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Furi–Martelli–Vignoli spectrum and Feng spectrum of nonlinear block operator matrices*

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We investigate the Furi–Martelli–Vignoli spectrum and the Feng spectrum of continuous nonlinear block operator matrices, and mainly describe the relationship between the Furi–Martelli–Vignoli spectrum (compared to the Feng spectrum) of the whole operator matrix and that of its entries. In addition, the connection between the Furi–Martelli–Vignoli spectrum of the whole operator matrix and that of its Schur complement is presented by means of Schur decomposition.

Keywords: nonlinear operator, spectrum, block operator matrix

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1. Introduction

Nonlinear operators are powerful tools for the study of nonlinear science. They are of great significance to the study of nonlinear functional analysis, differential equation and integral equation, and are widely applied in many fields such as mathematics, physics, biology and engineering.[1,2] In recent years, some topics of nonlinear operators have been studied in numerous articles.[3–6] In Ref. [7], the authors have discussed the solvability of nonlinear equations using some numerical characteristics of nonlinear operators. In Ref. [8], the authors have studied the uniqueness and existence of fixed points for a class of nonlinear operators, and applied their results to a class of integral equations. In view of the importance of nonlinear operators, spectra of nonlinear operators have been defined and studied by many researchers, and various definitions of spectra for nonlinear operators have been given.[9–18] One of them was introduced by Furi, Martelli and Vignoli in 1978.[15] This Furi–Martelli–Vignoli spectrum (FMV-spectrum, for short) of nonlinear operators shares some properties with the usual spectrum of bounded linear operators, such as it is always closed and sometimes even compact. In Ref. [16], another spectrum which is closely related to the FMV-spectrum was introduced by Feng in 1997. This Feng spectrum has similar topological properties as the FMV-spectrum and contains all the eigenvalues. In particular, these two spectra have various applications to integeral equations, boundary value problems, and bifurcation theory.[11,15,16]

Many problems arising in mathematical physics and biology may be solved by nonlinear block operator matrices. For example, Ben Amar, Jeribi and Krichen[19] introduced and studied the existence of solutions for a coupled system of differential equations under abstract boundary conditions of Rotenberg's model type, this last arises in growing cell populations. This problem is formulated by

$$\begin{pmatrix} -\upsilon\frac{\partial}{\partial\mu}-\sigma_1(\mu,\upsilon,\cdot) & R_{12} \\ R_{21} & -\upsilon\frac{\partial}{\partial\mu}-\sigma_2(\mu,\upsilon,\cdot) \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \lambda\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{1}$$

$$\psi_{i|_{\Gamma_0}} = K_i(\psi_{i|_{\Gamma_1}}), \quad i=1,2, \tag{2}$$

where $R_{ij}\psi_j(\mu,\upsilon) = \int_a^b r_{ij}(\mu,\upsilon,\upsilon',\psi_j(\mu,\upsilon'))\,\mathrm{d}\upsilon'$, $(i,j)\in\{(1,2),(2,1)\}$, $\mu\in[0,1]$, $\upsilon,\upsilon'\in[a,b]$ with $0\le a<b<\infty$. The functions $\sigma_i(\cdot,\cdot,\cdot)$ and $r_{ij}(\cdot,\cdot,\cdot)$ are nonlinear and λ is a complex number, $\Gamma_0=\{0\}\times[a,b]$, $\Gamma_1=\{1\}\times[a,b]$, and K_i are nonlinear operators from a suitable function space on Γ_1 to a similar one on Γ_0; $r_{ij}(\mu,\upsilon,\upsilon',\psi_j(\mu,\upsilon')) = k_{ij}(\mu,\upsilon,\upsilon')f(\mu,\upsilon',\psi_j(\mu,\upsilon'))$, $(i,j)\in\{(1,2),(2,1)\}$, where f is a measurable function defined by $f:[0,1]\times[a,b]\times\mathbb{C}\to\mathbb{C}, (\mu,\upsilon,u)\mapsto f(\mu,\upsilon,u)$ with $k_{ij}(\cdot,\cdot,\cdot), (i,j)\in\{(1,2),(2,1)\}$ are measurable functions from $[0,1]\times[a,b]\times\mathbb{C}$ to $\mathbb{C}$. The entries of block operator matrix associated with this system are nonlinear, and the block operator matrix can be represented by

$$T=\begin{pmatrix}A & B\\ C & D\end{pmatrix}. \tag{3}$$

A lot of mathematicians have studied several kinds of linear operator matrices, and discussed the spectral properties and related subjects of these operator matrices.[20–28] Recently, the operator (3) with nonlinear entries has attracted increasing attention. Researchers have studied some fixed point theorems for the operator (3) on Banach spaces and Banach algebras, and applied these results to some systems of nonlinear

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equations.[19,29–34] However, few results have been reported on the spectral theory for nonlinear operator matrices.

This paper is devoted to studying the Furi–Martelli–Vignoli spectrum and Feng spectrum of continuous nonlinear block operator matrices. We mainly describe the relationship between the Furi–Martelli–Vignoli spectrum (compared to the Feng spectrum) of the whole operator matrix and that of its entries. In addition, the connection between the Furi–Martelli–Vignoli spectrum of the whole operator matrix and that of its Schur complement is presented by means of Schur decomposition.

Throughout this paper, let X and Y be infinite dimensional complex Banach spaces, and $X\times Y$ be a product space equipped with the norm

$$\|(x,y)\| = \max\{\|x\|,\|y\|\}.$$

If F is a continuous operator from X to Y, we write $F\in C(X,Y)$ and, if $X=Y$, $F\in C(X)$. By $L(X,Y)$ denote the set of all bounded linear operators from X to Y, we abbreviate $L(X,X)$ by $L(X)$. The (Kuratowshi) measure of noncompactness $\alpha(M)$ of a bounded set $M\subset X$ is defined as the infimum of all $\varepsilon>0$ such that M can be covered by finitely many sets of diameter less than ε. Given $F\in C(X,Y)$ and a bounded set $M\subset X$, the upper and lower measures of noncompactness of F are defined as[35]

$$[F]_A = \inf\{k : \alpha(F(M)) \le k\alpha(M)\},$$
$$[F]_a = \sup\{k : \alpha(F(M)) \ge k\alpha(M)\}.$$

We can write $F\in U(X,Y)$ if $[F]_A<\infty$. In the case of $X=Y$ we can simply write $U(X)$ instead of $U(X,X)$. Observe that $[F]_A=0$ if and only if F is compact, i.e., the image $F(M)$ of each bounded set $M\subset X$ is precompact in Y. The upper and lower quasinorms of F are defined as[11]

$$[F]_Q = \limsup_{\|x\|\to\infty}\frac{\|F(x)\|}{\|x\|},\quad [F]_q = \liminf_{\|x\|\to\infty}\frac{\|F(x)\|}{\|x\|}.$$

If $[F]_Q<\infty$, we call F quasibounded. Using $Q(X,Y)$ denoting the set of all quasibounded operators from X to Y, as before, for $X=Y$ we can simply write $Q(X,X):=Q(X)$. Moreover, $[F]_q>0$ implies that F is coercive, i.e. ,

$$\lim_{\|x\|\to\infty}\|F(x)\| = \infty.$$

We also consider

$$[F]_B = \sup_{x\neq 0}\frac{\|F(x)\|}{\|x\|},\quad [F]_b = \inf_{x\neq 0}\frac{\|F(x)\|}{\|x\|}.$$

In the case of $[F]_B<\infty$, we can write $F\in B(X,Y)$ and call the operator F linearly bounded, and $B(X,X):=B(X)$ as before. From the definition and our general assumption $F(0)=0$ it follows that

$$[F]_b \le [F]_q \le [F]_Q \le [F]_B.$$

2. The Furi–Martelli–Vignoli spectrum of nonlinear block operator matrices

In this section, we first introduce some definitions and some auxiliary lemmas which will be used later.

Definition 2.1[36] A continuous operator $F: X\to Y$ can be stably solvable if for any continuous compact operator $G: X\to Y$ with $[G]_Q=0$, the equation $F(x)=G(x)$ has a solution $x\in X$.

Remark 2.1 Each stably solvable operator is surjective, but in general the converse is not true. For linear operators, however, surjectivity is equivalent to stable solvability.

Definition 2.2[15] A continuous operator $F: X\to X$ is FMV-regular if F is stably solvable, $[F]_a>0$ and $[F]_q>0$. The set

$$\rho_{\rm FMV}(F) = \{\lambda\in\mathbb{C} : F-\lambda I \text{ is FMV-regular}\}$$

is called the FMV-resolvent set of F, and its complement

$$\sigma_{\rm FMV}(F) = \mathbb{C}\setminus\rho_{\rm FMV}(F)$$

is called the FMV-spectrum of F. Also, the sets

$$\sigma_\delta(F) = \{\lambda\in\mathbb{C} : F-\lambda I \text{ is not stably solvabl}\},$$

$$\sigma_\pi(F) = \sigma_q(F)\cup\sigma_a(F)$$

are called the defect spectrum and approximate point spectrum of F, respectively, where

$$\sigma_q(F) = \{\lambda\in\mathbb{C} : [F-\lambda I]_q = 0\},$$
$$\sigma_a(F) = \{\lambda\in\mathbb{C} : [F-\lambda I]_a = 0\}.$$

Obviously

$$\sigma_{\rm FMV}(F) = \sigma_\delta(F)\cup\sigma_\pi(F) = \sigma_\delta(F)\cup\sigma_a(F)\cup\sigma_q(F).$$

Remark 2.2 In the case of a bounded linear operator F, $\sigma_\pi(F)$ is the approximate point spectrum of F, $\sigma_\delta(F)$ is the defect spectrum of F, and $\sigma_{\rm FMV}(F)$ is the usual spectrum of F.

Lemma 2.1[15] Let X and Y be Banach spaces and $M\subset X$, $N\subset Y$ be bounded subsets. Consider $X\times Y$ with the norm

$$\|(x,y)\| = \max\{\|x\|,\|y\|\}.$$

Then

$$\alpha(M\times N) = \max\{\alpha(M),\alpha(N)\}.$$

Lemma 2.2[15] Let X and Y be Banach spaces, $F,G\in C(X,Y)$. Then

(1) $[F]_a[G]_a \le [FG]_a \le [F]_A[G]_a$;
(2) $[F]_q[G]_q \le [FG]_q \le [F]_Q[G]_q$;
(3) $[F]_a - [G]_A \le [F+G]_a \le [F]_a + [G]_A$;
(4) $[F]_q - [G]_Q \le [F+G]_q \le [F]_q + [G]_Q$.

Lemma 2.3[11] Let X, Y and Z be Banach spaces, and let $F: X \to Y$ be stably solvable. Suppose that $G: Z \to X$ is continuous and surjective with a quasibounded right inverse $\widehat{G}: X \to Z$. Then $FG: Z \to Y$ is stably solvable.

Lemma 2.4[11] Let X, Y and Z be Banach spaces, and let $F: X \to Y$ be stably solvable. Suppose that $G: Y \to Z$ is continuous and surjective with a quasibounded right inverse $\widehat{G}: Z \to Y$. Then $GF: X \to Z$ is stably solvable.

Lemma 2.5[11] Let X be a Banach space, $F \in C(X)$. Suppose that F satisfies $[F]_A < 1$ and $[F]_Q < 1$. Then $I - F$ is surjective; in particular, F has a fixed point in X.

Next, we describe the relationship between the FMV-spectrum of the whole operator matrix and its entries.

Theorem 2.1 Let $F = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in C(X \times X)$ be a diagonal block operator matrix with $A(0) = D(0) = 0$. Then

(i) $\sigma_a(F) = \sigma_a(A) \cup \sigma_a(D)$;

(ii) $\sigma_q(F) = \sigma_q(A) \cup \sigma_q(D)$;

(iii) $\sigma_\delta(F) \supset \sigma_\delta(A) \cup \sigma_\delta(D)$;

(iv) $\sigma_{\mathrm{FMV}}(A) \cup \sigma_{\mathrm{FMV}}(D) \subset \sigma_{\mathrm{FMV}}(F) \subset \sigma_{\mathrm{FMV}}(A) \cup \sigma_{\mathrm{FMV}}(D) \cup \sigma_+(A) \cup \sigma_+(D)$,

where $\sigma_+(\cdot) = \{\lambda \in \mathbb{C} : \cdot - \lambda \text{ is not injective or } (\cdot - \lambda)^{-1} \text{ is not continuous}\}$.

Proof (i) Let $\lambda \in \sigma_a(A)$, then there exists a bounded sequence set $M_n \subset X$ with $\alpha(M_n) \neq 0$, $n \in \mathbb{N}$, such that

$$\frac{\alpha((A-\lambda)(M_n))}{\alpha(M_n)} \to 0 \quad \text{as } n \to \infty.$$

Set $Q_n = (M_n \ \ 0)^{\mathrm{T}} \subset X \times X, n \in \mathbb{N}$. Clearly, $\alpha(Q_n) = \alpha(M_n)$ and $\alpha((F-\lambda)(Q_n)) = \alpha((A-\lambda)(M_n))$. Thus

$$\frac{\alpha((F-\lambda)(Q_n))}{\alpha(Q_n)} = \frac{\alpha((A-\lambda)(M_n))}{\alpha(M_n)} \to 0 \quad \text{as } n \to \infty,$$

i.e., $\lambda \in \sigma_a(F)$. By a similar argument, we can show that $\sigma_a(D) \subset \sigma_a(F)$.

Conversely, let $\lambda \in \sigma_a(F)$, then there exists a bounded sequence set $Q_n = (M_n \ \ N_n)^{\mathrm{T}} \subset X \times X$ with $\alpha(Q_n) \neq 0$, $n \in \mathbb{N}$, such that

$$\frac{\alpha((F-\lambda)(Q_n))}{\alpha(Q_n)} \to 0 \quad \text{as } n \to \infty.$$

From Lemma 2.1, we know that $\alpha(Q_n) = \max\{\alpha(M_n), \alpha(N_n)\}$. Without loss of generality we may assume that $\alpha(Q_n) = \alpha(M_n)$ for infinitely many indices n, then

$$\frac{\alpha((A-\lambda)(M_n))}{\alpha(M_n)} \leq \frac{\max(\alpha(A-\lambda)(M_n), \alpha(D-\lambda)(N_n))}{\alpha(Q_n)} = \frac{\alpha((F-\lambda)(Q_n))}{\alpha(Q_n)}.$$

Thus

$$\frac{\alpha((A-\lambda)(M_n))}{\alpha(M_n)} \to 0 \quad \text{as } n \to \infty,$$

i.e., $\lambda \in \sigma_a(A)$. The proof for $\lambda \in \sigma_a(D)$ is similar. Therefore, $\sigma_a(F) = \sigma_a(A) \cup \sigma_a(D)$.

(ii) Let $\lambda \in \sigma_q(A)$, then there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with $\|x_n\| \to \infty$, such that

$$\frac{\|(A-\lambda)(x_n)\|}{\|x_n\|} \to 0 \quad \text{as } n \to \infty.$$

Set $z_n = (x_n \ \ 0)^{\mathrm{T}} \in X \times X$, then $\|z_n\| \to \infty$, and

$$\frac{\|(F-\lambda)(z_n)\|}{\|z_n\|} = \frac{\|(A-\lambda)(x_n)\|}{\|x_n\|} \to 0 \quad \text{as } n \to \infty,$$

i.e., $\lambda \in \sigma_q(F)$. Similarly, we can show that $\sigma_q(D) \subset \sigma_q(F)$.

Conversely, let $\lambda \in \sigma_q(F)$, then there exists a sequence $\{z_n\}_{n=1}^{\infty} = \{(x_n \ \ y_n)^{\mathrm{T}}\}_{n=1}^{\infty} \subset X \times X$ with $\|z_n\| \to \infty$, such that

$$\frac{\|(F-\lambda)(z_n)\|}{\|z_n\|} \to 0 \quad \text{as } n \to \infty.$$

Without loss of generality we may assume that $\|x_n\| \geq \|y_n\|$ for infinitely many indices n, then $\|x_n\| \to \infty$, and

$$\frac{\|(F-\lambda)(z_n)\|}{\|z_n\|} = \frac{\max\{\|(A-\lambda)(x_n)\|, \|(D-\lambda)(y_n)\|\}}{\max\{\|x_n\|, \|y_n\|\}} \geq \frac{\|(A-\lambda)(x_n)\|}{\|x_n\|}.$$

Thus

$$\frac{\|(A-\lambda)(x_n)\|}{\|x_n\|} \to 0 \quad \text{as } n \to \infty,$$

i.e., $\lambda \in \sigma_q(A)$. The proof for $\lambda \in \sigma_q(D)$ is similar. Consequently, $\sigma_q(F) = \sigma_q(A) \cup \sigma_q(D)$.

(iii) To show that $\sigma_\delta(F) \supset \sigma_\delta(A) \cup \sigma_\delta(D)$, it suffices to show that if F is stably solvable then so are A and B. If F is stably solvable, let $H_1: X \to X$ and $H_2: X \to X$ are continuous compact operators with $[H_1]_Q = 0$ and $[H_2]_Q = 0$. Define an operator $H: X \times X \to X \times X$ by

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}.$$

Clearly, H is a compact operator and $[H]_Q = 0$. Then

$$F \begin{pmatrix} x \\ y \end{pmatrix} = H \begin{pmatrix} x \\ y \end{pmatrix}$$

has a solution $(x \ \ y)^{\mathrm{T}} \in X \times X$, i.e.,

$$A(x) = H_1(x) \ \text{ and } \ D(y) = H_2(y),$$

which implies that both of A and D are stably solvable.

(iv) From (i)–(iii), it is clear that $\sigma_{\mathrm{FMV}}(A) \cup \sigma_{\mathrm{FMV}}(D) \subset \sigma_{\mathrm{FMV}}(F)$, now we prove that $\sigma_{\mathrm{FMV}}(F) \subset \sigma_{\mathrm{FMV}}(A) \cup \sigma_{\mathrm{FMV}}(D) \cup \sigma_+(A) \cup \sigma_+(D)$. Let $\lambda \notin \sigma_{\mathrm{FMV}}(A) \cup \sigma_{\mathrm{FMV}}(D) \cup \sigma_+(A) \cup \sigma_+(D)$, then $[A-\lambda]_a > 0$, $[A-\lambda]_q > 0$, $[D-\lambda]_a > 0$, $[D-\lambda]_q > 0$ and $A-\lambda$, $D-\lambda$ are homeomorphisms on X, therefore $[F-\lambda]_a > 0$, $[F-\lambda]_q > 0$ and $F-\lambda$ is a homeomorphism on $X \times X$. Let $H: X \times X \to X \times X$ is a continuous compact operator with $[H]_Q = 0$, then

$$[(F-\lambda)^{-1}H]_A \leq [(F-\lambda)^{-1}]_A [H]_A = 0$$

and

$$[(F-\lambda)^{-1}H]_Q \le [(F-\lambda)^{-1}]_Q[H]_Q = 0.$$

From Lemma 2.5, it follows that $(F-\lambda)^{-1}H$ has a fixed point in $X\times X$, and hence $F-\lambda$ is stably solvable. This together with $[F-\lambda]_a>0$ and $[F-\lambda]_q>0$ implies that $\lambda\notin\sigma_{\rm FMV}(F)$.

Corollary 2.1 Let $F=\begin{pmatrix} A & 0\\ 0 & D\end{pmatrix}\in C(X\times X)$ be a diagonal block operator matrix with $A(0)=D(0)=0$. Then

$$\sigma_{\rm FMV}(A)\cup\sigma_{\rm FMV}(D)=\sigma_{\rm FMV}(F)$$

if and only if

$$\sigma_+(A)\cup\sigma_+(D)\subset\sigma_{\rm FMV}(F).$$

In particular, if $\sigma_+(A)\cup\sigma_+(D)=\varnothing$, then $\sigma_{\rm FMV}(A)\cup\sigma_{\rm FMV}(D)=\sigma_{\rm FMV}(F)$.

Remark 2.3 For bounded linear diagonal operator matrices, obviously, $\sigma_+(A)\cup\sigma_+(D)\subset\sigma_{\rm FMV}(F)$.

Theorem 2.2 Let $F=\begin{pmatrix} A & B\\ 0 & D\end{pmatrix}\in C(X\times X)$ be an upper triangular block operator matrix with $B(0)=D(0)=0$. If $B\in U(X)\cap Q(X)$, then

(i) $\sigma_a(A)\subset\sigma_a(F)\subset\sigma_a(A)\cup\sigma_a(D)$;

(ii) $\sigma_q(A)\subset\sigma_q(F)\subset\sigma_q(A)\cup\sigma_q(D)$.

Proof (i) It is easy to see that $\sigma_a(A)\subset\sigma_a(F)$, and so we only need to prove that $\sigma_a(F)\subset\sigma_a(A)\cup\sigma_a(D)$. Let $\lambda\in\sigma_a(F)$, then $[F-\lambda]_a=0$. Evidently, the factorization formula

$$F-\lambda=\begin{pmatrix} I & 0\\ 0 & D-\lambda\end{pmatrix}\begin{pmatrix} I & B\\ 0 & I\end{pmatrix}\begin{pmatrix} A-\lambda & 0\\ 0 & I\end{pmatrix}$$

holds. Write $U=\left(\begin{smallmatrix} I & 0\\ 0 & D-\lambda\end{smallmatrix}\right)$, $R=\left(\begin{smallmatrix} I & B\\ 0 & I\end{smallmatrix}\right)$ and $V=\left(\begin{smallmatrix} A-\lambda & 0\\ 0 & I\end{smallmatrix}\right)$. Clearly, R is a homeomorphism and $[R]_a>0$. Since $[F-\lambda]_a=[URV]_a\ge[U]_a[R]_a[V]_a$, it follows that $[U]_a=0$ or $[V]_a=0$, and hence $\lambda\in\sigma_a(A)\cup\sigma_a(D)$. Thus, $\sigma_a(F)\subset\sigma_a(A)\cup\sigma_a(D)$.

(ii) The proof is analogous to (i), and its proof is omitted here.

Remark 2.4 From the proof of Theorem 2.2, we can see that if B is a bounded linear operator, Theorem 2.2 also holds. Indeed, since $R=\left(\begin{smallmatrix} I & B\\ 0 & I\end{smallmatrix}\right)$ is invertible, and hence $[R]_a>0$ and $[R]_q>0$ (Since for bounded linear operator L, $[L]_a>0$ if and only if L is left semi-Fredholm, $[L]_q>0$ if and only if L is injective with $R(L)$ closed).

It is easy to get the following corollary.

Corollary 2.2 Let $F=\begin{pmatrix} A & B\\ 0 & D\end{pmatrix}\in C(X\times X)$ be an upper triangular block operator matrix with $B(0)=D(0)=0$. If $B\in U(X)\cap Q(X)$, then

$$\sigma_\pi(A)\subset\sigma_\pi(F)\subset\sigma_\pi(A)\cup\sigma_\pi(D).$$

According to Corollary 2.2, we can estimate the distribution range of set $(\sigma_\pi(A)\cup\sigma_\pi(D))\backslash\sigma_\pi(F)$, see the following theorem.

Theorem 2.3 Let $F=\begin{pmatrix} A & B\\ 0 & D\end{pmatrix}\in C(X\times X)$ be an upper triangular block operator matrix with $B(0)=D(0)=0$. If $B\in U(X)\cap Q(X)$, then

$$\sigma_\pi(A)\cup\sigma_\pi(D)=\sigma_\pi(F)\cup(\sigma_\pi(D)\cap\Delta_1),$$

where $\Delta_1=\{\lambda\in\mathbb{C}:[A-\lambda]_a>0,[A-\lambda]_q>0$ and $A-\lambda$ is not stably solvable$\}$.

Proof It is easy to prove that

$$\sigma_\pi(A)\cup\sigma_\pi(D)\supset\sigma_\pi(F)\cup(\sigma_\pi(D)\cap\Delta_1).$$

Conversely, let $\lambda\in(\sigma_\pi(A)\cup\sigma_\pi(D))\backslash\sigma_\pi(F)$, then we have from Corollary 2.2 that $\lambda\in\sigma_\pi(D)\backslash\sigma_\pi(A)$, and so

$$\lambda\in\Delta_1\cup\rho_{\rm FMV}(A).$$

Suppose $\lambda\in\rho_{\rm FMV}(A)$, then $[A-\lambda]_a>0,[A-\lambda]_q>0$ and $A-\lambda$ is stably solvable, and hence $A-\lambda$ is surjective. From this, we can deduce that $[D-\lambda]_a>0$ and $[D-\lambda]_q>0$. In fact, let any bounded set $N\subset X$ with $\alpha(N)\ne0$. Since $-B(N)\in X=R(A-\lambda)$, then there exists a set $M\subset X$, such that

$$(A-\lambda)(M)=-B(N).$$

From $[A-\lambda]_q>0$, we infer that M is a bounded set. Thus, let $Q=(M\ \ N)^{\rm T}\in X\times X$, then Q is bounded with $\alpha(Q)\ne0$, and we have

$$\frac{\alpha((D-\lambda)(N))}{\alpha(N)}\ge\frac{\alpha((F-\lambda)(Q))}{\alpha(Q)}\ge[F-\lambda]_a>0,$$

which implies that $[D-\lambda]_a>0$. We can obtain $[D-\lambda]_q>0$ by a similar method. This contradicts $\lambda\in\sigma_\pi(D)$. Then $\lambda\in\Delta_1$, therefore

$$(\sigma_\pi(A)\cup\sigma_\pi(D))\backslash\sigma_\pi(F)\subset\sigma_\pi(D)\cap\Delta_1.$$

Remark 2.5 In Theorem 2.3, if further $A\in L(X)$, then

$$\sigma_\pi(A)\cup\sigma_\pi(D)=\sigma_\pi(F)\cup(\sigma_\pi(D)\cap\sigma_{r,1}(A)),$$

where $\sigma_{r,1}(A)=\{\lambda\in\mathbb{C}:A-\lambda$ is injective, $\overline{R(A-\lambda)}\ne X$, and $R(A-\lambda)$ is closed$\}$.

Proof From the proof of Theorem 2.3, it can be seen that when $\lambda\in\sigma_\pi(D)\backslash\sigma_\pi(A)$, we have

$$\lambda\in\sigma_{r,1}(A)\cup\rho(A).$$

Suppose $\lambda\in\rho(A)$, then $A-\lambda$ is invertible, and we have the following factorization

$$M=(F-\lambda)V,$$

where

$$M=\begin{pmatrix} A-\lambda & 0\\ 0 & D-\lambda\end{pmatrix},\quad V=\begin{pmatrix} I & -(A-\lambda)^{-1}B\\ 0 & I\end{pmatrix}.$$

Since $\lambda \notin \sigma_\pi(F)$, then

$$[M]_a = [(F-\lambda)V]_a \geq [F-\lambda]_a[V]_a > 0$$

and

$$[M]_q = [(F-\lambda)V]_q \geq [F-\lambda]_q[V]_q > 0,$$

therefore $[D-\lambda]_a > 0$ and $[D-\lambda]_q > 0$, so the desired result follows from the proof of Theorem 2.3.

According to Theorem 2.3 and Remark 2.5, we immediately have the following corollaries.

Corollary 2.3 Let $F = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in C(X \times X)$ be an upper triangular block operator matrix with $B(0) = D(0) = 0$. If $B \in U(X) \cap Q(X)$, then

$$\sigma_\pi(A) \cup \sigma_\pi(D) = \sigma_\pi(F)$$

if and only if

$$\sigma_\pi(D) \cap \Delta_1 \subset \sigma_\pi(F).$$

In particular, if $\sigma_\pi(D) \cap \Delta_1 = \varnothing$, then $\sigma_\pi(A) \cup \sigma_\pi(D) = \sigma_\pi(F)$.

Corollary 2.4 Let $F = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in C(X \times X)$ be an upper triangular block operator matrix with $B(0) = D(0) = 0$. If $A \in L(X)$ and $B \in U(X) \cap Q(X)$, then

$$\sigma_\pi(A) \cup \sigma_\pi(D) = \sigma_\pi(F)$$

if and only if

$$\sigma_\pi(D) \cap \sigma_{r,1}(A) \subset \sigma_\pi(F).$$

In particular, if $\sigma_\pi(D) \cap \sigma_{r,1}(A) = \varnothing$, then $\sigma_\pi(A) \cup \sigma_\pi(D) = \sigma_\pi(F)$.

Remark 2.6 For linear (not necessary bounded) upper triangular operator matrices, Theorem 2.3, Corollaries 2.3 and 2.4 are also valid, where $\Delta_1 = \sigma_{r,1}(A)$. Readers can see Refs. [24].

In Ref. [21], it was shown that if N and M are bounded linear operators, and satisfy $N^2 = 0$ and either $NM = 0$ or $MN = 0$, then

$$\sigma(M) = \sigma(M+N) \tag{4}$$

holds. In the following, we extended the result to the case that M is a nonlinear operator.

Theorem 2.4 Let $M \in C(X)$ and $N \in L(X)$. If $N^2 = 0$ and $NM = 0$, then

(i) $\sigma_a(M+N)\backslash\{0\} = \sigma_a(M)\backslash\{0\}$;

(ii) $\sigma_q(M+N)\backslash\{0\} = \sigma_q(M)\backslash\{0\}$;

(iii) $\sigma_\delta(M+N) = \sigma_\delta(M)$;

(iv) $\sigma_{\mathrm{FMV}}(M+N) = \sigma_{\mathrm{FMV}}(M)$.

Proof If $N = 0$, it is obvious that (i)–(iv) hold. Without loss of generality we may assume that $N \neq 0$. Let $\lambda \in \mathbb{C}\backslash\{0\}$, from $NM = N^2 = 0$, we have that $N - \lambda$ is invertible and

$$-\lambda^{-1}(N-\lambda)(M-\lambda) = N + M - \lambda. \tag{5}$$

Since $[M+N-\lambda]_a = [-\lambda^{-1}(N-\lambda)(M-\lambda)]_a \geq |\lambda|^{-1}[N-\lambda]_a[M-\lambda]_a$ and $[M-\lambda]_a = [-\lambda(N-\lambda)^{-1}(M+N-\lambda)]_a \geq |\lambda|[(N-\lambda)^{-1}]_a[M+N-\lambda]_a$, we conclude that $[M-\lambda]_a = 0$ if and only if $[M+N-\lambda]_a = 0$. Thus $\sigma_a(M+N)\backslash\{0\} = \sigma_a(M)\backslash\{0\}$. Similarly, we can obtain that $\sigma_q(M+N)\backslash\{0\} = \sigma_q(M)\backslash\{0\}$. Thus we complete the proof of (i) and (ii).

(iii) From $NM = 0$ and $N \neq 0$, we have

$$0 \in \sigma_\delta(M).$$

Also, from $N(M+N) = 0$ and $N \neq 0$, we know that

$$0 \in \sigma_\delta(M+N),$$

i.e., $0 \in \sigma_\delta(M) \cap \sigma_\delta(M+N)$. Next let $\lambda \in \mathbb{C}\backslash\{0\}$. By Eq. (5) and Lemma 2.4, we infer that $M+N-\lambda$ is stably solvable if and only if $M-\lambda$ is stably solvable. Thus

$$\sigma_\delta(M+N) = \sigma_\delta(M).$$

(iv) From (i)–(iii), we obtain that $\sigma_{\mathrm{FMV}}(M+N)\backslash\{0\} = \sigma_{\mathrm{FMV}}(M)\backslash\{0\}$. Moreover, similar to the proof of (iii), we can obtain $0 \in \sigma_{\mathrm{FMV}}(M) \cap \sigma_{\mathrm{FMV}}(M+N)$. Hence

$$\sigma_{\mathrm{FMV}}(M+N) = \sigma_{\mathrm{FMV}}(M).$$

Corollary 2.5 Let $F = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in C(X \times X)$ be an upper triangular block operator matrix with $A(0) = D(0) = 0$. If $B \in L(X)$ and $BD = 0$, then

(i) $\sigma_a(F)\backslash\{0\} = (\sigma_a(A) \cup \sigma_a(D))\backslash\{0\}$;

(ii) $\sigma_q(F)\backslash\{0\} = (\sigma_q(A) \cup \sigma_q(D))\backslash\{0\}$;

(iii) $\sigma_\delta(F) \supset \sigma_\delta(A) \cup \sigma_\delta(D)$;

(iv) $\sigma_{\mathrm{FMV}}(F) \supset \sigma_{\mathrm{FMV}}(A) \cup \sigma_{\mathrm{FMV}}(D)$.

Proof Let $M = \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right), N = \left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$, then $N^2 = 0$ and $NM = 0$. Thus, by Theorem 2.4, we know that $\sigma_k(F)\backslash\{0\} = \sigma_k(M)\backslash\{0\}$ (where $k \in \{a,q\}$) and $\sigma_s(F) = \sigma_s(M)$ (where $s \in \{\delta, FMV\}$). Thus, the desired results can be obtained immediately by applying Theorem 2.1.

Let us give an example to show that the statement (ii) of Corollary 2.5, in general, is not true when the spectral point is zero.

Example 2.1 Let X be an infinite dimensional complex Banach space with the maximum norm, $e = (1,0,0,\ldots) \in X$, and

$$A_1(x) = A_2(x) = D_1(x) = \|x\|e, \quad S_r(x) = (0, x_1, x_2, \cdots)$$

for $x = (x_1, x_2, \ldots) \in X$. Consider the block operator matrix

$$F = \begin{pmatrix} A_1 & 0 & S_r & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_1 \end{pmatrix},$$

where

$$A=\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},\ B=\begin{pmatrix} S_r & 0 \\ 0 & 0 \end{pmatrix},\ D=\begin{pmatrix} 0 & 0 \\ 0 & D_1 \end{pmatrix}.$$

Note that B is a bounded linear operator with $BD=0$. Moreover, by calculation, we have $[A]_q=1,[D]_q=0,[F]_q=\frac{1}{6}$. Thus, $0\in\sigma_q(A)\cup\sigma_q(D)$, but $0\notin\sigma_q(F)$.

Theorem 2.5 Let $M,N\in C(X)$.

(i) If $[N]_A=0$, then $\sigma_a(M+N)=\sigma_a(M)$.

(ii) If $[N]_Q=0$, then $\sigma_q(M+N)=\sigma_q(M)$.

(iii) If $[N]_A=0$ and $[N]_Q=0$, then $\sigma_\delta(M+N)=\sigma_\delta(M)$.

(iv) If $[N]_A=0$ and $[N]_Q=0$, then $\sigma_{\rm FMV}(M+N)=\sigma_{\rm FMV}(M)$.

Proof The proof is trivial, and its proof is omitted here.

Corollary 2.6 Let $F=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\in C(X\times X)$ be an upper triangular block operator matrix with $A(0)=D(0)=0$.

(i) If $[B]_A=0$, then $\sigma_a(F)=\sigma_a(A)\cup\sigma_a(D)$.

(ii) If $[B]_A=0$ and $[B]_Q=0$, then $\sigma_q(F)=\sigma_q(A)\cup\sigma_q(D)$.

(iii) If $[B]_Q=0$ and $[B]_A=0$, then $\sigma_\delta(F)\supset\sigma_\delta(A)\cup\sigma_\delta(D)$.

(iv) If $[B]_Q=0$ and $[B]_A=0$, then $\sigma_{\rm FMV}(F)\supset\sigma_{\rm FMV}(A)\cup\sigma_{\rm FMV}(D)$.

Proof Let $M=\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$, $N=\left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$, then $[N]_A=0$ and $[N]_Q=0$. Thus, by Theorem 2.5, we know that $\sigma_k(F)=\sigma_k(M)$ (where $k\in\{a,q,\delta,FMV\}$), and by applying Theorem 2.1, the desired results can be obtained immediately.

In the theory of bounded and unbounded linear block operator matrices, Schur complements are powerful tools to study the spectrum and various spectral properties, see the series of papers.[37–39] Given a linear block operator matrix

$$\mathscr{A}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

defined on the product Banach space $X_1\times X_2$ of two Banach spaces X_1,X_2. Then

$$\mathscr{A}-\lambda=\begin{pmatrix} I & 0 \\ C(A-\lambda)^{-1} & I \end{pmatrix}\begin{pmatrix} A-\lambda & 0 \\ 0 & S_1(\lambda) \end{pmatrix}\times\begin{pmatrix} I & (A-\lambda)^{-1}B \\ 0 & I \end{pmatrix},\quad \lambda\notin\sigma(A) \tag{6}$$

and

$$\mathscr{A}-\lambda=\begin{pmatrix} I & B(D-\lambda)^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} S_2(\lambda) & 0 \\ 0 & D-\lambda \end{pmatrix}\times\begin{pmatrix} I & 0 \\ (D-\lambda)^{-1}C & I \end{pmatrix},\quad \lambda\notin\sigma(D) \tag{7}$$

are called the Frobenius–Schur factorization, where

$$S_1(\lambda)=D-\lambda-C(A-\lambda)^{-1}B,\ \ \lambda\notin\sigma(A)$$

and

$$S_2(\lambda)=A-\lambda-B(D-\lambda)^{-1}C,\ \ \lambda\notin\sigma(D)$$

are called the Schur complements of $\mathscr{A}$.

Note that when A and C (or B and D) are bounded linear operators, the nonlinear block operator matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ can be decomposed into the forms of Eq. (6) (or (7)). This allows us to extend the notion of the Frobenius–Schur factorization to the nonlinear block operator matrix case.

In the following, we consider the relationship between the FMV-spectrum of the whole operator matrix and its Schur complement by means of Schur decomposition.

Theorem 2.6 Let $F=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in C(X\times X)$ be a block operator matrix. If $B\in U(X)\cap Q(X)$ and $A,C\in L(X)$ with $\rho(A)\neq\varnothing$, then for $\lambda\in\rho(A)$, the following cases will hold:

(i) $\lambda\in\sigma_a(F)$ if and only if $0\in\sigma_a(S_1(\lambda))$; (ii) $\lambda\in\sigma_q(F)$ if and only if $0\in\sigma_q(S_1(\lambda))$; (iii) if $0\in\sigma_\delta(S_1(\lambda))$, then $\lambda\in\sigma_\delta(F)$; (iv) if $0\in\sigma_{\rm FMV}(S_1(\lambda))$, then $\lambda\in\sigma_{\rm FMV}(F)$;

where $S_1(\lambda)=D-\lambda-C(A-\lambda)^{-1}B$.

Proof Since $\lambda\in\rho(A)$, then $F-\lambda$ has the factorization (6). Denoted it by

$$F-\lambda=URV,$$

where $U=\left(\begin{smallmatrix} I & 0 \\ C(A-\lambda)^{-1} & I \end{smallmatrix}\right)$, $R=\left(\begin{smallmatrix} A-\lambda & 0 \\ 0 & S_1(\lambda) \end{smallmatrix}\right)$ and $V=\left(\begin{smallmatrix} I & (A-\lambda)^{-1}B \\ 0 & I \end{smallmatrix}\right)$. Obviously, U and V are homeomorphisms on $X\times X$.

(i) Let $\lambda\in\sigma_a(F)$, then $[F-\lambda]_a=0$. Since $[U]_a>0$, $[V]_a>0$ and

$$[F-\lambda]_a=[URV]_a\geq[U]_a[R]_a[V]_a,$$

it follows that $[R]_a=0$, and hence $[S_1(\lambda)]_a=0$, i.e., $0\in\sigma_a(S_1(\lambda))$. Conversely, let $0\in\sigma_a(S_1(\lambda))$, then $[R]_a=0$. Since $[U^{-1}]_a>0$, $[V^{-1}]_a>0$ and

$$[R]_a=[U^{-1}(F-\lambda)V^{-1}]_a\geq[U^{-1}]_a[F-\lambda]_a[V^{-1}]_a,$$

then we have that $[F-\lambda]_a=0$, i.e., $\lambda\in\sigma_a(F)$.

(ii) The proof is similar to (i).

(iii) Since U,V,U^{-1} and V^{-1} are all quasibounded. Then, by Lemma 2.3 and Lemma 2.4, we obtain that $F-\lambda$ is stably solvable if and only if R is stably solvable. Thus, if $F-\lambda$ is stably solvable, then applying Theorem 2.1, we can infer that $S_1(\lambda)$ is stably solvable. So we complete the proof.

(iv) It follows immediately from (i)–(iii) and the definition of the FMV-spectrum.

Theorem 2.7 Let $F=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in C(X\times X)$ be a block operator matrix. If $C\in U(X)\cap Q(X)$ and $B,D\in L(X)$ with $\rho(D)\neq\varnothing$, then for $\lambda\in\rho(D)$, the following cases hold:

(i) $\lambda\in\sigma_a(F)$ if and only if $0\in\sigma_a(S_2(\lambda))$;

(ii) $\lambda\in\sigma_q(F)$ if and only if $0\in\sigma_q(S_2(\lambda))$;

(iii) if $0\in\sigma_\delta(S_2(\lambda))$, then $\lambda\in\sigma_\delta(F)$;

(iv) if $0\in\sigma_{\rm FMV}(S_2(\lambda))$, then $\lambda\in\sigma_{\rm FMV}(F)$;

where $S_2(\lambda)=A-\lambda-B(D-\lambda)^{-1}C$.

Proof The proof is analogous to that of Theorem 2.6, we only need to note that when $\lambda \in \rho(D)$, $F-\lambda$ has the factorization (7).

From the proofs of Theorems 2.6 and 2.7, we find the following theorem.

Theorem 2.8 Let $F=\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in C(X\times X)$ be an anti-triangular block operator matrix, if $B\in L(X), C\in U(X)\cap Q(X)$, then for $\lambda\neq 0$, the following cases hold:

(i) $\lambda\in\sigma_a(F)$ if and only if $0\in\sigma_a(T_1(\lambda))$;

(ii) $\lambda\in\sigma_q(F)$ if and only if $0\in\sigma_q(T_1(\lambda))$;

(iii) if $0\in\sigma_\delta(T_1(\lambda))$, then $\lambda\in\sigma_\delta(F)$;

(iv) if $0\in\sigma_{\rm FMV}(T_1(\lambda))$, then $\lambda\in\sigma_{\rm FMV}(F)$;

where $T_1(\lambda)=A-\lambda+\frac{1}{\lambda}BC, \lambda\neq 0$.

Proof Since $F-\lambda=\begin{pmatrix} A-\lambda & B \\ C & -\lambda I \end{pmatrix}$, λI is invertible as $\lambda\neq 0$. Then $F-\lambda$ has the following factorization

$$F-\lambda=\begin{pmatrix} I & -\frac{1}{\lambda}B \\ 0 & I \end{pmatrix}\begin{pmatrix} T_1(\lambda) & 0 \\ 0 & -\lambda I \end{pmatrix}\begin{pmatrix} I & 0 \\ -\frac{1}{\lambda}C & I \end{pmatrix}.$$

The rest of the proof is analogous to that in Theorem 2.6.

Theorem 2.9 Let $F=\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in C(X\times X)$ be an off-diagonal block operator matrix, if $B\in L(X), C\in U(X)\cap Q(X)$, then the following cases hold:

(i) $\sigma_a(F)\backslash\{0\}=\{\lambda\in\mathbb{C}:\lambda^2\in\sigma_a(BC)\}\backslash\{0\}$;

(ii) $\sigma_q(F)\backslash\{0\}=\{\lambda\in\mathbb{C}:\lambda^2\in\sigma_q(BC)\}\backslash\{0\}$;

(iii) $\sigma_\delta(F)\backslash\{0\}\supset\{\lambda\in\mathbb{C}:\lambda^2\in\sigma_\delta(BC)\}\backslash\{0\}$;

(iv) $\sigma_{\rm FMV}(F)\backslash\{0\}\supset\{\lambda\in\mathbb{C}:\lambda^2\in\sigma_{\rm FMV}(BC)\}\backslash\{0\}$.

Proof Let $\lambda\neq 0$, then

$$F-\lambda=\begin{pmatrix} I & -\frac{1}{\lambda}B \\ 0 & I \end{pmatrix}\begin{pmatrix} \frac{1}{\lambda}BC-\lambda & 0 \\ 0 & -\lambda I \end{pmatrix}\begin{pmatrix} I & 0 \\ -\frac{1}{\lambda}C & I \end{pmatrix}.$$

Similar to the proof of Theorem 2.6, we have

$$[F-\lambda]_a=0\Leftrightarrow[\frac{1}{\lambda}BC-\lambda]_a=0\Leftrightarrow[BC-\lambda^2]_a=0,$$

$$[F-\lambda]_q=0\Leftrightarrow[\frac{1}{\lambda}BC-\lambda]_q=0\Leftrightarrow[BC-\lambda^2]_q=0$$

and $F-\lambda$ is stably solvable $\Rightarrow \frac{1}{\lambda}BC-\lambda$ is stably solvable $\Leftrightarrow BC-\lambda^2$ is stably solvable.

Thus, we immediately have the assertions (i)–(iii) and (iv) hold.

Theorem 2.10 Let $F=\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in C(X\times X)$ be an off-diagonal block operator matrix, if $C\in L(X), B\in U(X)\cap Q(X)$, then the following cases hold:

(i) $\sigma_a(F)\backslash\{0\}=\{\lambda\in\mathbb{C}:\lambda^2\in\sigma_a(CB)\}\backslash\{0\}$;

(ii) $\sigma_q(F)\backslash\{0\}=\{\lambda\in\mathbb{C}:\lambda^2\in\sigma_q(CB)\}\backslash\{0\}$;

(iii) $\sigma_\delta(F)\backslash\{0\}\supset\{\lambda\in\mathbb{C}:\lambda^2\in\sigma_\delta(CB)\}\backslash\{0\}$;

(iv) $\sigma_{\rm FMV}(F)\backslash\{0\}\supset\{\lambda\in\mathbb{C}:\lambda^2\in\sigma_{\rm FMV}(CB)\}\backslash\{0\}$.

Finally, we end this section with an example to illustrate the previous results.

Example 2.2 Let X be an infinite dimensional complex Banach space, $e\in X$ with $\|e\|=1$, and

$$A(x)=D(x)=\|x\|e,\quad B(x)=\|x\|^{\frac{1}{2}}e$$

for $x\in X$. Consider the block operator matrix

$$F=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

Then by Theorem 2.3, we have $(\sigma_\pi(A)\cup\sigma_\pi(D))\backslash\sigma_\pi(F)=\sigma_\pi(D)\cap\Delta_1$. Furthermore, by Corollary 2.3, we have $\sigma_\pi(F)=\sigma_\pi(A)\cup\sigma_\pi(D)$, where $\Delta_1=\{\lambda\in\mathbb{C}:[A-\lambda]_a>0,[A-\lambda]_q>0$ and $A-\lambda$ is not stably solvable$\}$.

On the other hand, since A, B and D are compact operators, then $\sigma_a(A)=\sigma_a(D)=\sigma_a(F)=0$. Moreover, by calculation, we have[11]

$$\sigma_q(A)=\sigma_q(D)=\partial\overline{\mathbb{D}},\qquad \sigma_\delta(A)=\sigma_\delta(D)=\overline{\mathbb{D}},$$

where $\mathbb{D}=\{\lambda\in\mathbb{C}:|\lambda|<1\}$. In addition, we write F in the form $F=F_1+T$, where $F_1=\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ and $T=\left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$ are compact operators with $[T]_A=[T]_Q=0$. We claim that

$$\sigma_q(F)=\sigma_q(F_1)=\partial\overline{\mathbb{D}}.$$

In fact, the equality $[F_1]_Q=[F_1]_q=1$ follows from a straightforward calculation. Thus, $\sigma_q(F_1)\subseteq\partial\overline{\mathbb{D}}$. Moreover, for any $\lambda\in\partial\overline{\mathbb{D}}$, let $z_n=(ne/\lambda\ \ 0)^{\rm T}\in X\times X$, then $\|z_n\|=n$ with

$$[F_1-\lambda]_q\leq\lim_{n\to\infty}\frac{\|F_1(z_n)-\lambda z_n\|}{\|z_n\|}=\frac{\|ne-ne\|}{n}=0.$$

i.e.

$$\lambda\in\sigma_q(F_1).$$

In summary,

$$\sigma_\pi(A)=\sigma_\pi(D)=\sigma_\pi(F)=\{0\}\cup\partial\overline{\mathbb{D}}.$$

$\Delta_1=\{\lambda\in\mathbb{C}:[A-\lambda]_a>0,[A-\lambda]_q>0$ and $A-\lambda$ is not stably solvable$\}=\mathbb{D}\backslash\{0\}$.

Thus

$$\sigma_\pi(A)\cup\sigma_\pi(D)\backslash\sigma_\pi(F)=\varnothing,\qquad \sigma_\pi(D)\cap\Delta_1=\varnothing.$$

Therefore

$$(\sigma_\pi(A)\cup\sigma_\pi(D))\backslash\sigma_\pi(F)=\sigma_\pi(D)\cap\Delta_1.$$

Furthermore

$$\sigma_\pi(F)=\sigma_\pi(A)\cup\sigma_\pi(D).$$

This shows that Theorem 2.3 and Corollary 2.3 are valid.

3. The Feng spectrum of nonlinear block operator matrices

Let X and Y be infinite dimensional complex Banach spaces and denote by $\mathfrak{DBC}(X)$ the family of all open, bounded, connected subsets Ω of X with $0 \in \Omega$. Let $F : X \to Y$ be a continuous nonlinear operator.

Definition 3.1[11] A continuous operator $F : \overline{\Omega} \to Y$ is called epi on $\overline{\Omega}$, if $F(x) \neq 0$ on $\partial\Omega$, and for any compact operator $G : \overline{\Omega} \to Y$ satisfying $G(x) \equiv 0$ on $\partial\Omega$, the equation

$$F(x) = G(x)$$

has a solution in Ω. More generally, we call F a k-epi operator $(k \geq 0)$ on $\overline{\Omega}$ if the above property holds for all operators with $[G]_a \leq k$, not just compact operators.

For $F : \overline{\Omega} \to Y$ and $\Omega \in \mathfrak{DBC}(X)$ as before, we introduce

$$\nu_\Omega(F) := \inf\{k : k \geq 0,\ F \text{ is not } k\text{-epi on } \overline{\Omega}\}$$

and

$$\nu(F) := \inf_{\Omega \in \mathfrak{DBC}(X)} \nu_\Omega(F),$$

where $\nu(F)$ stands for the measure of solvability of F. Observe that $\nu(F) > 0$ if and only if there exists some $k > 0$, such that $F : \overline{\Omega} \to Y$ is k-epi for every $\Omega \in \mathfrak{DBC}(X)$.

Definition 3.2[11] A continuous operator $F : X \to X$ is F-regular if $[F]_a > 0$, $[F]_b > 0$ and $\nu(F) > 0$. The set

$$\rho_F(F) = \{\lambda \in \mathbb{C} : F - \lambda I \text{ is } F\text{-regular}\}$$

is called the Feng-resolvent set of F, and its complement

$$\sigma_F(F) = \mathbb{C} \setminus \rho_F(F)$$

is called the Feng-spectrum of F.

Obviously

$$\sigma_F(F) = \sigma_a(F) \cup \sigma_b(F) \cup \sigma_\nu(F),$$

where

$$\sigma_a(F) = \{\lambda \in \mathbb{C} : [F - \lambda I]_a = 0\},$$
$$\sigma_b(F) = \{\lambda \in \mathbb{C} : [F - \lambda I]_b = 0\},$$
$$\sigma_\nu(F) = \{\lambda \in \mathbb{C} : \nu(F - \lambda I) = 0\}.$$

Remark 3.1 Every F-regular operator $F : X \to X$ is surjective. Furthermore, $L \in L(X)$ is F-regular operator if and only if L is an isomorphism.

Definition 3.3[11] A continuous operator $F : X \to Y$ is called k-stably solvable $(k \geq 0)$ if for any continuous operator $G : X \to Y$ with $[G]_A \leq k$ and $[G]_Q \leq k$, the equation $F(x) = G(x)$ has a solution $x \in X$.

For $F \in C(X,Y)$, we can call the number

$$\mu(F) := \inf\{k : k \geq 0,\ F \quad \text{is not} k\text{-stably solvable}\}$$

the measure of stable solvability of F. Also, we can say that a continuous operator $F : X \to Y$ is strictly stably solvable if $\mu(F) > 0$, i.e., F is k-stably solvable for some positive k.

The next lemma provides a connection between FMV-regular operator and strictly stably solvable operators.

Lemma 3.1[11] Every FMV-regular operator $F : X \to Y$ is strictly stably solvable. More precisely, the estimate

$$\mu(F) \geq \min\{[F]_q,\ [F]_a\}$$

holds.

The next lemma connects the measure of solvability and the measure of stable solvability of F.

Lemma 3.2[11] For any continuous operator $F : X \to Y$, we have the estimate

$$\mu(F) \leq \nu(F).$$

In the following, we describe the relationship between the Feng-spectrum of the whole operator matrix and its entries.

Theorem 3.1 Let $F \in C(X \times X)$ be the diagonal block operator matrix as in Theorem 2.1. Then

$$\sigma_F(F) \subset \sigma_F(A) \cup \sigma_F(D) \cup \sigma_+(A) \cup \sigma_+(D),$$

where $\sigma_+(\cdot) = \{\lambda \in \mathbb{C} : \cdot - \lambda \text{ is not injective or } (\cdot - \lambda)^{-1} \text{ is not continuous}\}$.

Proof Let $\lambda \notin \sigma_F(A) \cup \sigma_F(D) \cup \sigma_+(A) \cup \sigma_+(D))$, then, by the proof of Theorem 2.1, we know that $F - \lambda$ is FMV-regular operator, therefore $\mu(F - \lambda) > 0$ by Lemma 3.1, and we have from Lemma 3.2 that $\lambda \notin \sigma_F(F)$.

Let $\sigma_\varphi(F) = \sigma_a(F) \cup \sigma_b(F)$, we have the following results which are similar to Theorem 2.3 and Corollaries 2.2–2.4, and can be proved in a similar way, so their proofs are omitted here.

Theorem 3.2 Let $F \in C(X \times X)$ be the upper triangular block operator matrix as in Corollary 2.2. If $B \in U(X) \cap B(X)$, then

$$\sigma_\varphi(A) \subset \sigma_\varphi(F) \subset \sigma_\varphi(A) \cup \sigma_\varphi(D).$$

Theorem 3.3 Let $F \in C(X \times X)$ be the upper triangular block operator matrix as in Theorem 2.3. If $B \in U(X) \cap B(X)$ and $A(0) = 0$, then

$$\sigma_\varphi(A) \cup \sigma_\varphi(D) = \sigma_\varphi(F) \cup (\sigma_\varphi(D) \cap \Delta_2),$$

where $\Delta_2 = \{\lambda \in \mathbb{C} : [A - \lambda]_a > 0, [A - \lambda]_b > 0 \text{ and } \nu(A - \lambda) = 0\}$.

Corollary 3.1 Let $F \in C(X \times X)$ be the upper triangular block operator matrix as in Theorem 2.3. If $B \in U(X) \cap B(X)$ and $A(0) = 0$, then

$$\sigma_\varphi(A) \cup \sigma_\varphi(D) = \sigma_\varphi(F)$$

if and only if

$$\sigma_\varphi(D) \cap \Delta_2 \subset \sigma_\varphi(F).$$

In particular, if $\sigma_\varphi(D)\cap\Delta_2=\varnothing$, then $\sigma_\varphi(A)\cup\sigma_\varphi(D)=\sigma_\varphi(F)$.

Theorem 3.4 Let $F\in C(X\times X)$ be the upper triangular block operator matrix as in Theorem 2.3. If $A\in L(X)$ and $B\in U(X)\cap B(X)$, then

$$\sigma_\varphi(A)\cup\sigma_\varphi(D)=\sigma_\varphi(F)\cup(\sigma_\varphi(D)\cap\sigma_{r,1}(A)),$$

where $\sigma_{r,1}(A)$ is defined as in Remark 2.5.

Corollary 3.2 Let $F\in C(X\times X)$ be the upper triangular block operator matrix as in Theorem 2.3. If $A\in L(X)$ and $B\in U(X)\cap B(X)$, then

$$\sigma_\varphi(A)\cup\sigma_\varphi(D)=\sigma_\varphi(F)$$

if and only if

$$\sigma_\varphi(D)\cap\sigma_{r,1}(A)\subset\sigma_\varphi(F)).$$

In particular, if $\sigma_\varphi(D)\cap\sigma_{r,1}(A)=\varnothing$, then $\sigma_\varphi(A)\cup\sigma_\varphi(D)=\sigma_\varphi(F)$.

4. Conclusions

Block operator matrix plays a significant role in system theory, quantum mechanics, hydrodynamics and magnetohydro-dynamics. In this paper, we obtain some connections between the FMV-spectrum (compared to the Feng spectrum) of the diagonal as well as upper triangular nonlinear operator matrix and that of their diagonal entries. In addition, the relationship between the FMV-spectrum of the whole 2×2 nonlinear operator matrix and that of its Schur complement is presented by means of Schur decomposition. This provides a theoretical basis for solving systems of nonlinear equations in mathematical physics.

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