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Thermodynamic properties of massless Dirac–Weyl fermions under the generalized uncertainty principle*

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The Dirac–Weyl equation characterized quasi-particles in the T3 lattice are studied under external magnetic field using the generalized uncertainty principle (GUP). The energy spectrum of the quasi-particles is found by the Nikiforov–Uvarov method. Based on the energy spectrum obtained, the thermodynamic properties are given, and the influence of the GUP on the statistical properties of systems is discussed. The results show that the energy and thermodynamic functions of massless Dirac–Weyl fermions in the T3 lattice depend on the variation of the GUP parameter.

Keywords: T3 lattice, massless Dirac–Weyl fermions, generalized uncertainty principle, thermodynamic property

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1. Introduction

It is well known that in well-known Heisenberg uncertainty principle, we can separately measure the position or momentum of a test particle with arbitrary accuracy in principle. However, various approaches to unifying quantum theory with gravity support the idea that near the Planck scale, the standard Heisenberg uncertainty principle should be replaced by the so-called generalized uncertainty principle that implies the existence of a minimal length.[1–9] Since Kempf and co-workers developed the theoretical framework of quantum mechanics based on a generalized uncertainty principle, the studies of Schrödinger equation, the Dirac equation and the DKP equation under the minimal length uncertainty have received much attention.[1,2,4,6,10] Some phenomena in black hole remnants,[11] the trans-Planckian problem of inflation,[12–14] and the cosmological constant problem[15–17] can be explained by generalized uncertainty relations. On the other hand, in recent years, many fields of physics have been influenced by the rapid development of topology, and a variety of lattices can be designed for utilizing.[18–23] In particular, research of the Kagome,[24–26] the Lieb and Perovskite lattices,[27] the honeycomb,[28] the square,[29,30] the checkerboard,[31] the pyrochlore,[32] the diamond,[33] the square-octogon[34] and the T3 lattice[23,35,36] have become one of the most active fields in condensed-matter physics. In such lattices, quasi-particles are characterized by the Dirac–Weyl equation. However, for Dirac-like particles with pseudo spin in these lattices, we have $s = 1$, rather than as $s = 1/2$ for graphene[23,36,37] and the algebraic properties of matrix in the Dirac–Weyl equation are different from Dirac matrix in the usual Dirac equation. The main purpose of this work is to obtain the exact energy levels of the Dirac-like particles under external magnetic field and the generalized uncertainty principle and to study the corresponding thermal properties including the partition function and mean internal energy, etc. This paper is organized as follows: In Section 2, we first use the Nikiforov–Uvarov (NU) method to solve the energy spectrum and the wave function of T3 lattice under the generalized uncertainty principle (GUP). Based on the energy spectrum, the partition function of T3 lattice is obtained and other thermodynamic functions of the system are further obtained. The variation curves of energy spectrum and thermodynamic functions of T3 lattice under the GUP are drawn and analyzed briefly. Section 3 is our summary.

2. Thermodynamic properties of the T3 lattice under the GUP

Considering the effect of the gravitation, in more than one dimension, the minimal measurable distance emerges inevitably, which can be encoded in the following form of the GUP:[1–8]

$$\Delta X_i \Delta P_j \geq \frac{\hbar}{2}\delta_{ij}[1+\beta((\Delta P)^2+\langle P\rangle^2)], \tag{1}$$

where β is the GUP parameter. The above inequality follows from the modified Heisenberg algebra

$$[X_i, P_j] = \mathrm{i}\hbar\delta_{ij}(1+\beta P^2). \tag{2}$$

In the absence of the second term on the right-hand side, the commutation relation (2) reduces to the usual Heisenberg' algebra. However, the above algebra does not admit a simple representation in position space. In momentum space repre-

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sentation, if we define

$$P_i = p_i, \quad X_i = \mathrm{i}\hbar(1+\beta p^2)\frac{\partial}{\partial p_i}, \tag{3}$$

where $p^2 = \sum_{i=0}^{3} p_i p_i$, and $[x_i, p_j] = \mathrm{i}\hbar\delta_{ij}$, $[p_i, p_j] = [x_i, x_j] = 0$, it is easy to show that Eq. (2) can be satisfied up to linear-order β if we substitute Eq. (3) into Eq. (2). Thus, under the generalized uncertainty principle, for arbitrary quantum systems the coordinates X_i and momentum P_j in the Hamiltonian must be replaced by $X_i = \mathrm{i}\hbar(1+\beta p^2)\frac{\partial}{\partial p_i}$ and $P_i = p_i$, respectively. Now let us consider the T3 lattice under the GUP with the minimal length. The T3 lattice, illustrated in Fig. 1, has a unit cell with three different lattice sites, one six-fold coordinated site H, called *hub*, and two three-fold coordinated sites A and B, called *rims*. All nearest-neighbor pairs are formed by a rim and a hub.[23,36] The corresponding low-energy Hamiltonian can be obtained by a long-wavelength approximation to the tight-binding Hamiltonian.

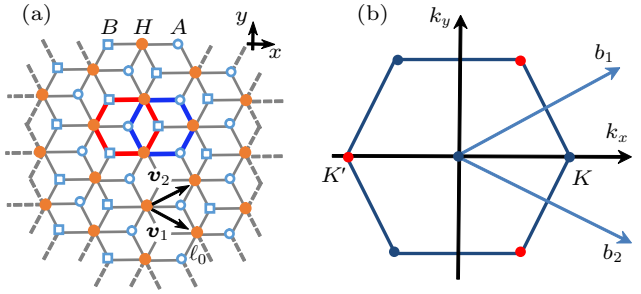


Fig. 1. (a) The T3 lattice, characterized by translation vectors, $\boldsymbol{v}_1 = (3/2, -\sqrt{3}/2)\ell_0$ and $\boldsymbol{v}_2 = (3/2, \sqrt{3}/2)\ell_0$, with lattice constant ℓ_0. Rim sites A and B have a lower connectivity of 3 compared to 6 of the hub site. (b) In the reciprocal space of T3 lattice, inverse lattice base vector $\boldsymbol{b}_1 = \frac{2\pi}{3\ell_0}(\sqrt{3}\boldsymbol{e}_x + \boldsymbol{e}_y)$, $\boldsymbol{b}_2 = \frac{2\pi}{3\ell_0}(\sqrt{3}\boldsymbol{e}_x - \boldsymbol{e}_y)$, where ℓ_0 is the distance between two adjacent lattice points. Moreover, the two non-equivalent Dirac points K and K' are located at $\boldsymbol{K} = \frac{4\pi}{3\sqrt{3}\ell_0}\boldsymbol{e}_x$ and $\boldsymbol{K}' = -\frac{4\pi}{3\sqrt{3}\ell_0}\boldsymbol{e}_x$.

As in graphene, in the T3 lattice, there are two inequivalent Dirac points, labeled K and K', for a given Dirac point the Hamiltonian reads[23,36,37]

$$H = \upsilon_{\mathrm{F}}\boldsymbol{S}\cdot\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right), \tag{4}$$

(in units where $c = \hbar = 1$), the v_{F} is the Fermi velocity, the $\boldsymbol{p} = -\mathrm{i}(\partial_x, \partial_y)$ is the momentum operator in the lattice xy-plane, while $\boldsymbol{A}(x,y)$ is the vector potential related to the magnetic field. $\boldsymbol{S}$ in Eq. (4) are 3×3 matrices and has the following form:

$$s_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$s_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix},$$

$$s_z = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{5}$$

These matrices satisfy angular momentum commutation relations. The corresponding eigenvalue equation of Hamiltonian in Eq. (4) reads

$$H\psi = E\psi, \tag{6}$$

where $\psi = (\psi_A(x,y), \psi_H(x,y), \psi_{\mathrm{B}}(x,y))^{\mathrm{T}}$ is three-component eigenvectors of Eq. (6) and T denotes the transpose. Letting $E' = \sqrt{2}E/\upsilon_{\mathrm{F}}$ then Eq. (6) becomes

$$\begin{aligned}
&\left[\left(p_x - \frac{e}{c}A_x\right) - \mathrm{i}\left(p_y - \frac{e}{c}A_y\right)\right]\Psi_H = E'\psi_A, \\
&\left[\left(p_x - \frac{e}{c}A_x\right) + \mathrm{i}\left(p_y - \frac{e}{c}A_y\right)\right]\Psi_H = E'\psi_B, \\
&\left[\left(p_x - \frac{e}{c}A_x\right) + \mathrm{i}\left(p_y - \frac{e}{c}A_y\right)\right]\Psi_A \\
&+\left[\left(p_x - \frac{e}{c}A_x\right) - \mathrm{i}\left(p_y - \frac{e}{c}A_y\right)\right]\Psi_B = E'\psi_H.
\end{aligned} \tag{7}$$

Using of Eq. (7) and considering magnetic field $\boldsymbol{B} = B_0\boldsymbol{e}_z$ whose magnetic vector potential is chosen as the symmetric gauge $\boldsymbol{A} = (-B_0Y/2,\ B_0X/2,\ 0)$, we can obtain the following equation satisfied by ψ_H:

$$\begin{aligned}
&\left[p_x^2 + p_y^2 + \frac{e^2B_0^2}{4c^2}X^2 + \frac{e^2B_0^2}{4c^2}Y^2\right. \\
&\left.-\frac{e^2B_0^2}{2c^2}(XP_y - YP_x)\right]\psi_H = \frac{E'^2}{2}\psi_H.
\end{aligned} \tag{8}$$

It is well known that in polar coordinates of momentum space we have

$$\begin{aligned}
&p_x = p\cos\theta, \quad p_y = p\sin\theta, \quad \frac{\partial}{\partial p_x} = \cos\theta\frac{\partial}{\partial p} - \frac{\sin\theta}{p}\frac{\partial}{\partial\theta}, \\
&\frac{\partial}{\partial p_y} = \sin\theta\frac{\partial}{\partial p} + \frac{\cos\theta}{p}\frac{\partial}{\partial\theta}.
\end{aligned} \tag{9}$$

Using the relationship (9) and setting $\psi_H = \varphi_H(p)\,\mathrm{e}^{\mathrm{i}m\theta}$, where m represents the magnetic quantum number, with the help of $X_i = \mathrm{i}\hbar(1+\beta p^2)\frac{\partial}{\partial p_i}$ $(X_1 = X,\ X_2 = Y)$, we can obtain

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}p^2} + \left(\frac{2\beta p}{1+\beta p^2} + \frac{1}{p}\right)\frac{\mathrm{d}}{\mathrm{d}p} + \frac{-(m^2\beta^4 + 4B_0^2 - 2m\beta^2)p^4 + 2(m + B_0^2E'^2 - m^2\beta^2)p^2 - m^2}{p^2(1+\beta p^2)^2}\right]\varphi_H = 0. \tag{10}$$

Next our task is to solve Eq. (10). Now, setting $s = p^2$, Eq. (10) becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \frac{\alpha_1 - \alpha_2 s}{s(1-\alpha_3 s)}\frac{\mathrm{d}}{\mathrm{d}s} + \frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{s^2(1-\alpha_3 s)^2}\right]\varphi_H = 0, \tag{11}$$

where

$$\xi_1 = \frac{m^2\beta^4}{4} - \frac{m\beta^2}{2} + B_0^2,$$
$$\xi_2 = \frac{1}{2}(m + B_0^2 E'^2 - m^2\beta^2), \quad \xi_3 = \frac{m^2}{4},$$
$$\alpha_1 = 1, \quad \alpha_2 = -\frac{3\beta}{2}, \quad \alpha_3 = -\beta, \quad \alpha_4 = 0,$$
$$\alpha_5 = \frac{\beta}{4}, \quad \alpha_6 = \frac{\beta^2}{16} + \xi_1, \quad \alpha_7 = -\xi_2, \quad \alpha_8 = \xi_3,$$
$$\alpha_9 = \beta\xi_2 + \beta^2\xi_3 + \xi_1 + \frac{\beta^2}{16}, \quad \alpha_{10} = 1 + 2\sqrt{\xi_3},$$
$$\alpha_{11} = 2(\sqrt{\beta\xi_2 + \beta^2\xi_3 + \xi_1 + \frac{\beta^2}{16}} - \beta\sqrt{\xi_3} - \beta),$$
$$\alpha_{12} = \sqrt{\xi_3},$$
$$\alpha_{13} = \frac{\beta}{4} - (\sqrt{\beta\xi_2 + \beta^2\xi_3 + \xi_1 + \frac{\beta^2}{16}} - \beta\sqrt{\xi_3}). \quad (12)$$

It can be easily found that Eq. (11) is of the form basic equation of the NU method.[38,39] Thus, we can straight-forwardly obtain the corresponding energy and wave function using the NU method (also see Appendix). Then the corresponding mathematical expression satisfied by energy spectra can be expressed as

$$-\frac{3\beta}{2}n + \left[\sqrt{\frac{\beta}{2}(m + B_0^2E'^2 - \beta^2m^2) + \frac{\beta^4m^2}{4} + \frac{\beta^2m^2}{4} - \frac{\beta^2m}{2} + B_0^2 + \frac{\beta^2}{16}} - \frac{\beta m}{2}\right](2n+1) - \frac{(2n+1)\beta}{4} - n(n-1)\beta$$
$$-\frac{1}{2}(m + B_0^2E'^2 - \beta^2m^2) - \frac{\beta m^2}{2} + m\sqrt{\frac{\beta}{2}(m + B_0^2E'^2 - \beta^2m^2) + \frac{\beta^4m^2}{4} + \frac{\beta^2m^2}{4} - \frac{\beta^2m}{2} + B_0^2 + \frac{\beta^2}{16}} = 0, \quad (13)$$

since β is a very small parameter, we only consider the first order term of β in Eq. (13). Hence, we find that the expression of energy reads

$$E = v_F\sqrt{\frac{4B_0(2n+1+m) - 2m + \beta(4n^2 + 4mn + 2m + 4n + 1)}{4B_0^2}}. \quad (14)$$

From Eq. (14), it is not difficult to find that in the case of $\beta = 0$, the Dirac particles in the T3 lattice still show a limited energy value even if $n = m = 0$, which is consistent with the conclusion in the T3 optical lattice under the condition of external magnetic field.[23] Now let us focus on the characteristic wave functions of the T3 lattice under the GUP. According to the NU method,[38–43] we can obtain the wave functions of any state of the T3 lattice:

$$\psi_n(p) = N_n p^{2\sqrt{\xi_3}}(1+\beta p^2)^{\delta_1} P_n^{(2\sqrt{\xi_3}, 2\delta_1 - \frac{1}{2})}(1+2\beta p^2), \quad (15)$$

where

$$\delta_1 = \frac{1}{4} - \frac{\sqrt{\beta\xi_2 + \beta^2\xi_3 + \xi_1 + \frac{\beta^2}{16}}}{\beta},$$

N_n is the normalization constant and $P_n^{(\alpha,\beta)}$ stands for Jacobi polynomials. Next, we use the relationship between the Jacobian polynomials and the hypergeometric functions ${}_2F_1$ [38–40,42]

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(1+\alpha)} \times {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-z}{2}\right). \quad (16)$$

Further, the characteristic wave function $\psi_n(p)$ of any state of the T3 lattice in Eq. (15) is expressed as[38–44]

$$\psi_n(p) = N_n\frac{\Gamma(n+2\sqrt{\xi_3}+1)}{\Gamma(2\sqrt{\xi_3}+1)n!}p^{2\sqrt{\xi_3}}(1+\beta p^2)^{\delta_1}{}_2F_1\left(-n, n+2\sqrt{\xi_3}+2\delta_1+\frac{1}{2}; 2\sqrt{\xi_3}+1; -\beta p^2\right), \quad (17)$$

where Γ is the Gamma function. Using Eq. (15) in Ref. [38], we obtain the analytic expression of normalized constant N_n as follows:

$$N_n = \sqrt{\frac{(4\sqrt{\xi_3}-1)n!(n+\sqrt{\xi_3}+\delta_1+\frac{1}{4})\Gamma(n+2\sqrt{\xi_3}+2\delta_1+\frac{1}{2})}{(n+\delta_1)\Gamma(n+2\sqrt{\xi_3}+\frac{3}{2})\Gamma(n+2\delta_1)}}. \quad (18)$$

In particular, when $n = 0$, the corresponding ground state wave function of the T3 lattice is

$$\psi_0(p) = N_0 p^{2\sqrt{\xi_3}}(1+\beta p^2)^{\delta_1}{}_2F_1(0, 2\sqrt{\xi_3}+2\delta_1+\frac{1}{2}; 2\sqrt{\xi_3}+1; -\beta P^2), \quad (19)$$

there

$$N_0=\sqrt{\frac{(4\sqrt{\xi_3}-1)(\sqrt{\xi_3}+\delta_1+\frac{1}{4})\Gamma(2\sqrt{\xi_3}+2\delta_1+\frac{1}{2})}{\delta_1\Gamma(2\sqrt{\xi_3}+\frac{3}{2})\Gamma(2\delta_1)}}. \quad (20)$$

Now we focus on the thermodynamic properties of the massless Dirac–Weyl Fermions in the T3 lattice. First of all, we know that the partition function Z expressed in Eq. (21) is the basis for obtaining all thermodynamic quantities,[45,46]

$$Z=\sum_n \mathrm{e}^{-QE_n}, \quad (21)$$

where $Q=\frac{1}{k_{\mathrm{B}}T}$, k_{B} is the Boltzmann constant. For the T3 lattice, the corresponding partition function reads

$$\begin{aligned} Z &= \sum_n \mathrm{e}^{-QE_n}, \\ E_n &= \frac{v_{\mathrm{F}}}{2B_0}\sqrt{\lambda n^2+\mu n+1} \\ &\quad \times\sqrt{4B_0(1+m)-2m+\beta(2m+1)}, \end{aligned} \quad (22)$$

where

$$\begin{aligned} \lambda &= \frac{4\beta}{4B_0(1+m)-2m+\beta(2m+1)}, \\ \mu &= \frac{4(\beta m+\beta+2B_0)}{4B_0(1+m)-2m+\beta(2m+1)}. \end{aligned}$$

Further, Eq. (21) can be rewritten as

$$\begin{aligned} Z &= \sum_n \mathrm{e}^{-\frac{1}{\Upsilon}\sqrt{\lambda n^2+\mu n+1}}, \\ \Upsilon &= \frac{2B_0\tau}{v_{\mathrm{F}}\sqrt{4B_0(1+m)-2m+\beta(2m+1)}},\ \tau=k_{\mathrm{B}}T. \end{aligned} \quad (23)$$

Now setting $\chi=\frac{1}{\Upsilon}\sqrt{\lambda n^2+\mu n+1}$ and using the following relation:[46–49]

$$\mathrm{e}^{-\chi}=\frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}s\chi^{-s}G(s), \quad (24)$$

the partition function (23) can be expressed as

$$\begin{aligned} Z &= \sum_n \mathrm{e}^{-\frac{1}{\Upsilon}\sqrt{\lambda n^2+\mu n+1}} \\ &= \frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}s(\frac{1}{G})^{-s}\sum_n(\lambda n^2+\mu n+1)^{-\frac{s}{2}}G(s) \\ &= \frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}s(\frac{1}{G})^{-s}\Omega(s)G(s), \end{aligned} \quad (25)$$

where $G(s)$ and $\Omega(s)$ are the Euler and Epstein zeta functions, respectively,[47,48]

$$\Omega(s)=\sum_n \frac{1}{F(1,n)^{\frac{s}{2}}}, \quad (26)$$

with

$$F(1,n)=\lambda n^2+\mu n+1. \quad (27)$$

Setting $x=\frac{\mu}{2}$ and $y=\frac{\sqrt{4\lambda-\mu^2}}{2}$, then Eq. (26) can be rewritten as

$$\begin{aligned} \Omega(s) &= 2\lambda^{-\frac{s}{2}}\zeta(s)+\frac{2\lambda^{-\frac{s}{2}}y^{1-s}\sqrt{\pi}}{G(\frac{s}{2})}\zeta(s-1)G\left(\frac{s}{2}-\frac{1}{2}\right) \\ &\quad +\frac{2\lambda^{-\frac{s}{2}}y^{(\frac{1}{2}-\frac{s}{2})}\pi^{\frac{s}{2}}}{G(\frac{s}{2})}H\left(\frac{s}{2}\right). \end{aligned} \quad (28)$$

Then, the partition function (25) becomes

$$\begin{aligned} Z &= \frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}s\left(\frac{1}{\Upsilon}\right)^{-s}2\lambda^{-\frac{s}{2}}\zeta(s)G(s) \\ &\quad +\frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}s\left(\frac{1}{\Upsilon}\right)^{-s}\frac{2\lambda^{-\frac{s}{2}}y^{1-s}\sqrt{\pi}}{G\left(\frac{s}{2}\right)}\zeta(s-1) \\ &\quad \times G\left(\frac{s}{2}-\frac{1}{2}\right)G(s)+\frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}s\left(\frac{1}{\Upsilon}\right)^{-s} \\ &\quad \times\frac{2\lambda^{-\frac{s}{2}}y^{(\frac{1}{2}-\frac{s}{2})}\pi^{\frac{s}{2}}}{G(\frac{s}{2})}H\left(\frac{s}{2}\right)G(s). \end{aligned} \quad (29)$$

Now let us apply the remainder theorem, we obtain

$$Z=2\zeta(0)+\frac{2}{\sqrt{\lambda}}(\zeta(1)+\zeta(0))\Upsilon+\frac{2\pi\Upsilon^2}{\lambda y}. \quad (30)$$

It is worth noting that the last integral in Eq. (29) approaches zero because of the following relation:

$$\frac{1}{G(s)}=s\mathrm{e}^{rs}\prod_{n=0}^{\infty}\left\{\left(1+\frac{x}{n}\right)\mathrm{e}^{-\frac{x}{n}}\right\}, \quad (31)$$

where r is Euler's constant written as

$$r=\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k}-\log(n)\right). \quad (32)$$

Finally, the partition function of the massless Dirac–Weyl Fermions in the T3 lattice under the GUP is obtained as follows:

$$Z(\Upsilon,\lambda)=\frac{2\pi}{\lambda\sqrt{\lambda-1}}\Upsilon^2+\frac{1}{\sqrt{\lambda}}\Upsilon-1. \quad (33)$$

Based on the above partition function (33), we have other thermodynamic functions of the massless Dirac–Weyl Fermions:

$$F = -\Upsilon\ln(Z)=-\Upsilon\ln\left(\frac{2\pi}{\lambda\sqrt{\lambda-1}}\Upsilon^2+\frac{1}{\sqrt{\lambda}}\Upsilon-1\right), \quad (34)$$

$$\begin{aligned} U &= \Upsilon^2\frac{\partial\ln(Z)}{\partial r} \\ &= \frac{4\pi\Upsilon^3\sqrt{\lambda}+\lambda\Upsilon^2\sqrt{\lambda-1}}{2\pi\Upsilon^2\sqrt{\lambda}+\lambda\Upsilon\sqrt{\lambda-1}-\lambda\sqrt{\lambda^2-\lambda}}, \end{aligned} \quad (35)$$

$$S = \ln(Z) + \Upsilon\frac{\partial \ln(Z)}{\partial \Upsilon} = \ln\left(\frac{2\pi}{\lambda\sqrt{\lambda-1}}\Upsilon^2 + \frac{1}{\sqrt{\lambda}}\Upsilon - 1\right) + \frac{4\pi\Upsilon^2\sqrt{\lambda}+\lambda\Upsilon\sqrt{\lambda-1}}{2\pi\Upsilon^2\sqrt{\lambda}+\lambda\Upsilon\sqrt{\lambda-1}-\lambda\sqrt{\lambda^2-\lambda}}, \tag{36}$$

$$C = 2\Upsilon\frac{\partial \ln(Z)}{\partial \Upsilon} + \Upsilon^2\frac{\partial^2 \ln(Z)}{\partial \Upsilon^2} = \Upsilon\frac{8\pi\Upsilon\sqrt{\lambda}+2\lambda\sqrt{\lambda-1}}{\Upsilon^2 2\pi\sqrt{\lambda}+\lambda\Upsilon\sqrt{\lambda-1}-\lambda\sqrt{\lambda^2-\lambda}}$$
$$+\Upsilon^2\frac{4\pi(2\pi\Upsilon^2\lambda+\lambda\Upsilon\sqrt{\lambda^2-\lambda}-\lambda\sqrt{\lambda^3-\lambda^2})-16\pi^2\Upsilon^2\lambda-8\pi\Upsilon\lambda\sqrt{\lambda^2-\lambda}-\lambda^2(\lambda-1)}{(2\pi\Upsilon^2\sqrt{\lambda}+\lambda\Upsilon\sqrt{\lambda-1}-\lambda\sqrt{\lambda^2-\lambda})^2}. \tag{37}$$

Before drawing the curves of energy spectrum and thermodynamic functions of the T3 lattice under the GUP, we first make a brief numerical analysis of the analytical expression of the thermodynamic functions. For the asymptotic behavior, we consider two special cases, namely high temperature and low temperature:[50–52]

(1) For a very high temperature range, the modified canonical partition function in Eq. (33) is approximately

$$Z(\tau) \simeq \frac{2\pi\kappa^2}{\lambda\sqrt{\lambda-1}}\tau^2, \tag{38}$$

where $\kappa = \frac{2B_0}{v_F\sqrt{4B_0(1+m)-2m+\beta(2m+1)}}$. Now we can easily obtain the average energy U and the specific heat C of the T3 lattice for a very high temperature,

$$U = \tau^2\frac{\partial \ln(Z)}{\partial \tau} \simeq 2k_B T, \tag{39}$$

$$C \simeq \frac{\partial U}{\partial T} \simeq 2k_B. \tag{40}$$

(2) For a very low temperature range, we have

$$Z \simeq -1, \tag{41}$$

and consequently both average energy U and specific heat C tend to zero:

$$U \to 0, \quad C \to 0. \tag{42}$$

Next, we draw a curve in Fig. 2, which is the variation of the energy spectrum E of the T3 lattice under the GUP with the principal quantum number n at different values of β. The variation curves of thermodynamic functions with variable τ ($k_B T$) under different values of β and m are plotted from Figs. 3–7.

Figure 2 presents the reduced spectrum of energy E and quantum number n of the T3 lattice under different values of β. The results show that the energy of T3 lattice increases with the increasing n. In addition, the GUP parameter has a significant influence on the energy spectrum, which is mainly shown as the energy spectrum increases with the increase of β. When the GUP is absent ($\beta = 0$), even if $n = m = 0$, the particles still display a limited energy value, which is similar to the energy spectrum of T3 optical lattice under the condition of external magnetic field in Ref. [23].

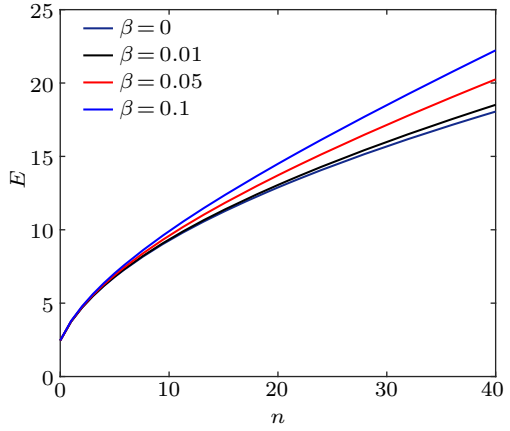


Fig. 2. The energy E of the T3 lattice in the context of the GUP as a function of the principal quantum number n for different values of β ($B_0 = v_F = m = c = \hbar = 1$).

It can be seen from Figs. 3 and 4 that the partition function Z and free energy F are negatively correlated with τ, β and m. Furthermore, at extremely low temperature, the partition function satisfies the asymptotic behavior $Z \simeq -1$.

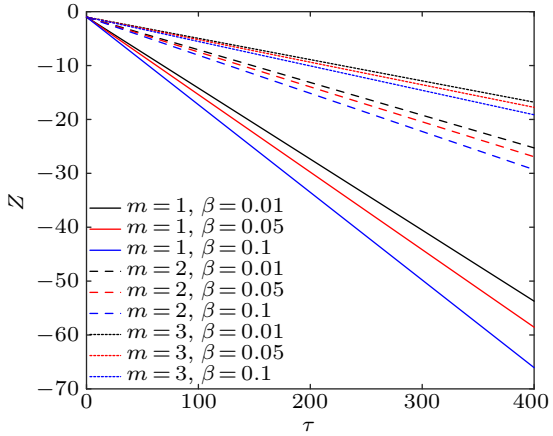


Fig. 3. The partition function Z of the T3 lattice in the context of the GUP as a function of τ ($k_B T$) for different values of β and m ($B_0 = 0.1$, $v_F = c = \hbar = 1$).

According to Fig. 5, we can know that there is a positive correlation between the average energy U and τ. When τ remains constant, the larger the m becomes, the smaller the average energy is. On the other hand, when β is different, the average energy variation curve tends to be almost the same straight line, which shows that the influence of β on the average energy is very small. In addition, the average energy tends to $U = 2k_B T$ in a very high temperature range and to zero $U \to 0$ in a very low temperature range.

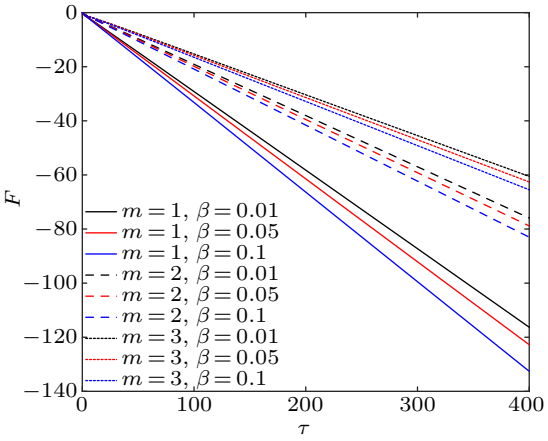


Fig. 4. The free energy F of the T3 lattice in the context of the GUP as a function of τ ($k_{\rm B}T$) for different values of β and m ($B_0 = 0.1$, $v_{\rm F} = c = \hbar = 1$).

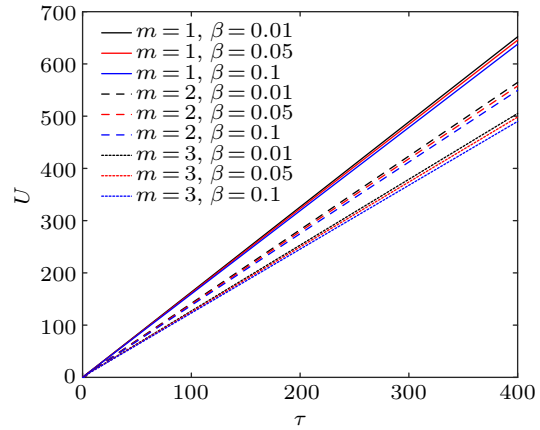


Fig. 5. The average energy U of the T3 lattice in the context of the GUP as a function of τ ($k_{\rm B}T$) for different values of β and m ($B_0 = v_{\rm F} = c = \hbar = 1$).

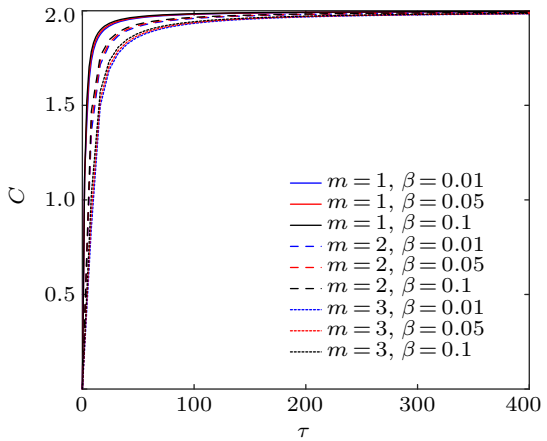


Fig. 6. The specific heat C of the T3 lattice in the context of the GUP as a function of τ ($k_{\rm B}T$) for different values of β and m ($B_0 = v_{\rm F} = c = \hbar = 1$).

Figures 6 and 7 show the relationship between specific heat C and entropy S with τ, respectively. It is not difficult to see that they all increase with the increase of τ. We find that the specific heat and entropy vary unobviously when β changes. It is worth mentioning that in Fig. 6, when the temperature is extremely high, all the specific heat curves coincide with each other and reach a fixed value $C = 2k_{\rm B}$, which is completely consistent with the asymptotic expression of specific heat in the extremely high temperature range. However, the heat capacity tends to zero $C \to 0$ in the extremely low temperature range, which satisfies the asymptotic behavior of specific heat at low temperature.

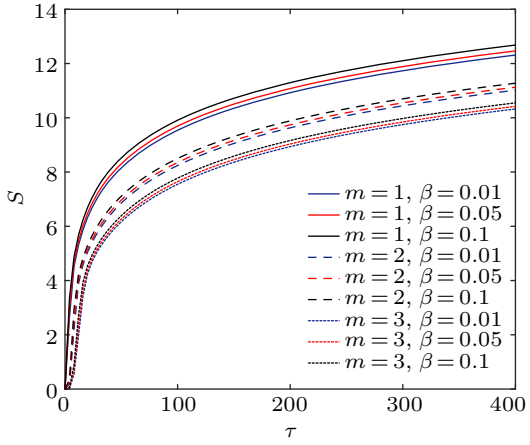


Fig. 7. The entropy S of the T3 lattice in the context of the GUP as a function of τ ($k_{\rm B}T$) for different values of β and m ($B_0 = 0.1$, $v_{\rm F} = c = \hbar = 1$).

3. Conclusion

In the framework of the modified version of quantum mechanics due to the generalized uncertainty principle (GUP), the Dirac–Weyl equation characterized quasi-particles in the T3 lattice are studied under external magnetic field using the GUP. The corresponding eigenvalue equation is analytically solved by the Nikiforov–Uvarov (NU) method, then the corresponding energy spectrum is extracted. By computing the modified canonical partition function, the thermodynamic properties of the system have been investigated. The influence of GUP parameter on thermodynamic properties of the T3 lattice is analyzed theoretically and numerically.

Appendix A

We briefly review the NU method which has been widely used to solve various differential equations.[38–44,53–56] Let us consider the following differential equation

$$\left\{\frac{{\rm d}^2}{{\rm d}s^2}+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{\rm d}{{\rm d}s}+\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{s^2(1-\alpha_3 s)^2}\right\}\psi=0. \quad \text{(A1)}$$

The corresponding eigenfunctions are

$$\psi(s)=s^{\alpha_{12}}(1-a_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}P_n^{(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-2\alpha_3 s), \quad \text{(A2)}$$

and the energy spectrum of a system can be obtained from

$$\alpha_2 n-(2n+1)\alpha_5+(2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}) +n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0, \quad \text{(A3)}$$

where

$$\alpha_4=\frac{1}{2}(1-\alpha_1),\quad \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),$$

$$\alpha_6=\alpha_5^2+\xi_1,\quad \alpha_7=2\alpha_4\alpha_5-\xi_2,$$
$$\alpha_8=\alpha_4^2+\xi_3,\quad \alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6,$$
$$\alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8},$$
$$\alpha_{11}=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}),$$
$$\alpha_{12}=\alpha_4+\sqrt{\alpha_8},\quad \alpha_{13}=\alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}),$$
$$P_n^{(\alpha,\beta)}(x)=\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)}\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)}\left(\frac{x-1}{2}\right)^m. \tag{A4}$$

$P_n^{(\alpha,\beta)}$ are Jacobi polynomials.

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