ISSN 1674-1056
http://cpb.iphy.ac.cn
http://iopscience.iop.org/cpb
中国物理 B
Chinese Physics
中国物理
Chinese Physics B

Introducing the general condition for an operator in curved space to be unitary

Jafari Matehkolaee Mehdi

Citation: Chin. Phys. B, 2021, 30 (**8**): 080301. DOI: 10.1088/1674-1056/abe300
Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Universal quantum circuit evaluation on encrypted data using probabilistic quantum homomorphic encryption scheme

Jing-Wen Zhang(张静文), Xiu-Bo Chen(陈秀波), Gang Xu(徐刚), and Yi-Xian Yang(杨义先)
Chin. Phys. B, 2021, 30 (**7**): 070309. DOI: 10.1088/1674-1056/ac003b

Creation and annihilation phenomena of electron and positron pairs in an oscillating field

M Jiang(江淼), D D Su(苏丹丹), N S Lin(林南省), and Y J Li(李英骏)
Chin. Phys. B, 2021, 30 (**7**): 070306. DOI: 10.1088/1674-1056/abe378

Gazeau-Klauder coherent states examined from the viewpoint of diagonal ordering operation technique

Dušan Popov, Romeo Negrea, Miodrag Popov
Chin. Phys. B, 2016, 25 (**7**): 070301. DOI: 10.1088/1674-1056/25/7/070301

From fractional Fourier transformation to quantum mechanical fractional squeezing transformation

Lv Cui-Hong, Fan Hong-Yi, Li Dong-Wei
Chin. Phys. B, 2015, 24 (**2**): 020301. DOI: 10.1088/1674-1056/24/2/020301

Disordered quantum walks in two-dimensional lattices

Zhang Rong(张融), Xu Yun-Qiu(徐韵秋), Xue Peng(薛鹏)
Chin. Phys. B, 2015, 24 (**1**): 010303. DOI: 10.1088/1674-1056/24/1/010303

Introducing the general condition for an operator in curved space to be unitary

Jafari Matehkolaee Mehdi†

Faculty of Physics, Semnan University, Semnan, 35131-19111, *Iran*

(Received 25 December 2020; revised manuscript received 19 January 2021; accepted manuscript online 4 February 2021)

We investigate the general condition for an operator to be unitary. This condition is introduced according to the definition of the position operator in curved space. In a particular case, we discuss the concept of translation operator in curved space followed by its relation with an anti-Hermitian generator. Also we introduce a universal formula for adjoint of an arbitrary linear operator. Our procedure in this paper is totally different from others, as we explore a general approach based only on the algebra of the operators. Our approach is only discussed for the translation operators in one-dimensional space and not for general operators.

Keywords: operator theory, algebraic methods, unitary operator, translation operator

PACS: 03.65.−w, 03.65.Fd, 02.30.Tb **DOI:** 10.1088/1674-1056/abe300

1. Introduction

A unitary operator is a linear operator in Hilbert space which its properties was basically discussed in details in Refs. [1,2]. This operator is very important in quantum mechanics because the scalar product of vectors under the operation of this operator is invariant. Also, it can transform two sets of base kets, both satisfying orthonormality and completeness, with two different representations.[3] The concept of translation in quantum mechanics is described by a unitary operator and in general, its generator is an anti-Hermitian operator.[3] The translation operator has also been considered in mathematical literatures. For example, there is one integral representation of the quantum translation operator.[4] Some attempts have been made to represent the matrix form of the translation operators.[5,6]

We are now asking this question: Is there any general requirement for an operator to be unitary (other than the formal properties mentioned in the textbooks)?

There is obviously a certain way to construct the unitary operators.[7] However, the standard method in this reference, based on the Moravek–Joseph theorem, is completely different with our procedure in this paper.

There are so many texts and surveys about generalized translation operators.[8–10] In these surveys, the researchers usually start with the explicit definition of a translation operator in generalized space such as in Ref. [8]. Also the phrase generalized translation operators have been used in the title of the paper.[11] However, this paper is strictly in mathematical language and totally unrelated to quantum mechanics and the contents of our paper.

The rest of this paper is arranged as follows. In Section 2, we argue straightforwardly the unitary operator in general form. We endeavor to introduce a general formula for the unitary operators in terms of the position operator. Then after, we consider some examples of scaling operators to justify our definition. In Section 3, as an ultimate purpose, we pursue previous procedure in generalized space. In this section, by using the unitary operator, we choose a special case which leads automatically to the concept of the translation operator in curved space. Note that, in this paper we use quite extensively the so-called bra–ket formalism. We are aware from the principal problems connected with Dirac's bra–ket formalism,[12] so we try here to avoid them and use correctly the formalism.

2. Unitary operator in Euclidean space

We begin with an operator T such that

$$T|x\rangle = |f(x)\rangle, \tag{1}$$

where f is a smooth, invertible, and differentiable function and $f(x)$ is the amount of the function f at the point x, so that $f(x)$ is a point (like x) itself. Also, it should be noting that $|x\rangle$ or $|f(x)\rangle$ are not in the Hilbert space because their length is infinite. But we know that the vectors of the Hilbert space can be expanded in terms of $|x\rangle$'s.[3]

The operator T is a linear operator but not generally unitary one. We want to show that there would be a straight and general strategy to recognize that whether an operator is unitary or not: $(TT^{\dagger} = 1)$.

The reader may ask why we did not write Eq. (1) in terms of the wave function. This is possible and is represented by the following equation:

$$T\psi(x) = \left|(f^{-1}(x))'\right| \psi(f^{-1}(x)). \tag{2}$$

The proof of this equation is given in Appendix B. For every

†Corresponding author. E-mail: jafarimatehkolaee@semnan.ac.ir

© 2021 Chinese Physical Society and IOP Publishing Ltd

operator T to be unitary, it is necessary that

$$\langle f(x)|\, f(y)\rangle = \langle x|\, y\rangle\,. \tag{3}$$

In general, there would be a parameter ε in the operator T and the smooth function f, such that $T(\varepsilon)\,|x\rangle = |f(\varepsilon,x)\rangle$. Now we can say that for every f there is a function h such that $\langle f(x)|\, f(y)\rangle = h(x,\varepsilon)\,\langle x|\, y\rangle$. Note that, we can write $\langle f(x)|\, f(y)\rangle = \delta(f(x),f(y)) = h(x,y,\varepsilon)\delta(x,y)$, since $h(x,y,\varepsilon)\delta(x,y) = h(x,x,\varepsilon)\delta(x,y)$ so by denoting $h(x,x,\varepsilon)$ to $h(x,\varepsilon)$, we can conclude $\langle f(x)|\, f(y)\rangle = h(x,\varepsilon)\,\langle x|\, y\rangle$.

In this case, the operator $\tilde{T}$ is unitary under the condition that

$$\tilde{T}\,|x\rangle = \frac{1}{\sqrt{h(x,\varepsilon)}}\,|f(x)\rangle \tag{4}$$

or

$$\tilde{T} = T\frac{1}{\sqrt{h(Q,\varepsilon)}} \tag{5}$$

with Q being the position operator and given by $Q\,|x\rangle = x\,|x\rangle$. According to the necessity of the condition (3), it is straightforward and easy to obtain Eq. (5).

For the representation of Eq. (5) we highly need to the adjoint of $T(\varepsilon)$ and it seems it depends on the exact form of the $|f(\varepsilon,x)\rangle$ but we will show in the next section that there is a general approach for the adjoint of $T(\varepsilon)$.

The author in Ref. [13] has pointed out a scaling operator as $T(\varepsilon)f(x) = \varepsilon f(\varepsilon^2 x)$. Based on Eq. (5) it is easy to show that this operator is unitary. Now we can consider another example: $T(\varepsilon)\,|x\rangle = |\exp(\varepsilon)x\rangle$, the operator T is not unitary, since $\langle x|\, T^{\dagger}T\,|y\rangle = \langle f(x)|\, f(y)\rangle = \exp(-\varepsilon)\,\langle x|\, y\rangle$ but the operator $\tilde{T} = (\exp(\varepsilon/2)T)$ is unitary because $\langle x|\, \tilde{T}^{\dagger}\tilde{T}\,|y\rangle = \langle f(x)|\, f(y)\rangle = \langle x|\, y\rangle$. Also, it is easy to verify that the translation operator in quantum mechanics, $T(\varepsilon)\,|x\rangle = |x+\varepsilon\rangle$ is unitary.

3. Unitary operator in non-Euclidean space

Now we continue the discussion in generalized space. In fact, we are looking for the answer to original question, already mentioned in introduction. In this section, we try to illustrate the details of calculation of any equation.

At first, we consider manifold M with an atlas consisting of only one chart. This manifold is such that there exists a global one to one correspondence between the points of M and $\mathbb{R}$. The normalized eigenvectors of position satisfy the relation $\langle x\,|y\rangle = \delta\,(x,y)$. Note that, $\delta\,(x,y)$ is the "generalized delta function" for the manifold and is defined by the conditions $\delta\,(x,y) = 0$ for $x \neq y$ and $\int \zeta(x)\delta(x,y)h(x) = h(y)$. Here, ζ is the invariant volume element $\zeta(x) = g(x)d^n x$ and g is the square root of the absolute value of the determinant of the metric matrix. Note that in all parts of this paper, g denotes the metrics except in Eqs. (8) and (19) in which g denotes a function. Similar to Section 2, the operator T corresponding to the diffeomorphism f is defined as $T\,|x\rangle = |f(x)\rangle$. Note that, x is the member of manifold M and also f is a function so that $f : M \to M$.

From the definition of T, we conclude that $\langle x|\, T^{\dagger}\,|y\rangle = \delta(f(x),y)$, so by defining z through $z = f(x)$ we get

$$\begin{aligned}\zeta(z) &= g(z)d^n z\\ &= g(z)\left|\det(\frac{\partial z}{\partial x})\right| d^n x\\ &= \left(\frac{g(f(x))}{g(x)}\right)|\det[D(f)(x)]|\,\zeta(x).\end{aligned} \tag{6}$$

Now we can write

$$\zeta[f(x)] = \left(\frac{f^*g}{g}\right)(x)\zeta(x), \tag{7}$$

where f^* is the pullback of f and we have

$$(f^*g)_{\mu\nu}(x) = \left\{g_{\alpha\beta}\,[f(x)]\right\}\frac{\partial f^{\alpha}}{\partial x^{\mu}}\frac{\partial f^{\beta}}{\partial x^{\nu}}. \tag{8}$$

Using Eq. (7), we find the following expression:

$$\begin{aligned}\int \zeta(x)\delta(f(x),y)h(x) &= \int \zeta(z)(f^{-1}(z))\delta(z,y)h(f^{-1}(z))\\ &= \left(\frac{g}{f^*g}\right)(f^{-1}(y))h(f^{-1}(y)).\end{aligned} \tag{9}$$

Note that, since f is a diffeomorphism so it is invertible and both of f and its inverse are differentiable.

Equation (9) gives the following equation:

$$\delta(f(x),y) = \delta(x,f^{-1}(y))\left(\frac{g}{f^*g}\right)(f^{-1}(y)). \tag{10}$$

Therefore we obtain

$$\langle x|\, T^{\dagger}\,|y\rangle = \langle x|\, f^{-1}(y)\rangle\left(\frac{g}{f^*g}\right)(f^{-1}(y)). \tag{11}$$

Now we can extract the following equation from the above equation:

$$T^{\dagger}\,|y\rangle = \left|f^{-}(y)\right\rangle\left(\frac{g}{f^*g}\right)\left(f^{-1}(y)\right). \tag{12}$$

According to the first definition, $T\,|x\rangle = |f(x)\rangle$, equation (12) could be a universal formula for the adjoint of the operator T. For the straight calculation of adjoint of an operator in curved space, readers can refer to Appendix A.

By inserting T on both sides of the above equation we conclude that

$$TT^{\dagger}\,|y\rangle = |y\rangle\left(\frac{g}{f^*g}\right)(f^{-1}(y)). \tag{13}$$

This shows that $TT^{\dagger}$ is not identity and hence T is not unitary, unless f is volume-preserving, that is (f^*g) would be the same as g. In any case, equation (13) can be written as

$$TT^{\dagger} = \left(\frac{g}{f^*g}\right)(f^{-1}(Q)), \tag{14}$$

where Q is the position operator and we have defined

$$B(Q)\,|x\rangle = B(x)\,|x\rangle\,. \tag{15}$$

Finally, the unitary operator $\tilde{T}$ is represented by

$$\tilde{T} = \left\{ \left(\frac{g}{f^* g} \right) (f^{-1}(Q)) \right\}^{-1/2} T. \tag{16}$$

Or in other form, we can write

$$\tilde{T} = T \left\{ \left(\frac{g}{f^* g} \right) (Q) \right\}^{-1/2}. \tag{17}$$

We recall that equation (17) is the generalization of Eq. (5) in curved space. Now we consider a special and more general example. For an infinitesimal diffeomorphism f_ε, we assume

$$f_\varepsilon = x + \varepsilon V(x) + o(\varepsilon). \tag{18}$$

Note that, in this equation $o(\varepsilon)$ tends to zero faster than ε and, as we mentioned before, x and f are members of the manifold M. Their coordinates are, of course, scalars, same as the components of $V(x)$, while $V(x)$ is itself a vector. The definition (8) gives the following equation:

$$(f^* g)_{\mu\nu} = g_{\mu\nu} + \varepsilon \left[(\nabla_\mu V)_\nu + (\nabla_\nu V)_\mu \right] + o(\varepsilon), \tag{19}$$

where ∇ is the covariant derivative corresponding to the Levi–Civita connection in relation to the metric g. So by using Eq. (19) we can obtain immediately

$$f^* g = [1 + 2\varepsilon \nabla V + o(\varepsilon)]\, g. \tag{20}$$

Defining T_ε in relation to f_ε as a generalized translation operator, we have

$$T_\varepsilon |x\rangle = |x + \varepsilon V(x)\rangle + o(\varepsilon). \tag{21}$$

In Eq. (21), $|x\rangle$ is a well-localized state around x that can be changed to another well-localized state around $x + \varepsilon V(x)$ leaves all the physical properties unchanged. The parameter ε determines the mixing between the displacement and the original position state. Notice if we consider $V(x) = 1$ and $o(\varepsilon) \to 0$, then equation (21) shows exactly the translation operator in standard quantum mechanics. Also, the translation operator we have introduced here is much more general than the generalized translation operator introduced in Ref. [8]. According to Eqs. (18)–(20), after some calculations we get the unitary form of translation operator (21) in the form of

$$T_\varepsilon^\dagger |x\rangle = |x - \varepsilon V(x)\rangle \left[1 - \varepsilon (\nabla V)(x)\right] + o(\varepsilon). \tag{22}$$

Hence, we obtain

$$T_\varepsilon T_\varepsilon^\dagger = [1 - \varepsilon (\nabla V)(Q)] + o(\varepsilon). \tag{23}$$

Therefore, the unitary form of the translation operator is as follows:

$$\tilde{T}_\varepsilon = \left[1 + \frac{\varepsilon}{2} (\nabla V)(Q) \right] T + o(\varepsilon). \tag{24}$$

Also in the other form we can write

$$\tilde{T}_\varepsilon = T \left[1 + \frac{\varepsilon}{2} (\nabla V)(Q) \right] + o(\varepsilon). \tag{25}$$

Note that, by putting $\varepsilon = 0$ in Eqs. (21) and (22) or Eq. (23) we clearly conclude that T_0 and $\tilde{T}_0$ are identity.

Now we can define the generators K and $\tilde{K}$ corresponding to T_ε and $\tilde{T}_\varepsilon$ as follows:

$$K = \lim_{\varepsilon \to 0} \frac{T_\varepsilon - 1}{\varepsilon} \quad \text{and} \quad \tilde{K} = \lim_{\varepsilon \to 0} \frac{\tilde{T}_\varepsilon - 1}{\varepsilon}. \tag{26}$$

It is clear that

$$K |x\rangle = V^\alpha \partial_\alpha |x\rangle, \tag{27}$$

$$\tilde{K} |x\rangle = \left[V^\alpha \partial_\alpha + \frac{1}{2} (\nabla V) \right] |x\rangle, \tag{28}$$

where $\tilde{K}$ is an anti-Hermitian generator.

As an obvious example for Eq. (18), the translation operator has been introduced in Ref. [14] by using a diagonal metric

$$T(\mathrm{d}x) |x\rangle = \left| x + g_{xx}^{1/2} \mathrm{d}x \right\rangle, \tag{29}$$

in which the diagonal metric is equals to

$$\mathrm{d}s^2 = g_{xx} \mathrm{d}x^2 + g_{yy} \mathrm{d}y^2 + g_{zz} \mathrm{d}z^2. \tag{30}$$

According to Eq. (27) and by choosing $V = g_{xx}^{-1/2}$, the divergence of V vanishes and then

$$\tilde{K} = g_{ii}^{-1/2} \partial_i. \tag{31}$$

Equation (31) is completely consistent with the generator obtained in Ref. [14].

In Ref. [8], by considering a well-localized state around x that can be changed to another well-localized state around $x + a + \gamma a x$, a translation operator have been introduced as follows:

$$T_\gamma(a) |x\rangle = |x + a + \gamma a x\rangle, \tag{32}$$

where parameter γ is the inverse of a characteristic length.

We can simply show that this operator is not unitary and instead the operator $\tilde{T}_\gamma = (1/\sqrt{1 + \gamma a})\, T$ is unitary.

4. Conclusion

Equation (5) indicates the general form of the unitary operator in Euclidean space. Note that, any example can be used to test of Eq. (5) and not necessarily scaling operators. Equation (16) or (17) is a general form of formula to represent a unitary operator in a generalized space. It is interesting that, the particular example we are considering here clearly led to the translation operator in non-Euclidean space. Our results clearly show a universal formula for adjoint of an arbitrary linear operator, *i.e.*, Eq. (12).

The non-traditional approach of this article, invites researchers to elaborate a general procedure in the face of unitary operators and their related generators. In particular, to extend our work, the dimensions of space can be generalized from one dimension.

Appendix A

The inner product is defined as

$$\langle\phi|\,\psi\rangle = \int \zeta(x)\bar{\phi}(x)\psi(x). \tag{A1}$$

Similar to Section 3, here also ζ is the invariant volume element $\zeta(x) = g(x)d^n x$, and g is the square root of the absolute value of the determinant of the metric matrix.

The operator O is defined through

$$\langle x|\,O|\,\psi\rangle = \psi[h(x)]. \tag{A2}$$

So

$$\langle\phi|\,O|\,\psi\rangle = \int \zeta(x)\bar{\phi}(x)\psi[h(x)]. \tag{A3}$$

Similar to Section 3, defining z through $z = h(x)$ and using

$$\zeta\left[h^{-1}(z)\right] = \left[\frac{\left(h^{-1}\right)^* g}{g}\right](z)\zeta(z), \tag{A4}$$

where h^* is the pullback of h, one arrives at

$$\langle\phi|\,O|\,\psi\rangle = \int \zeta(z)\left[\frac{\left(h^{-1}\right)^* g}{g}\right](z)\bar{\phi}[h^{-1}(z)]\psi[z]. \tag{A5}$$

Equation (5) shows that

$$\langle x|\,O^\dagger\big|\,\phi\rangle = \left[\frac{\left(h^{-1}\right)^* g}{g}\right](x)\bar{\phi}[h^{-1}(x)]. \tag{A6}$$

This is correct if h is one-to-one. If that is not the case, then there are several inverses for h.

Denoting these by h_i^{-1}'s, hence we can write

$$\langle x|\,O^\dagger\big|\,\phi\rangle = \sum_i \left[\frac{\left(h_i^{-1}\right)^* g}{g}\right](x)\phi[h_i^{-1}(x)]. \tag{A7}$$

Appendix B

Consider an operator T such that

$$T\,|x\rangle = |f(x)\rangle\,. \tag{B1}$$

In terms of the wave function, this equation is equal to

$$T\psi(x) = \left|(f^{-1}(x))'\right|\psi(f^{-1}(x)), \tag{B2}$$

where $\psi(x)$ is the desired wave function.

Proof In position space, wave function is represented by the action of the bra position on the ket state, that is,

$$\psi(x) = \langle x|\psi\rangle\,. \tag{B3}$$

So we can write

$$T\psi(x) = \langle x|\,T\,|\psi\rangle\,. \tag{B4}$$

This means that in order to find the action of T on the wave function, we need the action of T on the bra position. Hence, we can write

$$\langle x|\,T\,|y\rangle = \langle x|\,f(y)\rangle = \delta(x, f(y)). \tag{B5}$$

According to the property of the delta function we have

$$\delta(x, f(y)) = \frac{1}{|f'(y)|}\delta(f^{-1}(x), y). \tag{B6}$$

Therefore by putting Eq. (B6) into Eq. (B5), we obtain

$$\langle x|\,T = \left|(f^{-1}(x))'\right|\left\langle f^{-1}(x)\right|, \tag{B7}$$

where we have used the relation $\left[1/f'(f^{-1}(x))\right] = (f^{-1}(x))'$ in the above equation. Finally, from Eq. (B7), we obtain Eq. (2) or Eq. (B2):

$$T\psi(x) = \left|(f^{-1}(x))'\right|\psi(f^{-1}(x)). \tag{B8}$$

Acknowledgement

The author is deeply grateful to Professor Mohammad Reza Sarkardei for helpful discussions and Professor Mohammad Khorrami for precious comments.

References

[1] Akhiezer N I and Glazman I M 1981 *Theory of Linear operators in Hilbert Spaces*, Vol one (London: Pitman)
[2] Weidmann J 1980 *Linear operators in Hilbert Spaces* (New York: Springer-Verlag)
[3] Sakurai J J 1985 *Modern Quantum Mechanics*, Ed. San Fu Tuan (Benjamin/Cummings Menlo Park, Ca)
[4] Miskinis P 2012 *Math. Model. Anal.* **17** 100
[5] Danos M and Maximon L C 1965 *J. Math. Phys.* **6** 766
[6] Fitter E and Steinborn E O 1980 *J. Math. Phys.* **21** 2725
[7] Chan C L 1970 *Am. J. Phys.* **38** 468
[8] Costa Filho R N, Almedia M P, Farias G A and Anderade J S Jr 2011 *Phys. Rev. A* **84** 050102
[9] Costa Filho R N, Alencar G, Skagerstam B S and Anderade J S Jr 2013 *EuroPhys. Lett.* **101** 10009
[10] Da Costa B G and Borges E P 2014 *J. Math. Phys.* **55** 062105
[11] Xu Y 2005 *J. Comput. Appl. Math.* **178** 489
[12] Gieres F 2000 *Rep. Prog. Phys.* **63** 1893
[13] Peres A 2002 *Quantum theory: concepts and methods, Fundamental Theories of Physics* (New York: Kluwer Academic Publishers)
[14] Costa Filho R N, Braga J P M, Lira J H S and Andrade J S Jr 2016 *Phys. Lett. B* **755** 367