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Entanglement properties of GHZ and W superposition state and its decayed states*

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We use the generalized Wootters formula, the positive partial transpose (PPT) criterion and the matched entanglement witness, to detect entanglement of three-qubit GHZ and W superposition state and its decayed states. It shows that the results of the generalized Wootters formula in the part near the W state are tight. In the other parts, the PPT criterion is superior to the generalized Wootters formula. Furthermore, we investigate the relationship between entanglement and coherence.

Keywords: quantum entanglement, Wootters formula, matched entanglement witness, coherence

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1. Introduction

Quantum entanglement plays a crucial role in quantum information and quantum computation. Thus, judging whether a quantum state is entangled has become a prominent topic in quantum information. In Ref. [1] the entanglement property of the mixed state of three-qubit W state and GHZ state has been studied, and the superposition state has rarely been investigated. In this paper, we study the entanglement properties of the superposition of three-qubit GHZ state and W state and its decayed states.

Many entanglement criteria have been proposed for bipartite systems[2–6], such as PPT criterion,[7] which is necessary and sufficient for 2×2 and 2×3 bipartite systems to be separable, but it is no longer sufficient in the higher dimensions. Therefore, PPT criterion is often used as an auxiliary verification method. In multipartite systems,[8–13] there is no simple criterion. The entanglement witness (EW)[14–17] is a common method for entanglement detection of multipartite systems, and the construction of EW becomes a task for entanglement detection.

Another method for identifying entanglement is formulated in terms of Wootters formula. It is restricted to a two-qubit system, and its application in multipartite systems is limited. Thus the generalized Wootters formula was proposed, and it is not only applicable to higher dimensional bipartite systems but also to multipartite systems.

It is well known that both entanglement and coherence[18,19] originate from the superposition principle. Therefore, it is a very meaningful work to study the relationship between them.[20–24]

This paper has two purposes. Firstly, we use the the generalized Wootters formula, PPT criterion and matched entanglement witness, to detect entanglement of the superposition state of three-qubit GHZ state and W state and its decayed states. We find that the generalized Wootters formula can be used as a tight criterion for the full separability of quantum states near the W state. In the other parts, the PPT criterion is superior to the generalized Wootters formula. Secondly, we show the connection of entanglement and coherence for the above superposition state.

2. The generalized Wootters formula

The concurrence is a useful tool to characterize quantum correlations in various physical systems. Concurrence was first proposed by Wootters in 1998.[25] Wootters formula of a two-qubit state ρ is

$$C(\rho)=\max\{0,\lambda_1-\lambda_2-\lambda_3-\lambda_4\}, \tag{1}$$

where C is named concurrence. The entanglement of formation is a monotonically increasing function of C,[26,27] λ_i $(i=1,2,3,4)$ are the eigenvalues of the Hermitian matrix $R\equiv\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ in descending order, where $\tilde{\rho}=(\sigma_2\otimes\sigma_2)\rho^*(\sigma_2\otimes\sigma_2)$, with ρ^* being the complex conjugate of ρ, $\sigma_2=-\mathrm{i}|0\rangle\langle1|+\mathrm{i}|1\rangle\langle0|$ is a Pauli operator. Alternatively, λ_i $(i=1,2,3,4)$ are the square root eigenvalues of the non-Hermitian matrix $\rho\tilde{\rho}$. Note that each λ_i is a non-negative real number. When $C(\rho)$ is positive, ρ is entangled. In 2012, Chen *et al.*[28] extended the formula to higher dimensional bipartite systems and three-qubit system. For the bipartite pure quantum state of any dimension on $\mathcal{H}_A\otimes\mathcal{H}_B$ expressed as $\rho_{AB}=|\psi\rangle\langle\psi|$, the pure state $|\psi\rangle$ can be expressed in a product basis as $|\psi\rangle=\sum_{i=1}^{m}\sum_{j=1}^{n}\psi_{ij}|ij\rangle$. We can define on the subspace $\mathcal{H}_A$ (with dimension m) the generators of the special orthogonal group $SO(m)$ as $L_\alpha=|i\rangle\langle j|-|j\rangle\langle i|$, there are

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$m(m-1)/2$ such generators. Similarly, the special orthogonal group $SO(n)$ on the subspace $\mathcal{H}_B$ (with dimension n) has $n(n-1)/2$ generators S_β. Defining $J = L_\alpha \otimes S_\beta$ we can obtain the expression of the concurrence as

$$C(\rho) \geq \max\{0, \lambda_{mn}^{(1)} - \sum_{i>1} \lambda_{mn}^{(i)}\}, \tag{2}$$

where $\lambda_{mn}^{(i)}$ is the square root eigenvalues of the matrix $\chi = \rho J \rho^* J^*$ in descending order, when $C = 0$, then ρ is separable. Therefore, the criterion for the separability of the quantum state can be transformed to solve the eigenvalues of the χ matrix. In the three-qubit system, let $|\psi\rangle \in \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2$, it can be written as $|\psi\rangle = \sum_{A=1}^{2}\sum_{B=1}^{2}\sum_{C=1}^{2} \psi_{ABC}|ABC\rangle$. We may rewrite the state as $|\psi\rangle = \sum_{A=1}^{2}\sum_{BC=1}^{4} \psi_{A|BC}|A\rangle|BC\rangle$, where A, B, and C denote the three qubits. The orthogonal group $SO(2)$ is obtained in subspace $\mathcal{H}^2$, there is only one generator for $L^A = |0\rangle\langle 1| - |1\rangle\langle 0|$. In the subspace $\mathcal{H}^4$, the orthogonal group $SO(4)$ is obtained, there are 6 generators. We need to select appropriate generators to construct the J matrix according to different states. Considering the interchange of qubits, we can reach $J = J^{A|BC} + J^{B|AC} + J^{C|AB}$, where $J^{A|BC} = L^A \otimes S^{BC}$.

For the three-qubit Green–Horne–Zeilinger (GHZ) state $|\mathrm{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ mixed with white noise: $\rho^G(p) = p|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + \frac{1-p}{8} I_8$, where I_8 is an 8×8 identity matrix, we take a single-qubit operator $L^A = |0\rangle\langle 1| - |1\rangle\langle 0|$, and take a two-qubit operator $S^{BC} = |00\rangle\langle 11| - |11\rangle\langle 00|$. Finally we construct J_{GHZ} as

$$\begin{aligned} J_{\mathrm{GHZ}} \;=\; & 3|000\rangle\langle 111| + 3|111\rangle\langle 000| - |001\rangle\langle 110| \\ & - |110\rangle\langle 001| - |101\rangle\langle 010| - |010\rangle\langle 101| \\ & - |011\rangle\langle 100| - |100\rangle\langle 011|. \end{aligned} \tag{3}$$

When $p = 1$, the state $\rho^G(p)$ is the pure GHZ state, we can obtain that $\rho^G(p)$ is entangled when $p > 1/5$.

For the three-qubit W state $|W\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$ mixed with white noise: $\rho^W(p) = p|W\rangle\langle W| + \frac{1-p}{8} I_8$. In this case, we use again the operator $L^A = |0\rangle\langle 1| - |1\rangle\langle 0|$, while for the two-qubit operator, we use $S^{BC} = |00\rangle\langle 10| - |10\rangle\langle 00|$. Finally we construct J_{W},

$$\begin{aligned} J_{\mathrm{W}} \;=\; & |000\rangle\langle 011| + |011\rangle\langle 000| + |000\rangle\langle 101| + |101\rangle\langle 000| \\ & + |000\rangle\langle 110| + |110\rangle\langle 000| - |001\rangle\langle 010| \\ & - |010\rangle\langle 001| - |001\rangle\langle 100| - |100\rangle\langle 001| \\ & - |010\rangle\langle 100| - |100\rangle\langle 010|. \end{aligned} \tag{4}$$

When $p = 1$, the state $\rho^W(p)$ is the pure W state, we can obtain that $\rho^W(p)$ is entangled when $p > \sqrt{3}/(8+\sqrt{3}) \approx 0.178$.

The quantum state considered in this paper is the superposition state of the three-qubit W state and the GHZ state under the white noise environment, it is

$$\begin{aligned} \rho \;=\; & p|\psi\rangle\langle\psi| + \frac{1-p}{8} I_8 \\ =\; & p(a|W\rangle + b|\mathrm{GHZ}\rangle)(a\langle W| + b\langle \mathrm{GHZ}|) + \frac{1-p}{8} I_8, \end{aligned} \tag{5}$$

where $a^2 + b^2 = 1$. The above formula is further extended to cover the case of decaying,

$$\begin{aligned} \rho \;=\; & w|W\rangle\langle W| + g|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + \tau\sqrt{gw}(|W\rangle\langle\mathrm{GHZ}| \\ & + |\mathrm{GHZ}\rangle\langle W|) + \frac{1-g-w}{8} I_8, \end{aligned} \tag{6}$$

where g and w are probabilities, τ represents the decaying factor (the degree of decay), and its value ranges from 0 to 1. When $\tau = 0$, the state in Eq. (6) represents the mixed state of the three-qubit GHZ state and W state with white noise, and when $\tau = 1$, it represents the superposition state of the three-qubit GHZ state and the W state mixed with white noise.

By mixing J_{W} and J_{GHZ}, we obtain $J = \alpha J_{\mathrm{W}} + \beta J_{\mathrm{GHZ}}$, where α and β are probabilities and $\alpha + \beta = 1$, its explicit form in computational basis is

$$J = \begin{bmatrix} 0 & 0 & 0 & \alpha & 0 & \alpha & \alpha & -3\beta \\ 0 & 0 & -\alpha & 0 & -\alpha & 0 & \beta & 0 \\ 0 & -\alpha & 0 & 0 & -\alpha & \beta & 0 & 0 \\ \alpha & 0 & 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & -\alpha & -\alpha & \beta & 0 & 0 & 0 & 0 \\ \alpha & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ \alpha & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ -3\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We convert the square root eigenvalue for $\chi = \rho J \rho^* J^*$ into the absolute eigenvalues $|\lambda_i| (i = 1, \ldots, 8)$ of ρJ (ρ^* and J^* represent the complex conjugates of ρ and J; ρ and J are real matrices), we get four eigenvalues $\lambda_{1,2,3,4} = \frac{c}{2}(\alpha \pm \sqrt{\alpha^2 + 4\beta^2})$, where $c = \frac{1-g-w}{8}$, the remaining 4 eigenvalues need to be calculated numerically. The separable criterion is given by $C = 0$. If the criterion coincides with the numerical criterion of the matched entangled witness, the generalized Wootters formula can be used as a tight criterion for the full separability of the superposition state of the three-qubit GHZ state and the W state.

3. Matched entanglement witness

The state (6) can also be decomposed with the products of Pauli matrices and the identity matrix σ_0 of 2×2, namely,

$$\begin{aligned} \rho \;=\; & \frac{1}{8}(\sigma_0\sigma_0\sigma_0 + R_1(\sigma_1\sigma_0\sigma_0 + \sigma_0\sigma_1\sigma_0 + \sigma_0\sigma_0\sigma_1) \\ & + R_2(\sigma_3\sigma_0\sigma_0 + \sigma_0\sigma_3\sigma_0 + \sigma_0\sigma_0\sigma_3) \\ & + R_3(\sigma_1\sigma_1\sigma_0 + \sigma_0\sigma_1\sigma_1 + \sigma_1\sigma_0\sigma_1) \\ & + R_4(\sigma_0\sigma_2\sigma_2 + \sigma_2\sigma_2\sigma_0 + \sigma_2\sigma_0\sigma_2) \\ & + R_5(\sigma_0\sigma_3\sigma_3 + \sigma_3\sigma_3\sigma_0 + \sigma_3\sigma_0\sigma_3) \\ & + R_6(\sigma_1\sigma_1\sigma_1) + R_7(\sigma_1\sigma_1\sigma_3 + \sigma_1\sigma_3\sigma_1 + \sigma_3\sigma_1\sigma_1) \\ & + R_8(\sigma_2\sigma_2\sigma_1 + \sigma_2\sigma_1\sigma_2 + \sigma_1\sigma_2\sigma_2) \\ & + R_9(\sigma_3\sigma_3\sigma_1 + \sigma_3\sigma_1\sigma_3 + \sigma_1\sigma_3\sigma_3) \end{aligned}$$

$$
\begin{aligned}
&+R_{10}(\sigma_2\sigma_2\sigma_3+\sigma_2\sigma_3\sigma_2+\sigma_3\sigma_2\sigma_2)+R_{11}(\sigma_3\sigma_3\sigma_3)\\
&+R_{12}(\sigma_0\sigma_1\sigma_3+\sigma_0\sigma_3\sigma_1+\sigma_1\sigma_3\sigma_0\\
&+\sigma_1\sigma_0\sigma_3+\sigma_3\sigma_0\sigma_1+\sigma_3\sigma_1\sigma_0),
\end{aligned}
\tag{7}
$$

where σ_1, σ_2, σ_3 are Pauli matrices, $\sigma_1=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $\sigma_2=\begin{bmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{bmatrix}$, $\sigma_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$, the parameters of R_i are the rename of characteristic function[29]

$$
R_{jkl}=\mathrm{Tr}(\rho\sigma_j\otimes\sigma_k\otimes\sigma_l),
\tag{8}
$$

where $j,k,l=0,1,2,3$.

We have $(R_1,\ldots,R_{12})=(\frac{\sqrt{6gw}\tau}{3},\frac{w}{3},\frac{2w+\sqrt{6gw}\tau}{3},\frac{2w-\sqrt{6gw}\tau}{3},g-\frac{w}{3},g,\frac{2w-\sqrt{6gw}\tau}{3},-g,\frac{\sqrt{6gw}\tau}{3},\frac{2w+\sqrt{6gw}\tau}{3},-w,\frac{\sqrt{6gw}\tau}{3})$. The characteristic function vector is defined as $\boldsymbol{R}\equiv(3R_1,3R_2,3R_3,3R_4,3R_5,R_6,3R_7,3R_8,3R_9,3R_{10},R_{11},6R_{12})$.

A Hermitian operator $\hat{W}$ is called an entanglement witness if for any separable ρ_s, there is $\mathrm{Tr}(\rho_s\hat{W})\geq 0$, and for at least one entangled state ρ there is $\mathrm{Tr}(\rho\hat{W})<0$. When $\mathrm{Tr}(\rho_s\hat{W})=0$ is achieved we call $\hat{W}$ optimal entanglement witness. Let $\hat{W}=\Lambda I-\hat{M}$, where I is identity matrix with the same dimension as $\hat{M}$. If we define

$$
\Lambda=\max_{\rho_s}\mathrm{Tr}(\rho_s\hat{W}),
\tag{9}
$$

then $\hat{W}$ is optimal. The Hermitian operator $\hat{M}$ corresponding to the quantum state (7) can be written as

$$
\begin{aligned}
\hat{M}\ =\ &M_1(\sigma_1\sigma_0\sigma_0+\sigma_0\sigma_1\sigma_0+\sigma_0\sigma_0\sigma_1)\\
&+M_2(\sigma_3\sigma_0\sigma_0+\sigma_0\sigma_3\sigma_0+\sigma_0\sigma_0\sigma_3)\\
&+M_3(\sigma_1\sigma_1\sigma_0+\sigma_0\sigma_1\sigma_1+\sigma_1\sigma_0\sigma_1)\\
&+M_4(\sigma_0\sigma_2\sigma_2+\sigma_2\sigma_2\sigma_0+\sigma_2\sigma_0\sigma_2)\\
&+M_5(\sigma_0\sigma_3\sigma_3+\sigma_3\sigma_3\sigma_0+\sigma_3\sigma_0\sigma_3)+M_6(\sigma_1\sigma_1\sigma_1)\\
&+M_7(\sigma_1\sigma_1\sigma_3+\sigma_1\sigma_3\sigma_1+\sigma_3\sigma_1\sigma_1)\\
&+M_8(\sigma_2\sigma_2\sigma_1+\sigma_2\sigma_1\sigma_2+\sigma_1\sigma_2\sigma_2)\\
&+M_9(\sigma_3\sigma_3\sigma_1+\sigma_3\sigma_1\sigma_3+\sigma_1\sigma_3\sigma_3)\\
&+M_{10}(\sigma_2\sigma_2\sigma_3+\sigma_2\sigma_3\sigma_2+\sigma_3\sigma_2\sigma_2)+M_{11}(\sigma_3\sigma_3\sigma_3)\\
&+M_{12}(\sigma_0\sigma_1\sigma_3+\sigma_0\sigma_3\sigma_1+\sigma_1\sigma_3\sigma_0\\
&+\sigma_1\sigma_0\sigma_3+\sigma_3\sigma_0\sigma_1+\sigma_3\sigma_1\sigma_0).
\end{aligned}
\tag{10}
$$

Given a state ρ, we may adjust parameters M_i so that

$$
\mathcal{L}=\frac{\Lambda}{\mathrm{Tr}(\rho\hat{M})}=\frac{\Lambda}{\boldsymbol{M}\cdot\boldsymbol{R}}
\tag{11}
$$

is minimized. While choosing M_i we keep both Λ and $\boldsymbol{M}\cdot\boldsymbol{R}$ positive, $\boldsymbol{M}\equiv(M_1,M_2,\ldots,M_{12})$. We have the following conclusion, if a state satisfies

$$
\mathcal{L}_{min}=\min_{\hat{M}}\mathcal{L}\geq 1,
\tag{12}
$$

then it is separable, otherwise, it is entangled.

Any three-qubit fully separable state can be written as

$$
\rho_s=\sum_i p_i\rho_A^{(i)}\otimes\rho_B^{(i)}\otimes\rho_C^{(i)},
\tag{13}
$$

where p_i are probabilities, ρ_s can be written as the mixture of pure states $|\psi_s\rangle=|\psi_A\rangle|\psi_B\rangle|\psi_C\rangle$, $|\psi_j\rangle=\cos\frac{\theta_j}{2}|0\rangle+\sin\frac{\theta_j}{2}\mathrm{e}^{i\varphi_j}|1\rangle$ $(j=A,B,C)$, Eq. (9) then is simplified as

$$
\Lambda=\max_{|\psi_s\rangle}\langle\psi_s|\hat{M}|\psi_s\rangle=\max_{\theta_A,\theta_B,\theta_C,\varphi_A,\varphi_B,\varphi_C}\mathrm{Tr}(\hat{M}|\psi_s\rangle\langle\psi_s|).
\tag{14}
$$

We use Bloch representation expression to obtain

$$
\begin{aligned}
|\psi_A\rangle\langle\psi_A|&=\frac{1}{2}(Ix_0+\sigma_1x_1+\sigma_2x_2+\sigma_3x_3),\\
|\psi_B\rangle\langle\psi_B|&=\frac{1}{2}(Iy_0+\sigma_1y_1+\sigma_2y_2+\sigma_3y_3),\\
|\psi_C\rangle\langle\psi_C|&=\frac{1}{2}(Iz_0+\sigma_1z_1+\sigma_2z_2+\sigma_3z_3),
\end{aligned}
\tag{15}
$$

where $x_0=y_0=z_0=1$, $x_1=\sin\theta_A\cos\varphi_A$, $x_2=\sin\theta_A\sin\varphi_A$, $x_3=\cos\theta_A$, $y_1=\sin\theta_B\cos\varphi_B$, $y_2=\sin\theta_B\sin\varphi_B$, $y_3=\cos\theta_B$, and similar expressions for z_k can be represented with θ_C and φ_C. Then

$$
\begin{aligned}
\mathrm{Tr}(\hat{M}|\psi_s\rangle\langle\psi_s|)\ &=\ \frac{1}{8}\sum_{ijk}\sum_{lmn}M_{ijk}x_ly_mz_n\,\mathrm{Tr}[(\sigma_i\sigma_l)\otimes(\sigma_j\sigma_m)\otimes(\sigma_k\sigma_n)]\\
&=\sum_{ijk}M_{ijk}x_iy_jz_k=\left(\sum_{ij}M_{ij0}x_iy_j\right)z_0+\left(\sum_{ij}M_{ij1}x_iy_j\right)z_1+\left(\sum_{ij}M_{ij2}x_iy_j\right)z_2+\left(\sum_{ij}M_{ij3}x_iy_j\right)z_3\\
&\leq\left(\sum_{ij}M_{ij0}x_iy_j\right)+\sqrt{\left(\sum_{ij}M_{ij1}x_iy_j\right)^2+\left(\sum_{ij}M_{ij2}x_iy_j\right)^2+\left(\sum_{ij}M_{ij3}x_iy_j\right)^2},
\end{aligned}
\tag{16}
$$

where $M_{ijk}=\frac{1}{8}\mathrm{Tr}(\hat{M}\sigma_i\otimes\sigma_j\otimes\sigma_k)$. Correspondingly, M_l is the rename of some M_{ijk}.

Finally, the maximum value of Λ can be obtained according to the following expression:

$$
\Lambda=\max_{\theta_A,\theta_B,\varphi_A,\varphi_B}[\Lambda_1+\sqrt{(\Lambda_2)^2+(\Lambda_3)^2+(\Lambda_4)^2}],
\tag{17}
$$

where

$$
\Lambda_1\ =\ M_1(x_1+y_1)+M_2(x_3+y_3)+M_3x_1y_1
$$

$$
\begin{aligned}
&+M_4x_2y_2+M_5x_3y_3+M_{12}(x_1y_3+x_3y_1),\\
\Lambda_2 &= M_1+M_3(x_1+y_1)+M_6x_1y_1+M_7(x_1y_3+x_3y_1)\\
&+M_8x_2y_2+M_9x_3y_3+M_{12}(y_3+x_3),\\
\Lambda_3 &= M_4(x_2+y_2)+M_8(x_1y_2+x_2y_1)+M_{10}(x_2y_3+x_3y_2),\\
\Lambda_4 &= M_2+M_5(x_3+y_3)+M_7x_1y_1+M_9(x_1y_3+x_3y_1)\\
&+M_{10}x_2y_2+M_{11}x_3y_3+M_{12}(y_1+x_1).
\end{aligned}
$$

The maximum value of Λ is only related to $\theta_A, \theta_B, \varphi_A, \varphi_B$, and in the process of calculating the witness for the superposition state, we find that θ_i and φ_i $(i = A, B)$ have a certain relationship. Λ is achieved when $\theta_A = \theta_B$, and $\varphi_A = \varphi_B$ or $\varphi_A = -\varphi_B$. The maximum value of Λ calculated by Eq. (17) is a function of M_i. After random searching of M_i to find the minimum value of $\mathcal{L}$, we can obtain the matched entanglement witness.

4. Comparison of three methods

It can be clearly seen from Figs. 1 and 2 that there is no unified generalized Wootters formula and PPT criterion that completely explain the full separability of the superposition state of the three-qubit GHZ state and the W state.

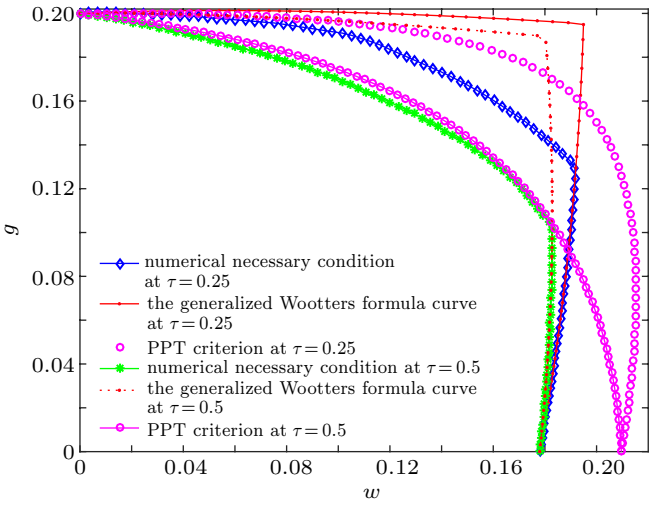


Fig. 1. Necessary conditions of separability for $\rho = w|W\rangle\langle W| + g|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + \tau\sqrt{gw}(|W\rangle\langle\mathrm{GHZ}| + |\mathrm{GHZ}\rangle\langle W|) + \frac{1-g-w}{8}I_8$ at $\tau = 0.25$ and $\tau = 0.5$. The numerical necessary condition is the numerical calculation of matched entanglement witness.

The generalized Wootters formula we constructed can only be used to explain the part close to the W state, in this part the numerically calculated results of matched entanglement witness coincide with the generalized Wootters formula. As the decay factor decreases, the range of coincidence increases, the superposition state is with the least coincidence range. The approximate range that the two curves overlap in the superposition state is $\Theta \in (0^\circ,\ 14^\circ)$; when $\tau = 0.75$, the range is $\Theta \in (0^\circ,\ 20^\circ)$; when $\tau = 0.5$, the range is $\Theta \in (0^\circ,\ 29^\circ)$; when $\tau = 0.25$, the range is $\Theta \in (0^\circ,\ 33^\circ)$, where $\Theta = \arctan(g/w)$. We also find that the PPT criterion is superior to the generalized Wootters formula except for the part near the W state.

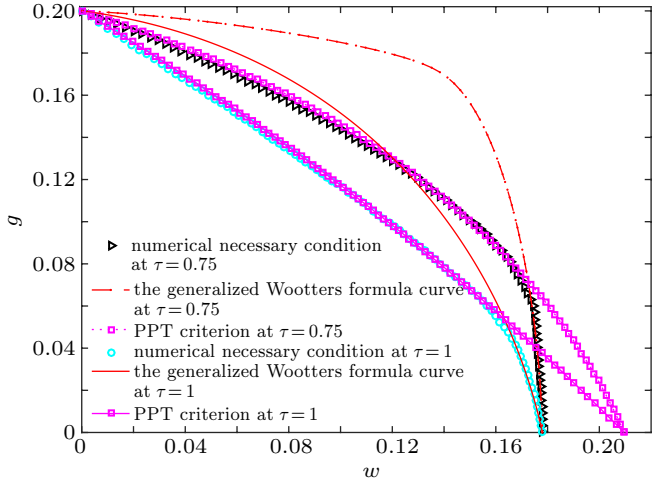


Fig. 2. Necessary conditions of separability for $\rho = w|W\rangle\langle W| + g|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + \tau\sqrt{gw}(|W\rangle\langle\mathrm{GHZ}| + |\mathrm{GHZ}\rangle\langle W|) + \frac{1-g-w}{8}I_8$ at $\tau = 0.75$ and $\tau = 1$. The numerical necessary condition is the numerical calculation of matched entanglement witness.

5. The relationship between entanglement and coherence

Quantum coherence depends on the choice of basis. In the d-dimensional Hilbert space, we choose an orthogonal basis $\{|i\rangle\}_{i=1}^{d}$, if a quantum state can be expressed as a diagonal matrix in the basis then it is called an incoherent state (otherwise it is a coherent state),

$$\delta = \sum_{i=1}^{d} \delta_i |i\rangle\langle i|, \tag{18}$$

where $\delta_i \geq 0$ is the diagonal element of the density matrix. Similarly, if we select $|\mathrm{GHZ}\rangle, |W\rangle$ as the basis in Eq. (8), then the off-diagonal element is $e = \tau\sqrt{gw}$.

The l_1-norm of coherence[30] is an intuitive coherence measure based on the off-diagonal elements of the density matrix. It is defined as

$$C_{l_1}(\rho) = \min_{\sigma\in\mathcal{I}} \|\rho - \sigma\|_{l_1} = \sum_{i\neq j} |\rho_{ij}|, \tag{19}$$

where $\mathcal{I}$ is the set of incoherent states. It exactly corresponds to the sum of the absolute values of all off-diagonal elements of the quantum state density operator. According to formula (19), the expression of coherence measure of the superposition state can be written as

$$C_{l_1}(\rho) = 2|e| = 2|\tau\sqrt{gw}|. \tag{20}$$

Note that the mixed state of the three-qubit GHZ state and W state (incoherent state) is σ, we denote the difference of white noise tolerance between the target state ρ_i and σ as D, where $i = 2,3,4,5$ correspond to the states with $\tau = 0.25$, $\tau = 0.5$, $\tau = 0.75$, $\tau = 1$, respectively; the $\mathcal{L}_{\min}$ for state σ is denoted as r_1, and the $\mathcal{L}_{\min}$ for the target state ρ_i is denoted as r_i, then the expression for D can be written as

$$D = (1 - r_i) - (1 - r_1) = r_1 - r_i. \tag{21}$$

The graphs of Θ–D and Θ-coherence are given in Figs. 3 and 4.

It can be seen from the figures that the relationship between D and coherence is that they are positively correlated.

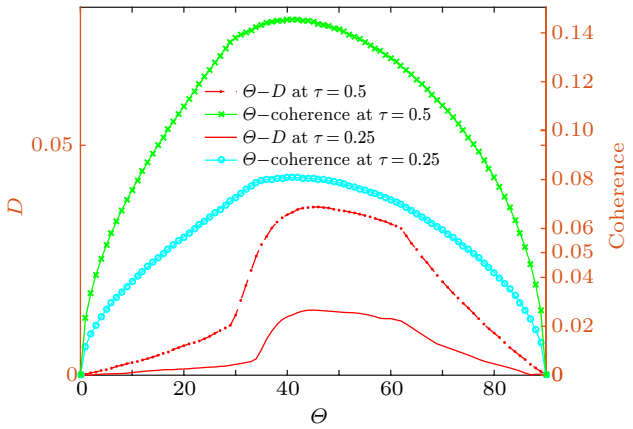


Fig. 3. The coherence and entanglement property of $\tau = 0.5$ and $\tau = 0.25$.

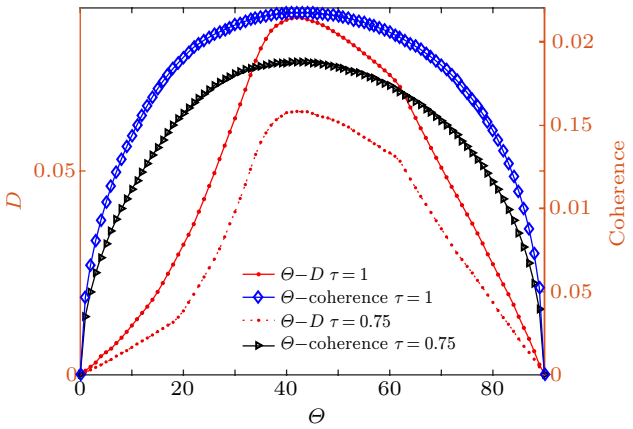


Fig. 4. The coherence and entanglement property of $\tau = 0.75$ and $\tau = 1$.

6. Conclusion

We have conducted a separable condition study on the superposition state of the three-qubit GHZ state and the W state in a noisy environment and extended it to its decayed states ($\tau = 0.25, 0.5, 0.75$). After a series of calculations, we obtain the numerical results of separability by the matched entangled witness. Then we compare the numerical results of separability by the matched entanglement witness with the generalized Wootters formula and PPT criterion. The comparison results can be seen from Figs. 1 and 2. We draw a conclusion that the generalized Wootters formula can be used as a tight criterion for the separability of the superposition state of the three-qubit GHZ state and the W state in the part close to the W state and its decayed states in white noise environment. The PPT criterion is superior to the generalized Wootters formula except for the part near the W state. In the study of coherence, we find that entanglement and coherence are positively correlated.

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