

Scheme to measure the expectation value of a physical quantity in weak coupling regime

Jie Zhang(张杰), Chun-Wang Wu(吴春旺), Yi Xie(谢艺), Wei Wu(吴伟), and Ping-Xing Chen(陈平形)

Citation: Chin. Phys. B, 2021, 30 (**3**): 033201. DOI: 10.1088/1674-1056/abd772
Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Precision measurements with cold atoms and trapped ions

Qiuxin Zhang(张球新), Yirong Wang(王艺蓉), Chenhao Zhu(朱晨昊), Yuxin Wang(王玉欣), Xiang Zhang(张翔), Kuiyi Gao(高奎意), Wei Zhang(张威)

Chin. Phys. B, 2020, 29 (**9**): 093203. DOI: 10.1088/1674-1056/aba9c6

Transparently manipulating spin-orbit qubit via exact degenerate ground states

Kuo Hai(海阔), Wenhua Zhu(朱文华), Qiong Chen(陈琼), Wenhua Hai(海文华)

Chin. Phys. B, 2020, 29 (**8**): 083203. DOI: 10.1088/1674-1056/ab99b1

Enhanced optical molasses cooling for Cs atoms with largely detuned cooling lasers

Di Zhang(张迪), Yu-Qing Li(李玉清), Yun-Fei Wang(王云飞), Yong-Ming Fu(付永明), Peng Li(李鹏), Wen-Liang Liu(刘文良), Ji-Zhou Wu(武寄洲), Jie Ma(马杰), Lian-Tuan Xiao(肖连团), Suo-Tang Jia(贾锁堂)

Chin. Phys. B, 2020, 29 (**2**): 023203. DOI: 10.1088/1674-1056/ab5fc6

Controlling Rydberg excitation process with shaped intense ultrashort laser pulses

Xiao-Yun Zhao(赵晓云), Chun-Cheng Wang(王春成), Shi-Lin Hu(胡师林), Wei-Dong Li(李卫东), Jing Chen(陈京), Xiao-Lei Hao(郝小雷)

Chin. Phys. B, 2019, 28 (**8**): 083202. DOI: 10.1088/1674-1056/28/8/083202

Ionization energies and term energies of the ground states $1s^2 2s$ of lithium-like systems

Li Jin-Ying, Wang Zhi-Wen

Chin. Phys. B, 2014, 23 (**1**): 013201. DOI: 10.1088/1674-1056/23/1/013201

SPECIAL TOPIC — Quantum computation and quantum simulation

Scheme to measure the expectation value of a physical quantity in weak coupling regime*

Jie Zhang(张杰)[1,2], Chun-Wang Wu(吴春旺)[1,2], Yi Xie(谢艺)[1,2], Wei Wu(吴伟)[1,2], and Ping-Xing Chen(陈平形)[1,2,†]

[1] Department of Physics, College of Liberal Arts and Sciences, National University of Defense Technology, Changsha 410073, China
[2] Interdisciplinary Center for Quantum Information, National University of Defense Technology, Changsha 410073, China

(Received 15 October 2020; revised manuscript received 13 November 2020; accepted manuscript online 30 December 2020)

In quantum mechanics, the expectation value of an operator can be measured by using the projective measurement, if the coupling between the measured system and pointer is strong enough. However in the weak coupling regime, the pointer can not show all the eigenvalue of the physical quantity directly due to the overlapping among the pointer states, which makes the measurement of the expectation value difficult. In this paper, we propose an expectation value measurement method in the weak coupling regime inspired by the weak measurement scheme. Compared to the projective measurement, our scheme has two obvious advantages. Experimentally we use the internal state and motional state of a single trapped $^{40}Ca^{+}$ to establish the measurement scheme and realize the proof of principle demonstration of the scheme.

Keywords: weak measurement, expectation value, trapped ions

PACS: 32.80.Qk, 03.65.Wj, 03.75.Be **DOI:** 10.1088/1674-1056/abd772

1. Introduction

Quantum measurement, involving a wide range of content of quantum mechanics, is difficult to be understood but also one of the most interesting basic research directions.[1] The study of quantum measurement can be summarized as two aspects, one is the understanding of the nature of quantum measurement, such as how quantum states collapse,[2] and the other is the application of quantum measurement. Here we focus on the latter. The applications of quantum measurement include three aspects: to obtain the information of the measured quantum state;[3] to obtain the value of the physical quantity of the measured system;[4,5] and to sense the information of the external environment,[6,7] such as sensing the intensity of the external magnetic field, etc. A basic model of quantum measurement is von Neumann measurement, which regards the measurement as the interaction between the measured system and measuring pointer.[2] If the interaction is strong enough so that the different states of the pointer corresponding to the eigenvalues of the measured system are orthogonal to each other, one can get the measurement results from the pointer states, which can be described by projective measurement. If the interaction is too weak to discriminate the pointer states completely, the measurement is modeled by the weak measurement scheme, which was proposed by Aharonov, Albert, and Vaidman (AAV) in 1988.[8] AAV also showed that the weak value amplification (WVA) effect[9–15] can be observed by an appropriate postselection in the weak measurement, which would be useful in quantum sensing.[16,17] Here is the question that under the weak measurement condition (the pointer states are not orthogonal to each other), how can we obtain the expectation value? In this paper we propose a scheme to measure the expectation value under weak coupling condition inspired by, but not using, WVA technique.

The structure of this paper is organized as follows. In Section 2, we will introduce the theoretical model of our measurement scheme. A proof of principle of our scheme using trapped ion is shown in Section 3. Finally, we give the discussion and conclusion of this paper.

2. Theoretical model of the measurement scheme

In this section, we introduce the theoretical model of the new measurement scheme. Suppose a general measurement described by the von Neumann's measurement model, and the interation between the measured system and measuring pointer can be written as

$$H_{\rm I} = g\hat{A}\hat{p}, \tag{1}$$

where g is the coupling strength, $\hat{A}$ and $\hat{p}$ are the operators of the measured system and measuring pointer, respectively. The evolution of the system at time t with the initial state $|\psi\rangle = \sum_n C_n|\psi_n\rangle$ is given by

$$\begin{aligned} |\Psi(t)\rangle &= \exp\left(-\frac{\rm i}{\hbar}g\hat{A}\hat{p}t\right)|\psi\rangle\otimes|\phi(x)\rangle \\ &= \sum_n C_n|\psi_n\rangle\exp\left(-\frac{\rm i}{\hbar}gA_n\hat{p}t\right)|\phi(x)\rangle \\ &= \sum_n C_n|\psi_n\rangle|\phi(x-gtA_n)\rangle, \end{aligned} \tag{2}$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 12004430 and 61632021) and the National Basic Research Program of China (Grant No. 2016YFA0301903).

†Corresponding author. E-mail: pxchen@nudt.edu.cn

© 2021 Chinese Physical Society and IOP Publishing Ltd

where $|\phi(x)\rangle$ is the initial state of the pointer, A_n is the eigenvalue of system operator $\hat{A}$ corresponding to the eigenstate $|\psi_n\rangle$, and $\hat{p} = -\mathrm{i}\hbar\partial/\partial x$ is the operator of the pointer.

Without loss of generality, we assume $\langle\phi(x)|x|\phi(x)\rangle = 0$, then $\langle\phi(x-gtA_n)|x|\phi(x-gtA_n)\rangle = gtA_n$, which indicates that $\phi(x-gtA_n)$ describes a wave packet with a shift of gtA_n. If the coupling strength is strong enough, the shifted pointer state meets

$$\langle\phi(x-gtA_n)|\phi(x-gtA_m)\rangle = \delta_{nm}. \tag{3}$$

The eigenvalue of system A_n and the probability of each eigenvalue can be read out by the measurement on the pointer states, and the expectation value of the operator $\bar{A} = \sum_n |C_n|^2 A_n$ is also obtained in this case.

However, in the case of small coupling strength or limited evolution time due to limited coherence time, there are overlaps among the pointer states and Eq. (3) does not hold, the expectation value cannot be read out directly by the measurement on $\hat{A}$. To overcome this dilemma, we propose a new measurement scheme, which includes 3 steps.

(i) We introduce another set of basis $\{\psi_1', \psi_2', \ldots\}$ for the measured system and it meets the condition

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \cdots \\ \psi_n \end{pmatrix} = \frac{1}{\sqrt{n}} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta_{11}} & \mathrm{e}^{\mathrm{i}\theta_{12}} & \cdots & \mathrm{e}^{\mathrm{i}\theta_{1n}} \\ \mathrm{e}^{\mathrm{i}\theta_{21}} & \mathrm{e}^{\mathrm{i}\theta_{22}} & \cdots & \mathrm{e}^{\mathrm{i}\theta_{2n}} \\ & & \cdots & \\ \mathrm{e}^{\mathrm{i}\theta_{n1}} & \mathrm{e}^{\mathrm{i}\theta_{n2}} & \cdots & \mathrm{e}^{\mathrm{i}\theta_{nn}} \end{pmatrix} \begin{pmatrix} \psi_1' \\ \psi_2' \\ \cdots \\ \psi_n' \end{pmatrix}, \tag{4}$$

where the transformation matrix is unitary.

(ii) We apply a projective measurement on the measured system with basis $\{\psi_1', \psi_2', \ldots\}$ with results being the state $|\psi_j'\rangle$, the pointer state is

$$|\phi_j\rangle = \frac{1}{\sqrt{n}} \sum_n C_n \mathrm{e}^{\mathrm{i}\theta_{nj}} |\phi(x-gtA_n)\rangle, \tag{5}$$

with probability $P_j = |\langle\psi_j'|\Psi\rangle|^2 = |\langle\phi_j|\phi_j\rangle|^2$. Then the shift of the pointer can be calculated

$$\begin{aligned} \bar{x}_j &= \langle\phi_j|x_j|\phi_j\rangle \\ &= \frac{1}{n}\Big(\sum_n |C_n|^2 gtA_n + \sum_{n\neq m} C_n C_m^* \mathrm{e}^{\mathrm{i}\theta_{nj}-\mathrm{i}\theta_{mj}} \langle\phi_m|x_j|\phi_n\rangle\Big)/P_j. \end{aligned} \tag{6}$$

(iii) Making a sum over all possible state $|\phi_j\rangle$, we can get the total shift of the pointer

$$\begin{aligned} \sum_{j=1} \bar{x}_j P_j &= gt\bar{A} + \frac{1}{n}\sum_{n\neq m}\sum_j C_n C_m^* \mathrm{e}^{\mathrm{i}(\theta_{nj}-\theta_{mj})} \langle\phi_m|x_j|\phi_m\rangle \\ &= gt\bar{A}. \end{aligned} \tag{7}$$

Here we use the unitrary feature of the transformation matrix, which results in $\sum_j \mathrm{e}^{-\mathrm{i}\theta_{mj}} \mathrm{e}^{\mathrm{i}\theta_{nj}} = \delta_{nm}$. Since P_j is the post-seletion probability, and g and t are the constant parameters in the experiment, we can then read out the expectation value from the average shift of the pointer as shown in Eq. (7).

It can be seen that the scheme has two advantages over the projective method in the weak coupling rgime. We can find from Eq. (2) that the pointer state is a mixture of some wave packets with different shifts after the interaction process between the measured system and pointer. If we measure the mixed state directly we cannot distinguish all the shifts owing to the overlap of the wave packets and small shift under weak coupling, and then fail to get the expectation value of $\hat{A}$. In our scheme, some pointer shifts may be amplified owing to the last term of Eq. (6) and the post selection probability.[18] On the other hand, if the pointer is an even function, the second term of Eq. (6) would be zero, and the expectation value can be read by only one measurement.

3. Experimental setup and results

Now we take measurement of $\langle\hat{\sigma}_x\rangle$ as an example to demonstrate our measurement scheme experimentally. We use the internal state and the motional states of a single trapped $^{40}Ca^+$ ion to establish von Neumann's measurement scheme. The experimental setup for coupling the system and pointer is shown in Fig. 1. The Zeeman sublevels $S_{1/2}(m_J = -1/2)$ and $D_{1/2}(m_J = -1/2)$ are identified as the states $|\uparrow\rangle$ and $|\downarrow\rangle$, which act as the measured system and can be coherently operated by a narrow linewidth 729 nm laser beam. The axial motional state of the ion with secular frequency $\omega_z = 2\pi \times 1.4$ MHz is chosen as the pointer state. The coupling of the measured system and the pionter is realized by a bichromatic light field, which is generated by injecting a RF with two frequency compenonents with gap $2\omega_z$ and resonant with the blue and red sidebands of transition $S_{1/2}(m_J = -1/2) \leftrightarrow D_{1/2}(m_J = -1/2)$. In the Lamb–Dicke regime, the interaction Hamiltonian reads

$$\begin{aligned} \hat{H}_{\mathrm{bic}} = {} & \frac{\hbar\eta\Omega}{2}\left[\hat{\sigma}_x \sin\phi_+ + \hat{\sigma}_y \cos\phi_+\right] \\ & \otimes \left[-(\hat{a}^\dagger + \hat{a})\cos\phi_- + \mathrm{i}(\hat{a}^\dagger - \hat{a})\sin\phi_-\right], \end{aligned} \tag{8}$$

where η is the Lamb–Dicke parameter, Ω is the Rabi frequency, $\hat{a}^+$ and $\hat{a}$ are the creation and annihilation operators for motional states, $\phi_+ = (\phi_{\mathrm{red}} + \phi_{\mathrm{blue}})/2$ and $\phi_- = (\phi_{\mathrm{red}} - \phi_{\mathrm{blue}})/2$ are the relative phases between the red sideband laser phase ϕ_{red} and the blue sideband laser phase ϕ_{blue}.

The motional state is prepared in ground state $|\varphi_i(z)\rangle = (\frac{1}{2\pi\Delta_z{}^2})^{\frac{1}{4}} \exp(-\frac{z^2}{4\Delta_z{}^2})$ by laser cooling before each experiment, where $\Delta_z = \sqrt{\hbar/2m\omega_z}$ is the size of the ground state wave packet. The internal state of the ion is initialized to state $|\downarrow\rangle$ by optical pumping. The interaction Hamiltonian between the internal state and motional state

$$H_{\mathrm{bic}} = \eta\Omega\Delta_z\hat{\sigma}_x\hat{p} \tag{9}$$

is created by setting $\phi_+ = \phi_- = \pi/2$, here $\hat{\sigma}_x$ is the operator of the measured system, $\hat{p} = \frac{\mathrm{i}\hbar(\hat{a}^\dagger - \hat{a})}{2\Delta_z}$ is the momentum operator for the pointer, and $g = \eta\Omega\Delta_z$ is the coupling strength between the measured system and pointer.

To build the measurement scheme, we apply a 729 nm laser pulse to prepare the internal state in $|\Psi_0\rangle = \cos\theta|+\rangle +$

$\sin\theta|-\rangle$, which is the state for measuring the expectation value of operator $\hat{\sigma}_x$, and then the entangled state

$$|\Psi(t)\rangle = \cos\theta|+\rangle|\varphi_i(z-gt)\rangle - \sin\theta|-\rangle|\varphi_i(z+gt)\rangle \quad (10)$$

is created by applying Eq. (9) for a duration of t. It can be seen that the the spin state is the projective basis for states $|+\rangle$ and $|-\rangle$, and it is easy to find that the transformation unitary matrix from $\{|+\rangle, |-\rangle\}$ to $\{|\uparrow\rangle, |-\rangle\}$ is the Hadmard gate. Note that the detection laser beam at 397 nm only interacts with the $|\downarrow\rangle$ state for $^{40}Ca^{+}$ ion, therefore we can post select the state $|\uparrow\rangle$ without destroying the pointer state. When the post selected state is $|\uparrow\rangle$, the pointer state will be

$$|\varphi\rangle = \frac{1}{\sqrt{2}}\cos\theta|\varphi_\uparrow(z-gt)\rangle - \frac{1}{\sqrt{2}}\sin\theta|\varphi_\uparrow(z+gt)\rangle, \quad (11)$$

with the post selection probability $P_\uparrow = \frac{1}{2}(1 - \sin(2\theta)\,e^{-(gt/\Delta_z)^2/2})$.

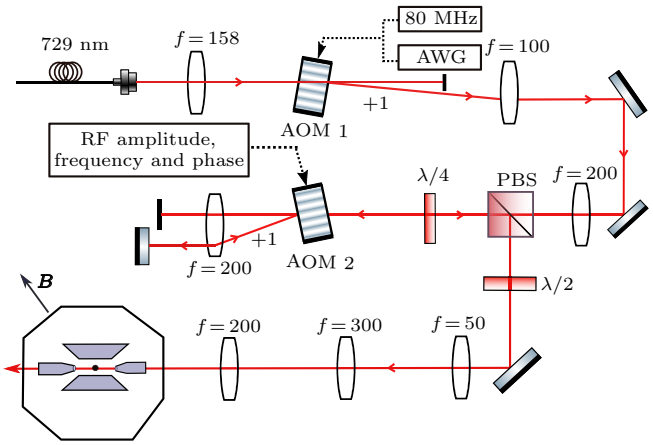


Fig. 1. Experimental setup for the expectation value measurement. AOM 1 is used to generate the bichromatic beam with arbitrary wavefrom generator (AWG), and AOM 2 is used to shift the laser frequency by about 270 MHz and control the laser power. The 729 beam goes through the two holes on the endcap electrodes and stimulates the axial motional modes with Lamb–Dicke parameters $\eta \approx 0.08$.

Note that in the trapped ion system, the pointer state is a Guassian wave packet (even function), therefore we find the relation

$$\langle\varphi(z-g\Delta_z)\,|\,z\,|\,\varphi(z+g\Delta_z)\rangle = 0. \quad (12)$$

This formula indicates that the second term in Eq. (6) is zero for all post selected states, and the central shift of the pointer for post selection state $|\psi_j\rangle$ is simplified to

$$\bar{z}_j = \langle\phi_j\,|\,z\,|\,\phi_j\rangle = \frac{1}{n}\Big(\sum_n |C_n|^2 gtA_n\Big)/P_j, \quad (13)$$

therefore the expectation value of the opeator can be read out by either of the pointer shifts. Experimentally we only need to take the measurement for the postselection state $|\uparrow\rangle$, and the central position of the pointer is

$$\langle z\rangle = -\frac{1}{2}gt\langle\hat{\sigma}_x\rangle/P_\uparrow. \quad (14)$$

The expectation value of operator $\hat{\sigma}_x$ can be obtained from the experimentally measured post selection probability and pointer shift, and it reads

$$\langle\hat{\sigma}_x\rangle = \frac{2\langle z\rangle P_\uparrow}{gt}. \quad (15)$$

In the trapped ion system, the only observable that can be measured directly is $\hat{\sigma}_z$. To get the central shift of the post selected wave packet, we use an indirect method, which was successfully used in the quantum simulation[19–22] to get its probability distribution, see detail in Refs. [20,23].

Figure 2(a) shows the measured shifts for the wave packet under various initial phase θ in the weak coupling regime, where the coupling parameter $gt = 0.1\Delta_z \ll \Delta_z$ is realized by setting the Rabi frequency $\Omega = 2\pi \times 20$ kHz and the laser pulse duration $t = 100$ μs. We can see a sharp amplification near the angle $\theta = \pi/4$, where the initial state is near $|\downarrow\rangle$ and the corresponding expectation value of operator $\hat{\sigma}_x$ is 0. This indicates that we can have a precise measurement near the zero point. Figure 2(b) shows the shifted wave packet with $g = 0.1$, $\theta = \pi/6$. The central position of the wave packet is extracted by fitting the data with a Gaussian function.

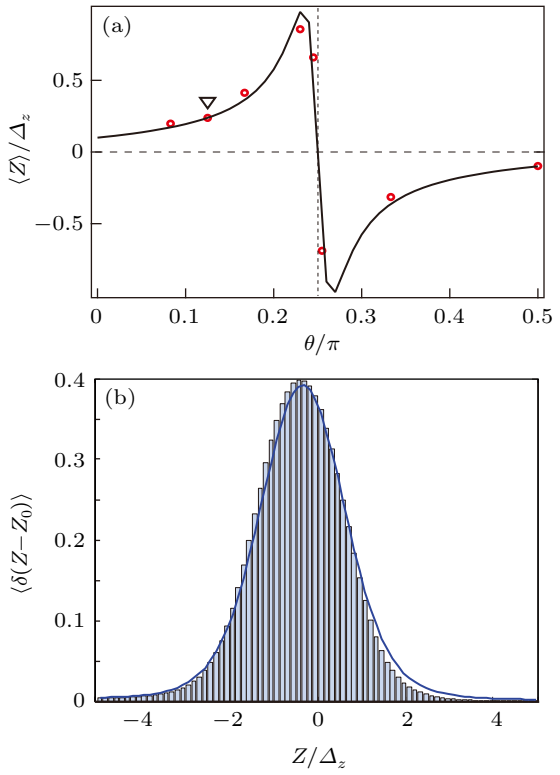


Fig. 2. (a) Central position of wave packet for different initial states with fixed coupling strength $g = 0.1$. The dots mean the experimental measurement and the solid line indicates the theoretical prediction. (b) The reconstructed wave packet for $g = 0.1$, $\theta = \pi/6$, which is marked with inverted triangle in (a). The histogram is the experimental results and the solid blue line is the fitted curve by a Gaussian function.

Applying Eq. (15), we obtain the expectation of operator $\hat{\sigma}_x$ under initial phase from 0 to $\pi/2$ as shown in Fig. 3. The red dots show the experimental results and the black curve indicates the theoretical predition of $\langle\hat{\sigma}_x\rangle = \cos(2\theta)$. It can be seen that the experimental result fits the theoretical prediction,

and the deviation comes from the instable experimental parameters and limited data points for reconstruciton of the wave packet.

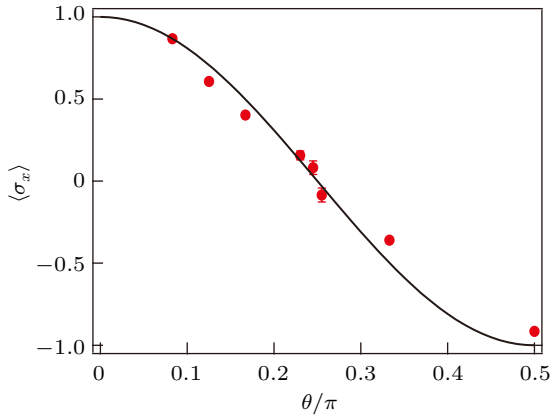


Fig. 3. Expectation value measurement for different initial states with coupling parameter $gt = 0.1\Delta_z$. The solid line indicates the theoretical prediction of operator $\langle\hat{\sigma}_x\rangle = \cos(2\theta)$ with initial state $\cos\theta|+\rangle + \sin\theta|-\rangle$ and the dots mean the experimental measurement.

4. Discussion and conclusion

It is interesting to compare the weak value with the expectation value in the weak coupling regime. Firstly, the expectation value has a different definition from the weak value as shown in Ref. [8]. Secondly, to get a useful weak value, one should choose an appropriate postselected state according to the initial state. On the contrary, to get the expectation value one may always choose the postselected state as shown in Eq. (4) in spite of the initial states.

In conclusion, we proposed a useful method for measuring the expectation value of a weak coupling system and demonstrated the scheme by taking advantage of a single trapped $^{40}Ca^{+}$. This scheme provides an effective method to extract the expectation value for the weak coupling system. The good agreement between the theoretical prediction and experimental results shows that the scheme may be useful in quantum sensing.

References

[1] Braginsky V B, Khalili F Y and Thorne K S 1995 *Quantum Measurement* (Cambridge: Cambridge University Press)

[2] Neumann J V 1932 *Mathematical Foundations of Quantum Mechanics* (Berlin: Springer Press)

[3] Thekkadath G S, Giner L, Chalich Y, Horton M J, Banker J and Lundeen J S 2016 *Phys. Rev. Lett.* **117** 120401

[4] Peres A 1995 *Quantum Theory: Concepts and Methods* (Dordrecht: Kluwer)

[5] Preskill J 1998 *Quantum Information and Computation* (lecture notes for physics 229) California Institute of Technology

[6] Araneda G, Walser S, *et al.* 2019 *Nat. Phys.* **15** 17

[7] Zou W J, Li Y H, Wang S C, Cao Y, Ren J G, Yin J, Peng C Z, Wang X B and Pan J W 2017 *Phys. Rev. A* **95** 042342

[8] Aharonov Y, Albert D Z and Vaidman L 1988 *Phys. Rev. Lett.* **60** 1351

[9] Romito A, Gefen Y and Blanter Y M 2008 *Phys. Rev. Lett.* **100** 056801

[10] Zilberberg O, Romito A and Gefen Y 2011 *Phys. Rev. Lett.* **106** 080405

[11] Groen J P, Riste D, Tornberg L, Cramer J, de Groot P C, Picot T, Johansson G and DiCarlo L 2013 *Phys. Rev. Lett.* **111** 090506

[12] Lu D, Brodutch A, Li J, *et al.* 2014 *New J. Phys.* **16** 053015

[13] Denkmayr T, Geppert H, Lemmel H, Waegell M, Dressel J, Hasegawa Y and Sponar S 2017 *Phys. Rev. Lett.* **118** 010402

[14] Shomroni I, Bechler O, Rosenblum S and Dayan B 2013 *Phys. Rev. Lett.* **111** 023604

[15] Zhang M and Zhu S Y 2015 *Phys. Rev. A* **92** 043825

[16] Hosten O and Kwiat P 2008 *Science* **319** 787

[17] Dixon P B, Starling D J, Jordan A N and Howell J C 2009 *Phys. Rev. Lett.* **102** 173601

[18] Xu L, Liu Z X, Datta A, Knee G C, Lundeen J S, Lu Y Q and Zhang L J 2020 *Phys. Rev. Lett.* **125** 080501

[19] Wallentowitz S and Vogel W 1995 *Phys. Rev. Lett.* **75** 2932

[20] Gerritsma R, Kirchmair G, *et al.* 2010 *Nature* **463** 68

[21] Zähringer F , Kirchmair G, Gerritsma R, Solano E, Blatt R and Roos C F 2010 *Phys. Rev. Lett.* **104** 100503

[22] Pan Y M, Zhang J, Cohen E, Wu C W, Chen P X and Davidson N 2020 *Nat. Phys.* **16** 1206

[23] Wu C W, Zhang J, Xie Y, Ou B Q, Chen T, Wu W and Chen P X 2019 *Phys. Rev. A* **100** 062111