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Complex network perspective on modelling chaotic systems via machine learning*

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Recent advances have demonstrated that a machine learning technique known as "reservoir computing" is a significantly effective method for modelling chaotic systems. Going beyond short-term prediction, we show that long-term behaviors of an observed chaotic system are also preserved in the trained reservoir system by virtue of network measurements. Specifically, we find that a broad range of network statistics induced from the trained reservoir system is nearly identical with that of a learned chaotic system of interest. Moreover, we show that network measurements of the trained reservoir system are sensitive to distinct dynamics and can in turn detect the dynamical transitions in complex systems. Our findings further support that rather than dynamical equations, reservoir computing approach in fact provides an alternative way for modelling chaotic systems.

Keywords: reservoir computing approach, complex networks, chaotic systems

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Recently, machine learning technique, as a critical branch of artificial intelligence, has attracted increasing attention.[1,2] A great variety of machine learning methods, such as Bayes learning,[3] neural networks,[2] and decision trees,[4] are proposed for dealing with the problems related to artificial intelligence. Among them, an intriguing one named reservoir computing approach has received considerable attention in time series domain.[5,6] A growing number of studies have demonstrated that this approach is competent for forecasting low-dimensional chaotic systems,[6] inferring unmeasured variables,[7] and even predicting spatiotemporally chaotic systems.[8] In this sense, reservoir computing approach provides an effective way for modelling and characterizing chaotic systems.

Beyond short-term prediction, there is a growing industry in revealing long-term behaviors of the trained reservoir system, for example, Lyapunov exponents,[9] correlation dimension,[10,11] and even attractor.[12] However, these works mainly study a trained reservoir system from a dynamical angle, while a wide range of statistics in complex network domain have been overlooked. In fact, complex network theory provides a new paradigm to understand and characterize dynamical systems.[13–15] It offers a range of powerful tools to describe a great variety of nonlinear systems. For example, recurrence networks in terms of recurrence allow us to classify dynamics and to detect dynamical transitions.[14] Therefore, network theory will bring us a new perspective on study long-term behaviors of the trained reservoir system in modelling chaotic systems.

In this paper, we study long-term behaviors of a trained reservoir system by virtue of network measurements for which we transform its prediction trajectory into recurrence networks. We find that a great variety of network statistics, such as degree distribution, the clustering coefficient, and the mixing pattern, induced for the trained reservoir system are almost the same as that of a considered chaotic system. Remarkably, we show that when learning an observed system in distinct dynamical regimes, their resultant network structures are consequently different. These distinctions can appropriately detect and identify the dynamical transitions in the learned system. Numerical results on two benchmark chaotic systems (*i.e.*, the Rössler system and the Hénon map) further support our findings.

We begin with introducing the basic framework of reservoir computing (RC) approach. Its architecture is usually composed of three components: an input layer coupled with an input vector $\boldsymbol{u}(t)$, a reservoir network in the middle layer consisting of N nodes, and an output layer coupled with an output vector $\boldsymbol{y}(t)$, as illustrated in Fig. 1(a). Here, we follow Jaeger's design and define the evolving equations of the reservoir vector $\boldsymbol{r}$ and the output vector $\boldsymbol{y}$ as follows:[6]

$$\boldsymbol{r}(t+1)=(1-\alpha)\boldsymbol{r}(t)+\alpha\tanh\left(\boldsymbol{A}\boldsymbol{r}(t)+\boldsymbol{W}_{\rm in}\begin{pmatrix}b_{\rm in}\\ \boldsymbol{u}(t)\end{pmatrix}\right), \quad (1)$$

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$$\boldsymbol{y}(t+1)=\boldsymbol{W}_{\text{out}}\begin{pmatrix} b_{\text{out}} \\ \boldsymbol{u}(t) \\ \boldsymbol{r}(t+1) \end{pmatrix}. \tag{2}$$

The reservoir parameters $\boldsymbol{A}$, $\boldsymbol{W}_{\text{in}}$, α, b_{in}, and b_{out} are usually given in advance before training. Specifically, the parameter α is chosen from the range $(0,1)$, while the elements of $\boldsymbol{W}_{\text{in}}$ are drwan from a uniform distribution $[-1,1]$. The matrix $\boldsymbol{A}$ is the adjacency matrix of the reservoir network and is built from a sparse random network. Empirically, we set $b_{\text{in}}=1$ and $b_{\text{out}}=1$.[5,16] In this context, the output weighted matrix $\boldsymbol{W}_{\text{out}}$ is the sole parameter needed to be determined in the training stage. For a chaotic system $\{\boldsymbol{s}(t)\}_{t=1}^{n}$ of interest, previous studies have demonstrated that the output weighted matrix can be expressed analytically as follows:[17–19]

$$\boldsymbol{W}_{\text{out}}=\boldsymbol{Y}\boldsymbol{X}^{\text{T}}(\boldsymbol{X}\boldsymbol{X}^{\text{T}}+\lambda\mathbb{I})^{-1}, \tag{3}$$

where $\mathbb{I}$ is an identity matrix and λ is the ridge regresion parameter. The k-th columns of the matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $[b_{\text{out}};\boldsymbol{s}(k);\boldsymbol{r}(k)]$ and $\boldsymbol{s}(k+1)$, respectively. After training stage, when $\boldsymbol{y}(t+1)$ is adopted as $\boldsymbol{u}(t+1)$, the reservoir computing system can run autonomously based on Eqs. (1) and (2).

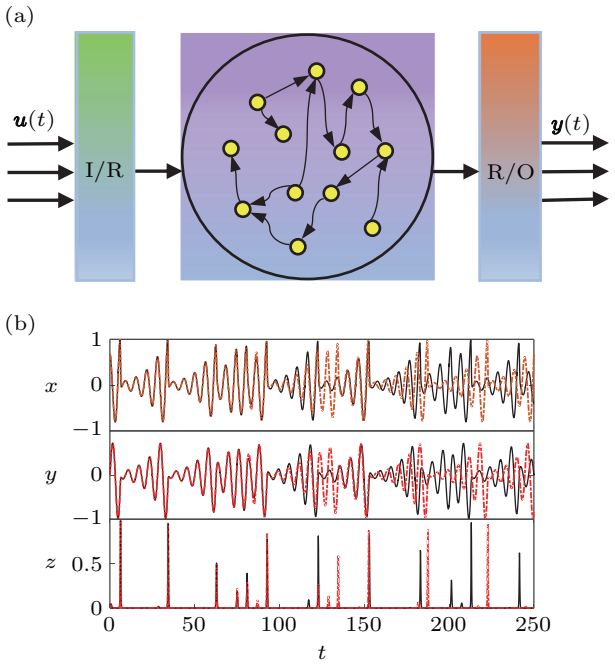


Fig. 1. (a) Schematic illustration of a reservoir computer with the input vector $\boldsymbol{u}(t)$ and the output vector $\boldsymbol{y}(t)$. (b) The prediction of the trained reservoir computer and the actual trajectories of the Rössler system.

We first apply the reservoir computing approach to the Rössler system in the chaotic regime given by

$$\mathrm{d}x/\mathrm{d}t = -y-z, \tag{4}$$

$$\mathrm{d}y/\mathrm{d}t = x+0.2y, \tag{5}$$

$$\mathrm{d}z/\mathrm{d}t = 0.2+z(x-9). \tag{6}$$

We calculate a numerical solution for this system based on the fourth-order Runge–Kutta method and obtain 2×10^4 points with time step $\Delta t=0.1$. For eliminating transient states, we discard the leading 5000 observations. We then use the first 2600 points with the input vector $\boldsymbol{u}=(x,y,z)$ for training. Here, we choose the reservoir parameters $n=2600, \alpha=0.5$, $N=500$, and $\lambda=1\times10^{-8}$. It is shown that the trained reservoir system produces short-term prediction correctly, as described in Fig. 1(b). Since the sensitive to the initial condition of the Rössler system, the prediction data gradually deviates from the actual Rössler trajectory, as expected. Nonetheless, we notice that after $t\approx110$, the long-term profile of the reservoir system resemble that of the Rössler system. This phenomenon hints that besides predicting short-term dynamics, the long-term behavior of a considered system seems to be consequently captured in the trained reservoir system.

We now explore the long-term behavior of a trained RC model from complex network perspective for which a wide range of network metrics can be employed. In fact, network science has recently been applied to understand and characterize chaotic systems of interest.[14,20] A number of approaches for transforming time series into networks have been reported, such as cycle network,[13] visibility graph,[21] and recurrence network.[14,22] Here, we adopt the recurrence method for mapping time series of chaotic systems and their learned RC models into networks. In particular, for a time series given by $\{S_i\}_{i=1}^{N}$, the transformed network is

$$A_{ij}=\Theta(\varepsilon-\parallel S_i-S_j\parallel)-\delta_{ij}, \tag{7}$$

where $\Theta(\cdot)$ is the Heaviside function, ε is a threshold value, and δ_{ij} is the Kronecker delta. Empirically, we choose a sufficiently small threshold value for which the resultant recurrence network is connected.[14] In the constructed network, we observe its local, intermediate, and global network properties in terms of degree distribution $P(k)$, the clustering coefficient C, the distribution of shortest path length $P(d)$ and the assortativity coefficient r, respectively. We then apply these network measurements to study the previous Rössler system and the corresponding RC model. Here, we select the threshold value $\varepsilon=0.13$ and the length of time series $l=5000$. Interestingly, we find that the degree distribution $P(k)$ of the RC model presents the similar profile as that of the Rössler system, see Fig. 2(a). This phenomenon also occurs on the distribution of shortest path length, as illustrated in Fig. 2(b). These results reveal that the topological feature of an observed chaotic system is preserved in the trained RC model. This is further supported by observing the average clustering coefficient and the assortativity coefficient for which $C=0.619$ and $r=0.841$ for the Rössler system, while $C=0.624$ and $r=0.829$ for the RC model. Remarkably, when examing the spatial distribution of the degree in (x,y) plane, they present almost identical pattern, see Figs. 2(c) and 2(d). Clearly, high values of k lie in the central region, and low values almost filled the other region. Our findings reveal that the RC model is identical with a chaotic system of interest from complex network perspective.

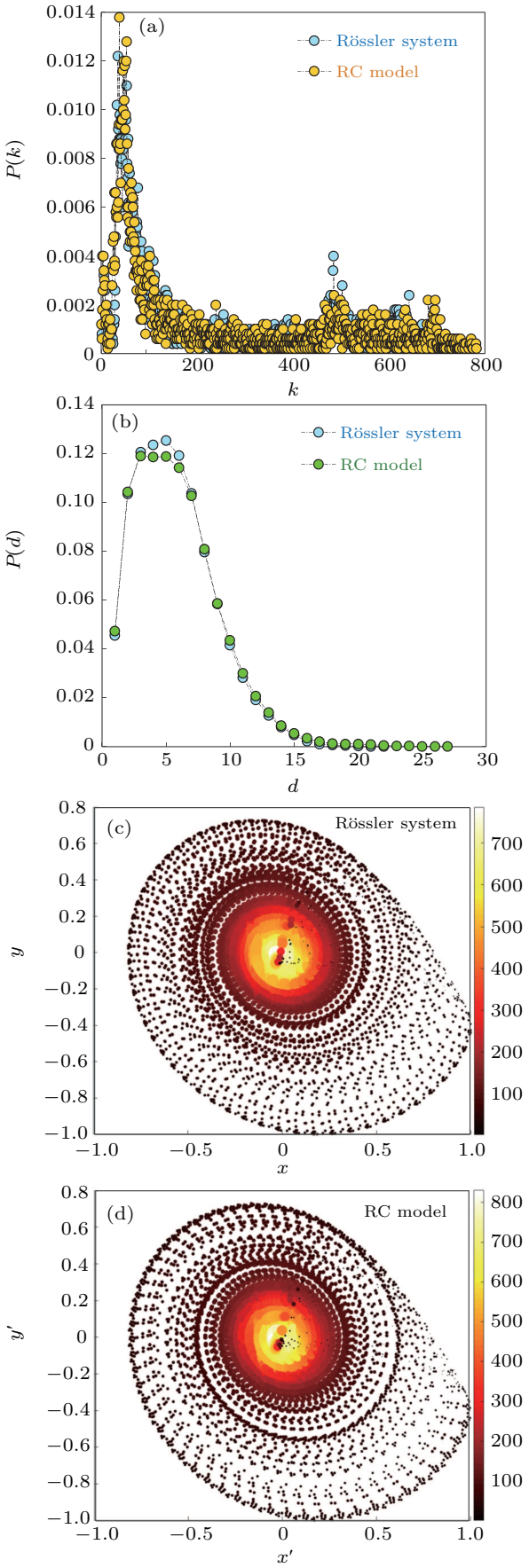


Fig. 2. Comparison of the Rössler system and its learned RC model with respect to different network statistics: (a) the degree distribution and (b) the distribution of shortest path length. Colour-coded representation of the degree k in the (x,y) plane for (c) the Rössler system and (d) the reservoir computer model.

We further confirm this interesting finding on the Hénon map given by

$$x_{n+1} = 1 + y_n - ax_n^2, \tag{8}$$

$$y_{n+1} = bx_n, \tag{9}$$

where $a = 1.4$ and $b = 0.3$. We generate the 2×10^4 observations from this map and use the first 2600 points with the input $\boldsymbol{u} = (x,y)$ for training. Here, we set the reservoir parameters $n = 2600, \alpha = 0.25$, $N = 500$, and $\lambda = 1 \times 10^{-8}$. After the training stage, we generate a trajectory of length $l = 5000$ of this reservoir system and transform it into recurrence network. Here, we choose the threshold value $\varepsilon = 0.03$. Figures 3(a) and 3(b) show the degree distribution and the distribution of shortest path length for the Hénon map and the RC model, respectively. Clearly, it is shown that profiles of $P(k)$ *versus* k for the Hénon map and the RC model present a similar tendency. This phenomenon is also established on the distribution of shortest path length. Moreover, we further observe that the spatial distribution of the degree in phase space for the RC model is almost identical with that for the Hénon map, see Figs. 3(c) and 3(d). These findings further confirm that beside prediction, the trained reservoir system captures long-term behavior of an observed chaotic system in terms of network statistics.

Finally, we show that the long-term behaviors of the trained RC model are different when learning distinct dynamical systems and these distinctions can identify the dynamical transitions of the complex system of interest. Here, we take the previous Hénon map as a benchmark example. We select $b = 0.3$ and $a \in [1, 1.4]$ with a step size $\Delta a = 0.005$. With the increase of a, the Hénon map undergoes from period-doubling route to chaos. For every a here, we record 2×10^4 successive values after discarding the leading 5000 data (to eliminate transient states). The bifurcation diagram gives an intuitive feeling of the dynamical transition of the Hénon map, see the top panel of Fig. 4(a). For each record, we use the first 2600 points with the input $\boldsymbol{u} = (x,y)$, $\alpha = 0.25$, and $N = 500$ for training the RC model. After training stage, we generate 1×10^4 data points from each trained RC system. Interestingly, we show that the trained RC systems can reproduce the bifurcation diagram of the Hénon map exactly, as illustrated in Fig. 4(a). We then calculate the clustering coefficients C of recurrence networks constructed from the Hénon map and the RC model. We find that they present an identical tendency, see Fig. 4(b). Meanwhile, we notice that they are sensitive to the presence of dynamical transitions of Hénon map indicating by the largest Lyapunov exponent $\lambda_{\max}$. Here, we calculate the largest Lyapunov exponent using the TISEAN software package.[23] Specifically, the maximal values of C (*i.e.*, $C = 1$) are calculated from the periodic regime, whereas the chaotic behavior results in a relatively smaller value.

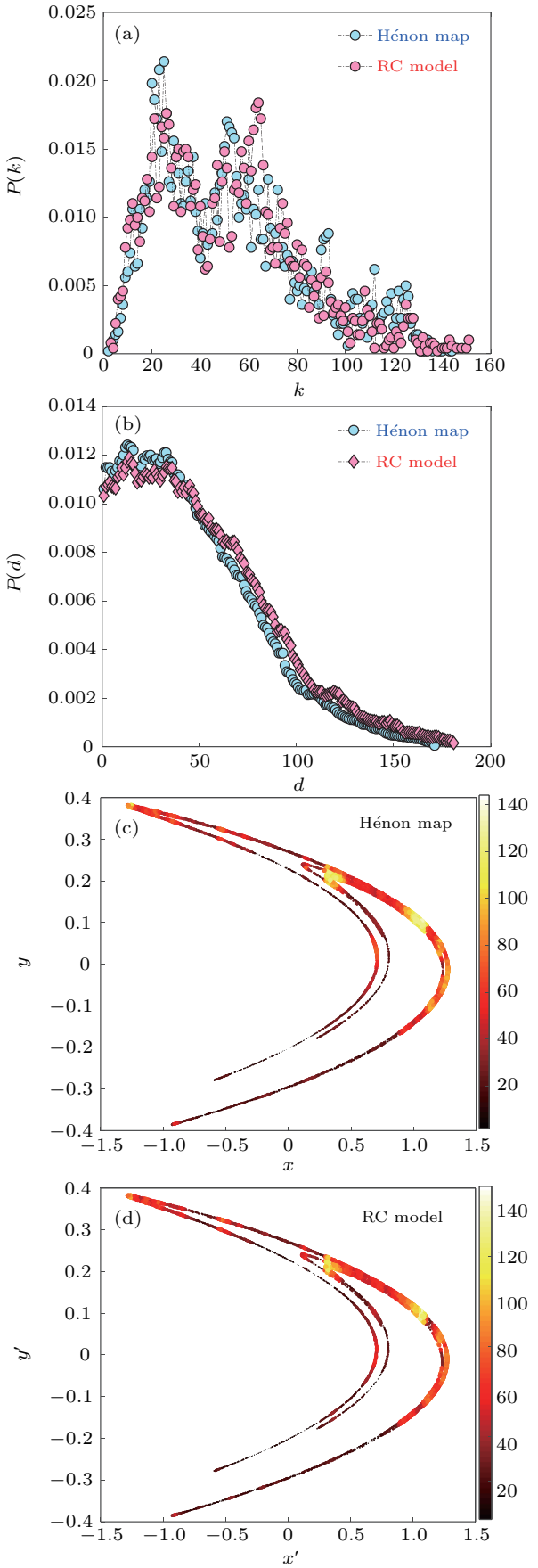


Fig. 3. Comparison of the Hénon map and its learned RC model with respect to different network statistics: (a) degree distribution and (b) distribution of shortest path length. Colour-coded representation of the degree k in phase space for (c) the Hénon map and (d) the reservoir computer model.

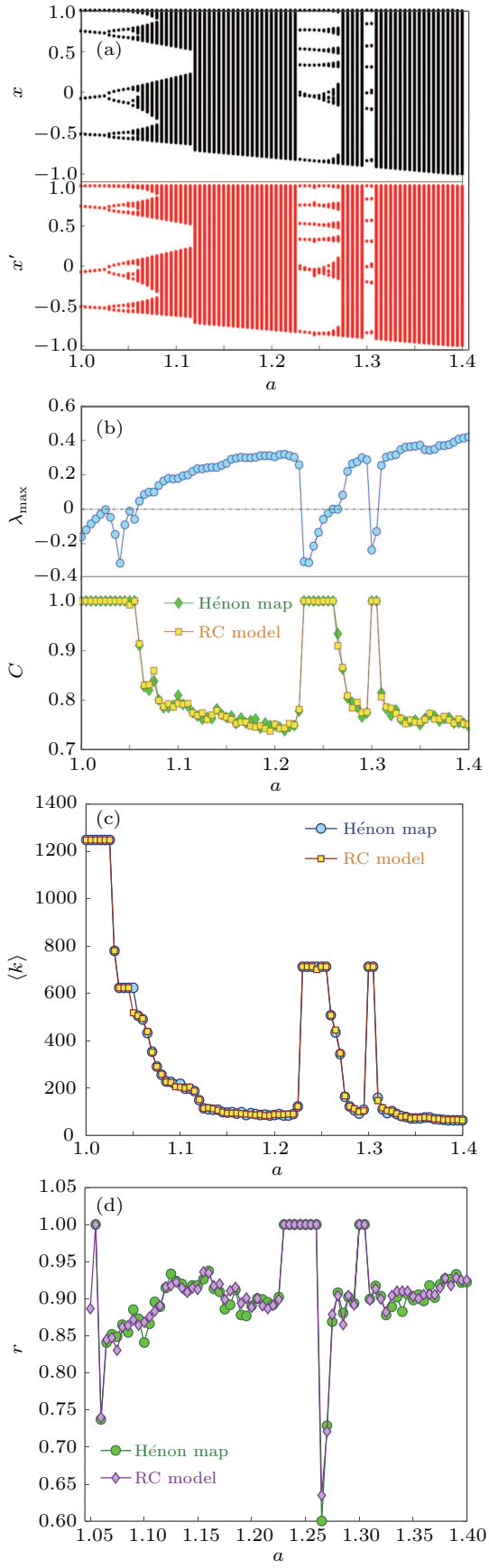


Fig. 4. (a) The bifurcation diagrams of the Hénon map (top panel) and the associated RC model (bottom panel). (b) The maximum Lyapunov exponent $\lambda_{\max}$ and the clustering coefficient C *versus* the parameter a. (c) The mean degree $\langle k\rangle$ and (d) the assortativity coefficient r of recurrence networks obtained with different a.

These results reveal that the topological feature of the trained RC model is not only the same with that of its learned system, but also can be used to dicriminate different dynamical regimes. This is further supported by observing the mean degree $\langle k \rangle$ and the assortativity coefficient r, where they match almost exactly between the Hénon map and the trained RC model, as illustrated in Figs. 4(c) and 4(d). Our findings uncover that from a complex network perspective, the RC model is indistinguishable from that of an observed chaotic system.

In summary, we studied the reservoir computing approach for modelling chaotic systems from a complex network perspective. By transforming their trajectories into recurrence networks, we find that a great variety of network measurements, such as degree distribution, the clustering coefficient, and the assortativity coefficient are almost identical between the trained reservoir system and its learned chaotic system of interest. Remarkably, we show that some statistics commonly used in network science generated from the RC model are sensitive to dynamical transitions and can be in turn used to detect dynamical changes in chaotic systems. Our findings are confirmed on two classical dynamical systems. Our work reveals that from a complex network perspective, reservoir computing approach provides an alternative way for modelling chaotic systems rather than conventional dynamical equations. Moreover, for convenience, the transformation method we have considered here is the recurrence network. The investigation of a broad range of transformation methods, for example cycle network,[13,24] and ordinal partition network,[25] calls for additional research effects.

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