. Phys.* **19** 103030
[21] Gong Z X, Maghrebi M F, Hu A, Foss-Feig M, Richerme P, Monroe C and Gorshkov A V 2016 *Phys. Rev. B* **93** 205115
[22] Gong Z X, Maghrebi M F, Hu A, Wall M L, Foss-Feig M and Gorshkov A V 2016 *Phys. Rev. B* **93** 041102
[23] Viyuela O, Vodola D, Pupillo G and Martin-Delgado M A 2016 *Phys. Rev. B* **94** 125121
[24] Lepori L, Vodola D, Pupillo G, Gori G and Trombettoni A 2016 *Ann. Phys. (NY)* **374** 35
[25] Alecce A and DellAnna L 2017 *Phys. Rev. B* **95** 195160
[26] Gori G, Paganelli S, Sharma A, Sodano P and Trombettoni A 2015 *Phys. Rev. B* **91** 245138
[27] Giuliano D, Paganelli S and Lepori L 2018 *Phys. Rev. B* **97** 155113
[28] Lepori L, Giuliano D and Paganelli S 2018 *Phys. Rev. B* **97** 041109
[29] Sabyasachi N and Arti G 2019 *Phys. Rev. B* **99** 224203
[30] Beatriz P G, Miguel B, Alvaro G L and Gloria P 2019 *Phys. Rev. B* **99** 035146
[31] Ahmadi N, Abouie J and Baeriswyl D 2020 *Phys. Rev. B* **101** 195117
[32] Xu X W, Li Y Z, Liu Z F and Chen A X 2020 *Phys. Rev. A* **101** 063839
[33] Wang H P and Tao R B 2015 *Chin. Phys. B* **24** 117301
[34] Wang H P, Gao T and Tao R B 2015 *Sci. Rep.* **5** 8679
[35] Dy K S and Brasher D James 1980 *Phys. Rev. B* **22** 4868
[36] Ho W, Cunningham S L, Weinberg W H and Dobrzynski L 1975 *Phys. Rev. B* **12** 3027
[37] Mostoller M and Rajagopal A K 1982 *Phys. Rev. B* **25** 6168
[38] Wang H P, Ren L, Qin L G and Qiu Y 2020 *J. Phys. Soc. Jpn.* **89** 074705
[39] Lee D H and Joannopoulos J D 1981 *Phys. Rev. B* **23** 4988
[40] Zhao Y Y, Li W and Tao R B 2012 *Physica B* **407** 724