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Xiaodong Liu(柳晓东) and Lipo Mo(莫立坡)

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Consensus problems on networks with free protocol*

Xiaodong Liu(柳晓东)[1] and Lipo Mo(莫立坡)[1,2,†]

[1] School of Mathematics and Statistics, Beijing Technology and Business University, Beijing 100048, China
[2] Key Laboratory of Power Big Data of Guizhou Province, Guizhou Institute of Technology, Guiyang 550003, China

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In the real world, rule makers can only restrict, not completely control the behavior of the governed, while the governed can only choose their behavior patterns under these restrictions. In this paper, we design a new control protocol called free protocol to describe this situation. First, we calculate consensus probabilities based on the information of the interaction networks. Then, sufficient conditions are obtained for all agents converging to a same value with probability one. Finally, numerical simulation results are given to verify the above results.

Keywords: multi-agent systems, random protocol, consensus with probability one

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1. Introduction

In recent decades, more and more studies for multi-agent systems have been carried out and many excellent results have been reported.[1–24] In Refs. [1–14], agents were assumed to keep cooperative relationships with each other. However, in Refs. [15–24], the antagonistic relation was considered. By studying the differences between the above literature, we consider the following four questions: Firstly, why should we give agents the freedom? Secondly, how do we describe the agents' requirement that they want to have a standby choice to deal with the change of the outside environment of the system? Thirdly, what control protocols will they be able to choose if we give agents the right to choose? Fourthly, what is the final outcome of the system if we give agents the rights to choose their relationship with neighbors?

As is known to all, humans cannot be controlled like a machine. In the social network, humans have always been controlled as agents. In order to better study the effects of human behavior on social networks, we should give agents the freedom as humans. That's the answer to the first question.

For the second question, we designed a new control protocol called the free protocol to describe the agent's requirement of having alternate protocols to be chosen. According to Refs. [16,17], the weight between agents can be explained in symbol and size. The symbol represents whether agents are cooperative or antagonistic between each other. The size represents the strength of information that the agent responds to its neighbors. The larger the size is, the more eager the agent is to keep the same or opposite state with its neighbor agent. Hence, we propose two simple control protocols for agents to choose.

For the third question, we consider how an agent makes a choice. When we make a decision, we always simulate what we are going to do at first. Listing and evaluating all the options, then we choose one of them to execute. In this paper, based on this behavior pattern, we assume all agents randomly select the two control protocols and count the probability distribution that the agent follows.

For the last question, we analyze all the possible patterns when agents are free to choose. If all agents select once on the whole time, we call it static selection. Instead, if there is an agent changing its choice frequently, we call it dynamic selection. When the information interaction between agents is single-orientation, we call it single-orientation communication. Otherwise, we call it double-orientation communication. When an agent chooses a policy independently, we call it unilateral pattern. In addition, when the protocol is chosen jointly by two communicating agents, we call it bilateral pattern.

The contributions of this paper are as follows: (i) Compared with previous papers such as Refs. [1–24], we consider the willingness of agents in a multi-agent system, which is more reasonable and realistic. (ii) Different from the noise in the previous works, such as Ref. [24], the method used in this paper combines probabilistic science with control science in a new perspective. (iii) The concept of the structurally balanced is used to calculate the probability of bipartite agreement in some cases firstly, then by the knowledge of probability, the sufficient conditions for achieving agreement with probability one are obtained.

2. Preliminaries

Suppose that all agents choose their relationship with their neighbors based one their own willing and the relationship between them is cooperative at first. Moreover, we use a

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†Corresponding author. E-mail: molipo@th.btbu.edu.cn

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random processes to represent the random selection of agents to the corresponding relationships.

The signed graph $\mathcal{G}$ is expressed as $\mathcal{G} = \{\mathcal{V}, \varepsilon, \Phi(A)\}$, where $\mathcal{V} = \{\upsilon_1, \ldots, \upsilon_n\}$ is a set of nodes, $\varepsilon \subseteq \mathcal{V} \times \mathcal{V}$ is a set of edges, $A = [a_{ij}] \in \boldsymbol{R}^{n\times n}$, $\Phi(A) = [Y_{ij}(t)a_{ij}]$, a_{ij} is the fixed size of weight between agents. We assume that $a_{ij} > 0$ if $(\upsilon_i, \upsilon_j) \in \varepsilon$, otherwise $a_{ij} = 0$. If $(\upsilon_i, \upsilon_j) \in \varepsilon$, then agent i can obtain the information of agent j. The neighbor set of agent i are defined as $N_i = \{j|(\upsilon_i, \upsilon_j) \in \varepsilon\}$. The degree matrix of $\mathcal{G}$ is defined as $D = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}$, where $d_i = \sum_{j\in N_i} a_{ij}$, $i = 1, 2, \ldots, n$. The Laplace matrix of $\mathcal{G}$ is defined as $L = D - \Phi(A)$.

Definition 1 A graph $\mathcal{G}(\Phi(A))$ is said to be structurally balanced if it admits a bipartition of the nodes $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, such that $\phi(a_{ij}) \in \boldsymbol{R}^+ \ \forall \upsilon_i$, $\upsilon_j \in \mathcal{V}_q$ $(q \in \{1, 2\})$ and $\phi(a_{ij}) \in \boldsymbol{R}^- \ \forall \upsilon_i \in \mathcal{V}_q$, $\upsilon_j \in \mathcal{V}_r$ $q \neq r(q, r \in \{1, 2\})$. Otherwise, it is said to be structurally unbalanced.

Definition 2 If $\lim_{t\to\infty}|x_i(t) - x_j(t)| = \delta$ a.s. $\forall i, j \in \mathcal{V}$, where δ is a random variable $P(\delta > 0) = 1$, the system is said to reach bipartite consensus.

Definition 3 If $P\{\lim_{t\to\infty}|x_i(t) - x_j(t)| = 0$ for all $i, j \in \mathcal{V}\} = 1$, the agents are said to reach consensus with probability one.

Lemma 1[16,17] If the graph $\mathcal{G}(\Phi(A))$ is either strongly connected graph or connected graph. Then the system admits a bipartite consensus solution if and only if $\mathcal{G}(\Phi(A))$ is structurally balanced.

Lemma 2[16] If the graph $\mathcal{G}(\Phi(A))$ is structurally balanced, then there exists $Q = \mathrm{diag}\{\sigma\}$ with $\sigma = [\sigma_1, \ldots, \sigma_n], \sigma_i \in \{\pm 1\}$ such that $Q\Phi(A)Q = A$.

Lemma 3[16] If the graph $\mathcal{G}(\Phi(A))$ is structurally unbalanced, then $\lambda_1(L) > 0$, i.e., $-L$ is Hurwitz stable.

3. System model and free protocol

Since we are going to give agents freedom, the first thing we need to analyze is the decisions they might make. From Refs. [16,17], we know that agents respond to neighbor information in only two ways, that is, to get closer or to move away. This is also why the relationship between agents is divided into cooperation and antagonism.

Consider a first order multi-agent system consisting of n agents, suppose that the dynamic of each agent is as follows:

$$\dot{x}_i(t) = u_i(t),$$

where $x_i(t), u_i(t) \in \boldsymbol{R}$ represent the position and control input of the agent i, respectively.

In this paper, we propose the following control protocol:

$$u_i(t) = -\sum_{j\in N_i}(a_{ij}x_i(t) - Y_{ij}(t)a_{ij}x_j(t)),$$

where $Y_{ij}(t)$ is a random variable depending on the choice made by the agent i.

Remark In social networks, we cannot control individuals in our society, such as robots, which does not mean we cannot limit their behavior. In this paper, suppose that all agents have the right to choose control protocols and we calculate the corresponding probability distribution.

4. Static selection

In social networks, social individuals often do not change their choices after they have identified their relationships with others, which we call the static selection. There are three situations based on the orientation of communication and the pattern they make decisions.

4.1. Single-orientation communication pattern

Single-orientation communication means that the exchange of information between agents is single orientation. Therefore, an agent receiving information from its neighbors can only unilaterally make cooperative or antagonistic decisions. Thus the graph of single-orientation communication pattern is equivalent to a special directed graph.

Suppose $a_{ij}a_{ji} = 0$, $\forall i, j = 1, \ldots, n$, $Y_{ij}(t) = Y_{ij}$ is time-invariant and $P(Y_{ij} = 1) = p_{ij}, P(Y_{ij} = -1) = 1 - p_{ij}, \forall (i, j) \in \varepsilon$, where $0 < p_{ij} < 1$ is the probability of agents choosing their protocols. Then the system equation is changed into

$$\dot{x}_i(t) = -\sum_{j\in N_i} a_{ij}(x_i(t) - Y_{ij}x_j(t)). \tag{1}$$

Theorem 1 Suppose that the directed graph $\mathcal{G}(\Phi(A))$ is strongly connected. The probability of the system (1) reaching the bipartite consensus is

$$P_1 = \sum_{l=1}^{n} \sum_{1\le j_1<\cdots<j_l\le n} \left[\prod_{k_1\in S_k^1}(1 - p_{k_1}) \prod_{k_2\in S_k^2} p_{k_2}\right]^{1/2},$$

where $S_k^1 = \{ij|a_{ij} > 0, \forall i \in J_l,\ j \in G_{n-l}$ or $i \in G_{n-l},\ j \in J_l\}$, $S_k^2 = \{ij|a_{ij} > 0,\ \forall i, j \in J_l$ or $i, j \in G_{n-l}\}$, for any choice of j_l, $1 \le l \le n$ with $1 \le j_1 < \cdots < j_l \le n$, and $1 \le g_1 < \cdots < g_{n-l} \le n$ represent the remaining integers in $[1, n]$, $J_l = \{j_1, j_2, \ldots, j_l\}$, $G_{n-l} = \{g_1, g_2, \ldots, g_{n-l}\}$.

Proof From Lemma 1, the system with a strongly connected directed graph admits a bipartite consensus solution if and only if $\mathcal{G}$ is structurally balanced. It follows from Lemma 2 that the system wants to be structurally balanced, it should be $Q \in \bigcup_{l=1}^{n} B_l$, $Q\Phi(A)Q = A$, where $B_l = \{Q$: there are l entries in σ to be l and other $n - l$ entries to be $-1\}$ and $A = [a_{ij}] \in \boldsymbol{R}^{n\times n}$.

Note that $Q\Phi(A)Q = -Q\Phi(A)(-Q)$, $Q = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\}$, $\sigma_i \in \{\pm 1\}$. Hence,

$$P_1 = \left[P\{Q \in \bigcup_{l=1}^{n} B_l, \quad Q\Phi(A)Q = A\}\right]^{1/2}.$$

For any $i \neq j$, $B_i \bigcap B_j = \emptyset$, hence

$$\left[P\{Q \in \bigcup_{l=1}^{n} B_l, Q\Phi(A)Q = A\}\right]^{1/2} = \left[\sum_{l=1}^{n} P\{Q \in B_l, Q\Phi(A)Q = A\}\right]^{1/2}.$$

Taking $B_l = \bigcup_{1\le j_1<\ldots<j_l\le n} b_l$, where $b_l = \{Q : C_{j_1}\cdots C_{j_l}C_{g_1}^c\cdots C_{g_{n-l}}^c\}$. Hence

$$\left[\sum_{l=1}^{n} P\{Q \in B_l, Q\Phi(A)Q = A\}\right]^{1/2} = \left[\sum_{l=1}^{n} P\{Q \in \bigcup_{1\le j_1<\ldots<j_l\le n} b_l, Q\Phi(A)Q = A\}\right]^{1/2}.$$

For any choice of j_l, g_l, the events b_l are different. Then

$$\left[\sum_{l=1}^{n} P\{Q \in \bigcup_{1\le j_1<\ldots<j_l\le n} b_l, Q\Phi(A)Q = A\}\right]^{1/2} = \left[\sum_{l=1}^{n} \sum_{1\le j_1<\ldots<j_l\le n} P\{Q = b_l, Q\Phi(A)Q = A\}\right]^{1/2}.$$

Note that $Q\Phi(A)Q = A$ is equivalent to $\sigma_i a_{ij}\sigma_j = Y_{ij}a_{ij}$, we have

$$\left[\sum_{l=1}^{n} \sum_{1\le j_1<\ldots<j_l\le n} P\{Q = b_l, Q\Phi(A)Q = A\}\right]^{1/2} = \left[\sum_{l=1}^{n} \sum_{1\le j_1<\ldots<j_l\le n} P\{Q = b_l, \sigma_k a_{kr}\sigma_r = Y_{kr}a_{kr},\ \forall 1 \le k,r \le n\}\right]^{1/2}.$$

From $a_{ij}a_{ji} = 0, \forall i,j = 1,\ldots,n$ and $P(Y_{ij} = 1) = p_{ij}, P(Y_{ij} = -1) = 1 - p_{ij}, \forall (i,j) \in \varepsilon$, we have

$$\left[\sum_{l=1}^{n} \sum_{1\le j_1<\ldots<j_l\le n} P\{Q = b_l, \sigma_k a_{kr}\sigma_r = Y_{kr}a_{kr},\ \forall 1 \le k,r \le n\}\right]^{1/2} = \sum_{l=1}^{n} \sum_{1\le j_1<\ldots<j_l\le n} \left[\prod_{k_1\in S_k^1}(1-p_{k_1}) \prod_{k_2\in S_k^2} p_{k_2}\right]^{1/2}.$$

The proof is completed.

4.2. Double-orientation communication and unilateral pattern

If the communication of information between agents is bidirectional but the choice of relations between agents is made independently, we call it double-orientation communication and unilateral pattern. In this case, although the graph appears to be an undirected graph, the existence of random parameters makes the graph equivalent to a special directed graph in nature.

Suppose that $\forall a_{ij} > 0$, $a_{ij}a_{ji} > 0$, $Y_{ij}(t) = Y_{ij}$ is time-invariant and $P(Y_{ij} = 1) = p_{ij}, P(Y_{ij} = -1) = 1 - p_{ij}, \forall (i,j) \in \varepsilon$, where $0 < p_{ij} < 1$ is the probability of agents choosing their protocols. Then the system equation is changed into

$$\dot{x}_i(t) = -\sum_{j\in N_i} a_{ij}(x_i(t) - Y_{ij}x_j(t)). \tag{2}$$

Theorem 2 Suppose that the undirected graph $\mathcal{G}(\Phi(A))$ is connected. The probability of the system (2) reaching the bipartite consensus is

$$P_2 = \sum_{l=1}^{n} \sum_{1\le j_1<\ldots<j_l\le n} \left[\prod_{k_3\in S_k^3}(1-p_{k_3}) \prod_{k_4\in S_k^4} p_{k_4}\right]^{1/2},$$

where $S_k^3 = \{ij | a_{ij} > 0, \forall i \in J_l,\ j \in G_{n-l} \text{ or } i \in G_{n-l},\ j \in J_l\}$, $S_k^4 = \{ij | a_{ij} > 0,\ \forall i,j \in J_l \text{ or } i,j \in G_{n-l}\}$, for any choice of j_l, $1 \le l \le n$ with $1 \le j_1 < \cdots < j_l \le n$, and $1 \le g_1 < \cdots < g_{n-l} \le n$ represent the remaining integers in $[1,n]$, $J_l = \{j_1,\ j_2,\ldots,\ j_k\}$, $G_{n-l} = \{g_1,\ g_2,\ldots,\ g_{n-l}\}$.

Proof The first part follows from Theorem 1, we have

$$P_2 = \left[\sum_{l=1}^{n} \sum_{1\le j_1<\cdots<j_l\le n} P\{Q = b_l, \sigma_k a_{kr}\sigma_r = Y_{kr}a_{kr},\ \forall 1 \le k,r \le n\}\right]^{1/2}.$$

From $\forall a_{ij} > 0$, $a_{ij}a_{ji} > 0$ and $P(Y_{ij} = 1) = p_{ij}, P(Y_{ij} = -1) = 1 - p_{ij}, \forall (i,j) \in \varepsilon$, then

$$\left[\sum_{l=1}^{n} \sum_{1\le j_1<\cdots<j_l\le n} P\{Q = b_l, \sigma_k a_{kr}\sigma_l = Y_{kr}a_{kr},\ \forall 1 \le k,r \le n\}\right]^{1/2} = \sum_{l=1}^{n} \sum_{1\le j_1<\cdots<j_l\le n} \left[\prod_{k_3\in S_k^3}(1-p_{k_3}) \prod_{k_4\in S_k^4} p_{k_4}\right]^{1/2}.$$

The proof is completed.

4.3. Double-orientation communication and bilateral pattern

If the information exchange between agents is bidirectional, and the relationship between agents is a decision jointly made by both parties, we call it double-orientation communication and bilateral pattern. In this case, the graph is also equivalent to an undirected graph.

Suppose that $\forall a_{ij} > 0$, $a_{ij}a_{ji} > 0$, $Y_{ij}(t) = Y_{ij}$ is time-invariant and $P(Y_{ij} = 1) = p_{ij}, P(Y_{ij} = -1) = 1 - p_{ij}, \forall (i,j) \in \varepsilon$, where $0 < p_{ij} < 1$ is the probability of agents choosing their protocols. Then the system equation is changed into

$$\dot{x}_i(t) = -\sum_{j\in N_i} a_{ij}(x_i(t) - Y_{ij}x_j(t)). \tag{3}$$

Theorem 3 Suppose that the undirected graph $\mathcal{G}(\Phi(A))$ is connected. The probability of the system (3) reaching the bipartite consensus is

$$P_3 = \sum_{l=1}^{n}\sum_{1\le j_1<\cdots<j_l\le n}\left[\prod_{k_5\in S_k^5}(1-p_{k_5})\prod_{k_6\in S_k^6}p_{k_6}\right]^{1/2},$$

where $S_k^5 = \{ij|ij \equiv ji,\ a_{ij} > 0,\ \forall i \in J_l,\ j \in G_{n-l}\}$, $S_k^6 = \{ij|ij \equiv ji,\ a_{ij} > 0,\ \forall i,j \in J_l \text{ or } i,j \in G_{n-l}\}$, for any choice of j_l, $1 \le l \le n$ with $1 \le j_1 < \cdots < j_l \le n$, and $1 \le g_1 < \cdots < g_{n-l} \le n$ represent the remaining integers in $[1,n]$, $J_l = \{j_1, j_2, \ldots, j_l\}$, $G_{n-l} = \{g_1, g_2, \ldots, g_{n-l}\}$.

Proof The first part follows from Theorem 1, we have

$$P_3 = \left[\sum_{l=1}^{n}\sum_{1\le j_1<\cdots<j_l\le n} P\{Q = b_l, \sigma_k a_{kr}\sigma_r = Y_{kr}a_{kr},\ \forall 1 \le k,r \le n\}\right]^{1/2}.$$

From $\forall a_{ij} > 0$, $a_{ij}a_{ji} > 0$ and $P(Y_{ij} = Y_{ji} = 1) = p_{ij}, P(Y_{ij} = Y_{ji} = -1) = 1 - p_{ij}, \forall (i,j) \in \varepsilon$, we have

$$\left[\sum_{l=1}^{n}\sum_{1\le j_1<\cdots<j_l\le n} P\{Q = b_l, \sigma_k a_{kr}\sigma_r = Y_{kr}a_{kr}, \quad \forall 1 \le k,r \le n\}\right]^{1/2} = \sum_{i=1}^{n}\sum_{1\le j_1<\cdots<j_l\le n}\left[\prod_{k_5\in S_k^5}(1-p_{k_5})\prod_{k_6\in S_k^6}p_{k_6}\right]^{1/2}.$$

The proof is completed.

5. Dynamic selection

In a social network, the attitude of agents towards other agents is always not constant in some situations. In this section, we mainly discuss the case of dynamic selection, where attitudes between agents change randomly over time.

According to the time point denoted by $h_0 = 0, h_1, h_2, \ldots$, at which the agent changes its attitude. We divide the time into some nonoverlapping time slots $t_0 - 0, t_1 = h_1, t_2 = h_2 - h_1, \ldots$.

Let $P(Y_{ij}(t) = 1) = p_{ij}^s, P(Y_{ij}(t) = -1) = 1 - p_{ij}^s, \forall (i,j) \in \varepsilon$, $\sum_{i=1}^{s} t_i < t \le \sum_{i=1}^{s+1} t_i$, $s \in \{0,1,2,\ldots\}$, where $0 < p_{ij}^s < 1$ is the probability. Then the system equation is changed into

$$\dot{x}_i(t) = -\sum_{j\in N_i} a_{ij}(x_i(t) - Y_{ij}(t)x_j(t)). \tag{4}$$

Theorem 4 Suppose that the directed graph $\mathcal{G}(\Phi(A))$ is strongly connected. Let $a > b > 0$, $sup\{t_k\} = a > 0$ and $inf\{t_k\} = b > 0$ then system (4) can reach consensus with probability one.

Proof It follows from Theorem 1 and $0 < p_{(i,j)}^s < 1$ that the probability of structurally balanced is

$$p^s(t) = \sum_{i=1}^{n}\sum_{1\le j_1<\cdots<j_i\le n}\left[\prod_{k_1\in S_k^1}(1-p_{k_1}^s)\prod_{k_2\in S_k^2}p_{k_2}^s\right]^{1/2} > 0,$$

where $\sum_{i=1}^{s} t_i < t \le \sum_{i=1}^{s+1} t_i$, $s \in \{0,1,2,\ldots\}$, for any choice of j_k, $1 \le k \le n$ with $1 \le j_1 < \cdots < j_k \le n$ and $1 \le g_1 < \cdots < g_{n-k} \le n$ represent the remaining integers in $[1,n]$ $J_k = \{j_1, j_2, \ldots, j_k\}$, $G_{n-k} = \{g_1, g_2, \ldots, g_{n-k}\}$.

From $sup\{t_k\} = a > b = inf\{t_k\} > 0$, the attitudes between agents would change infinitely. Let B denotes the times of $\mathcal{G}(\Phi(A))$ being structurally unbalanced. Then

$$P\{t \to \infty, B \to \infty\} = 1. \tag{5}$$

Let $X_{\max}(t) = \max\{|x_i(t)|\}$ and $s(t)$ is agent with the largest absolute value of state, i.e., $x_{s(t)}(t) = X_{\max}(t)$. Then

$$\begin{aligned}&\operatorname{sgn}(x_{s(t)})\dot{X}_{\max}(t)\\ &= -\sum_{j\in N_i} a_{ij}(\operatorname{sgn}(x_{s(t)})X_{\max}(t) - Y_{ij}(t)x_j(t))\end{aligned}$$

then,

$$\dot{X}_{\max}(t) = -\sum_{j\in N_i} a_{ij}(X_{\max}(t) - Y_{ij}(t)\operatorname{sgn}(x_{s(t)}(t))x_j(t)). \tag{6}$$

It is clear that $X_{\max}(t)$ is non-increasing.

Next, we discuss the corresponding probability according to the following two cases.

Case 1

There are structurally unbalanced graphs at $[0,\infty)$. According to Lemma 3 and Eq. (6), we have

$$P\{\forall \varepsilon, N > 0, \exists s > N\ s.t.\ X_{\max}(\sum_{k=1}^{s} t_k) < \varepsilon\} = 1.$$

By the definition of $X_{\max}(t)$, we can conclude that

$$P\{\lim_{t\longrightarrow\infty}|x_i(t) - x_j(t)| = 0\} = 1 \quad \forall i,j \in \mathcal{V}.$$

Case 2

There are no structurally unbalanced graphs at $[0,\infty)$. Since $P\{Y_{ij}(t)=1\}>0, \forall (i,j)\in\varepsilon$, we have

$$P\{\forall T>0, \exists t>T,\ s.t.\ \Phi(A)=A\}=1,$$

then

$$P\{\forall\varepsilon, N>0,\ \ \exists X_{\max}\left(\sum_{k=1}^{s}t_k\right)=\varepsilon,\ s>N\ s.t.$$

$$\sum_{k=1}^{s}t_i<t\le\sum_{k=1}^{s+1}t_i,\quad X_{\max}(t)\le X_{\max}\left(\sum_{k=1}^{s}t_k\right)*e^{-L}<\varepsilon\}=1.$$

From the definition of $X_{\max}(t)$, we can conclude that

$$P\{\lim_{t\longrightarrow\infty}|x_i(t)-x_j(t)|=0\}=1\ \ \forall i,j\in\mathcal{V}.$$

The proof is completed.

6. Simulations

Example 1 Here, we consider the case of Theorem 1. The interaction graph is shown in Fig. 1.

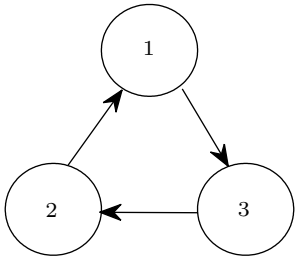


Fig. 1. Single-orientation communication pattern.

The experimental results are shown in Table 1.

Table 1. Simulation parameters.

Total number of trials	Number of bipartite consensus	Number of consensus
24	9	15
24	8	16
24	11	13
24	6	18
24	11	13
24	11	13
24	10	14
24	9	15
24	10	14
24	10	14

According to Theorem 1, the probability of the system reaching the bipartite consensus is $P_1=0.375$. Let us do the χ^2 test.

Its original hypothesis H_0: there is no difference between the probability of the system reaching the bipartite consensus and P_1.

Form $n=24$, $n_1=9$, $n_2=8$, $n_3=11$, $n_4=6$, $n_5=11$, $n_6=11$, $n_7=10$, $n_8=9$, $n_9=10$, $n_{10}=10$, the degree of freedom is $m=10-2=8$ and the confidence level is $\alpha=0.05$, we have

$$\chi^2=\Sigma_{i=1}^{10}\frac{(n_i-nP_1)^2}{nP_1}\approx 2.78.$$

Inquiring the χ^2 table, we note the significance of $2.78=\chi^2<\chi^2_{0.95}(8)=15.507$, which is not obvious, and we cannot reject the original hypothesis H_0, the actual probability distribution is not different from the probability distribution. Therefore, our conclusion is verified.

Example 2 Here, we consider the case of Theorem 2. The interaction graph is given in Fig. 2.

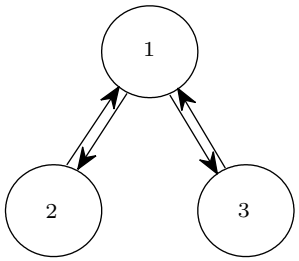


Fig. 2. Double-orientation communication and unilateral pattern.

The experimental results are listed in Table 2.

Table 2. Simulation parameters.

Total number of trials	Number of bipartite consensus	Number of consensus
48	9	39
48	10	38
48	7	41
48	9	39
48	9	39
48	10	38
48	12	36
48	10	38
48	7	41
48	14	34

According to Theorem 2, the probability of the system reaching the bipartite consensus is $P_2=3/16=0.1875$. Let us do the χ^2 test.

Its original hypothesis H_0: there is no difference between the probability of the system reaching the bipartite consensus and P_2.

Form $n=48$, $n_1=9$, $n_2=10$, $n_3=7$, $n_4=9$, $n_5=9$, $n_6=10$, $n_7=12$, $n_8=10$, $n_9=7$, $n_{10}=14$, the degree of freedom is $m=10-2=8$ and the confidence level is $\alpha=0.05$, we have

$$\chi^2=\Sigma_{i=1}^{10}\frac{(n_i-nP_2)^2}{nP_2}=5.$$

Through inquiring the χ^2 table, we note the significance of $5=\chi^2<\chi^2_{0.95}(8)=15.507$, which is not obvious, and cannot reject the original hypothesis H_0, the actual probability

distribution is not different from the probability distribution. Therefore, our conclusion is verified.

Example 3 Here, we consider the case of Theorem 3. The interaction graph is shown in Fig. 3.

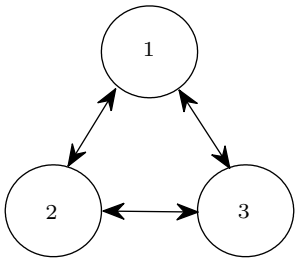


Fig. 3. Double-orientation communication and Bilateral pattern.

The experimental results are listed in Table 3.

Table 3. Simulation parameters.

Total number of trials	Number of bipartite consensus	Number of consensus
24	4	15
24	9	16
24	8	13
24	3	18
24	5	13
24	5	13
24	4	14
24	4	15
24	8	14
24	5	14

According to Theorem 3, the probability of the system reaches the bipartite consensus of $P_3 = 0.25$. Let us do the χ^2 test.

Its original hypothesis H_0: there is no difference between the probability of the system reaching the bipartite consensus and P_3.

From $n = 24$, $n_1 = 4$, $n_2 = 9$, $n_3 = 8$, $n_4 = 3$, $n_5 = 5$, $n_6 = 5$, $n_7 = 4$, $n_8 = 4$, $n_9 = 8$, $n_{10} = 5$, the degree of freedom is $m = 10 - 2 = 8$ and the confidence level is $\alpha = 0.05$, we have

$$\chi^2 = \sum_{i=1}^{10} \frac{(n_i - nP_3)^2}{nP_3} \approx 6.83.$$

Inquiring the χ^2 table, we note the significance of $6.83 = \chi^2 < \chi^2_{0.95}(8) = 15.507$, which is not obvious, and cannot reject the original hypothesis H_0, the actual probability distribution is not different from the probability distribution. Therefore, our conclusion is verified.

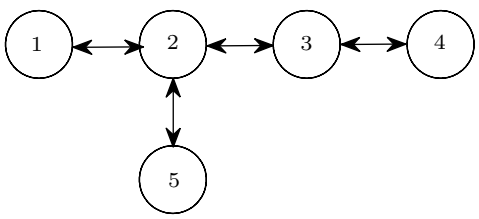


Fig. 4. Dynamic selection graph.

Example 4 Here, we consider the case of Theorem 4. The interaction graph is given in Fig. 4.

The experimental results are shown in Fig. 5.

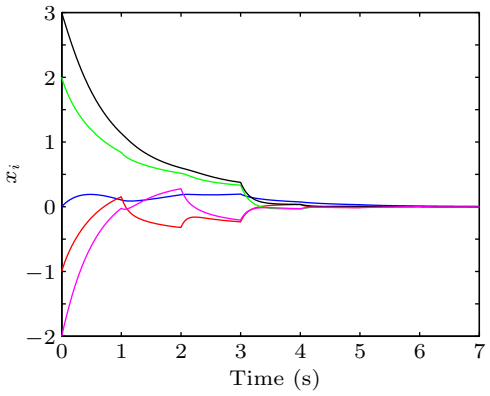


Fig. 5. One result of dynamic selection.

The results of 100 trials were consistent. Hence, we verify the correctness of the theoretical results.

7. Conclusion

In this paper, we have extended the concept of consensus to the autonomous network of agents, and the autonomy of agents is constructed with a random parameter. In the case of static selection, we could use the structure of the communication topology to calculate the probability of reaching consensus. Under the dynamic selection mechanism, a sufficient condition for consensus with probability one is given. In the future, we will analyze the combination of dynamics and statistics from another perspective. In order to make the agent in a higher degree of freedom, it is still necessary to achieve the consensus and to find the boundary of freedom.

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