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Lie transformation on shortcut to adiabaticity in parametric driving quantum systems*

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Shortcut to adiabaticity (STA) is a speedway to produce the same final state that would result in an adiabatic, infinitely slow process. Two typical techniques to engineer STA are developed by either introducing auxiliary counterdiabatic fields or finding new Hamiltonians that own dynamical invariants to constraint the system into the adiabatic paths. In this paper, an efficient method is introduced to naturally cover the above two techniques with a unified Lie algebraic framework and neatly remove the design difficulties and loose assumptions in the two techniques. A general STA scheme for different potential expansions concisely achieves with the aid of squeezing transformations.

Keywords: shortcut to adiabaticity, parametric driving, Lewis–Riesenfeld invariants

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1. Introduction

The quantum adiabatic technique is one of the promising strategies for quantum computation based on quantum adiabatic theorem.[1] "Shortcuts to adiabaticity" (STA) are control protocols that take quick routines to the same populations, or even the same final states reached by the slowly adiabatic processes.[2] A motivation to apply STA methods to quantum systems is to manipulate quantum evolution with fast coherent dynamics of high fidelity.[3] Now STA has become a typical technique in preparing or driving internal and external states in atomic, molecular, optical, and solid-state physics.[4–7]

Due to the time-consuming problem of adiabatic parametric driving, two equivalent strategies for STA engineering developed so far. One is the transitionless quantum driving method,[8–10] the other is the invariant-based inverse engineering technique.[11] The basic idea of transitionless quantum driving is to introduce an auxiliary interaction $\hat{H}_1(t)$ to the original Hamiltonian $\hat{H}_0(t)$,

$$\hat{H}_0(t) = \sum_n |n(t)\rangle \varepsilon_n(t) \langle n(t)|, \tag{1}$$

in order to make $\hat{H}_0(t)$ exactly follow the adiabatic evolution reversely driven by $\hat{H}_1(t)$ starting from $\hat{H}_0(0) = \sum_n |n(0)\rangle \varepsilon_n(0) \langle n(0)|$ through evolution operator

$$\hat{U}(t) = \sum_n |\psi_n(t)\rangle \langle n(0)|, \tag{2}$$

where

$$|\psi_n(t)\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t' \varepsilon_n(t')} |n(t)\rangle \tag{3}$$

is the adiabatic states starting with $|n(0)\rangle$ without the geometric phase for the *one* parametric driving system. The reverse auxiliary interaction $\hat{H}_1$ is given by Berry's formulation[12]

$$\begin{aligned} \hat{H}(t) &= \mathrm{i}\hbar(\partial_t \hat{U})\hat{U}^\dagger \\ &= \hat{H}_0(t) + \mathrm{i}\hbar \sum_n |\partial_t n(t)\rangle \langle n(t)| \\ &\equiv \hat{H}_0(t) + \hat{H}_1(t), \end{aligned} \tag{4}$$

where the auxiliary Hamiltonian $\hat{H}_1(t)$ is nondiagonal in the basis of $|n(t)\rangle$, and it requires a full knowledge of spectral properties of the instantaneous Hamiltonian $\hat{H}_0(t)$. However, in most practical cases, the auxiliary interaction calculated by instantaneous eigenstates is hard to carry out. Now we give an alternative method to do STA in a general parametric driving system with Lie transformations. Starting from a general Hamiltonian of harmonic oscillator,[13] three independent generators can be defined by

$$\hat{J}_+ = \frac{1}{2\hbar}\hat{x}^2, \quad \hat{J}_- = \frac{1}{2\hbar}\hat{p}^2, \quad \hat{J}_0 = \frac{\mathrm{i}}{4\hbar}(\hat{x}\hat{p} + \hat{p}\hat{x}), \tag{5}$$

where $\hat{J}_0$ is an anti-Hermitian operator for $\hat{J}_0^\dagger = -\hat{J}_0$. The three operators satisfy the SU(2) commutation relations

$$[\hat{J}_+, \hat{J}_-] = 2\hat{J}_0, \quad [\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm, \tag{6}$$

and we can introduce three independent unitary transformations based on SU(2) by[14]

$$\hat{U}_\pm(t) = \mathrm{e}^{\mathrm{i}\theta_\pm(t)\hat{J}_\pm}, \quad \hat{U}_0(t) = \mathrm{e}^{2\theta_0(t)\hat{J}_0}, \tag{7}$$

where the transformation parameters $\theta_\pm(t)$ and $\theta_0(t)$ are piecewise continuous real functions defined within a control time interval $[0, \tau]$. The above quantum transformations $\hat{U}_\pm(t)$ and $\hat{U}_0(t)$ correspond to classical canonical transformations

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called rotations and squeezing, respectively. These unitary operators maintain the phase volume of the transformed state and constitute our "Lie transformations" method to construct adiabatic Hamiltonian for STA designs.

2. Parametrically driven quantum harmonic oscillator

The first system, which receives much attention due to its fundamental and practical significance, is the parametric harmonic oscillator with fast harmonic trap expansions. It can be applied to the cold atoms involving an adiabatic tuning of the system after a cooling phase.[15] For a parametric driven harmonic oscillator, we consider the Hamiltonian

$$\hat{H}_0(t)=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2(t)\hat{x}^2=\hbar\omega(t)\left(\hat{a}_t^\dagger\hat{a}_t+\frac{1}{2}\right), \tag{8}$$

where $\hat{a}_t$ and $\hat{a}_t^\dagger$ are the annihilation and creation operators at time t defined by

$$\hat{a}_t=\sqrt{\frac{m\omega(t)}{2\hbar}}\left[\hat{x}+\frac{\mathrm{i}}{m\omega(t)}\hat{p}\right], \tag{9}$$

$$\hat{a}_t^\dagger=\sqrt{\frac{m\omega(t)}{2\hbar}}\left[\hat{x}-\frac{\mathrm{i}}{m\omega(t)}\hat{p}\right]. \tag{10}$$

Since the frequency depends on time, the instantaneous ladder operator $\hat{a}_t$ ($\hat{a}_t^\dagger$) will create (annihilate) different instantaneous states at different times adapting to different frequencies.

Now, we consider an STA control from $t=0$ to τ by Lie transformations. In this case, we first introduce squeezing transformation as

$$\hat{U}_0(t)=\mathrm{e}^{2(\ln r)\hat{J}_0}=\mathrm{e}^{\frac{1}{2}(\ln r)[\hat{a}^2-(\hat{a}^\dagger)^2]}, \tag{11}$$

where the subscript t of $\hat{a}$ and $\hat{a}^\dagger$ has been dropped because the squeezing combination $\hat{a}^2-(\hat{a}^\dagger)^2\propto\hat{x}\hat{p}+\hat{p}\hat{x}$ is dynamically independent of time. Here, the transformation parameter $r(t)=\sqrt{\frac{\omega(t)}{\omega(0)}}$ plays as a squeezing (adiabatic) scaling factor, which gives

$$\hat{U}_0(t)|n(0)\rangle=|n(t)\rangle, \tag{12}$$

and the instantaneous states of the final system satisfy

$$\hat{H}_0(t)|n(t)\rangle=\varepsilon_n(t)|n(t)\rangle. \tag{13}$$

A direct substitution of Eq. (12) into Eq. (4) yields the adiabatic Hamiltonian

$$\begin{aligned}\hat{H}(t)&=\sum_n\hat{U}_0|n(0)\rangle\varepsilon_n(t)\langle n(0)|\hat{U}_0^\dagger\\&\quad+\mathrm{i}\hbar\sum_n\dot{\hat{U}}_0|n(0)\rangle\langle n(0)|\hat{U}_0^\dagger\\&=r^2(t)\hat{U}_0(t)\hat{H}_0(t)\hat{U}_0^\dagger(t)+\mathrm{i}\hbar\dot{\hat{U}}_0(t)\hat{U}_0^\dagger(t)\\&=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2(t)\hat{x}^2-\frac{\dot{r}}{2r}(\hat{x}\hat{p}+\hat{p}\hat{x}),\end{aligned} \tag{14}$$

where the initial Hamiltonian $\hat{H}_0(0)=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega_0^2\hat{x}^2$ with $\omega_0\equiv\omega(0)$. Then the auxiliary Hamiltonian $\hat{H}_1(t)$ is given by

$$\hat{H}_1(t)=-\frac{\dot{r}}{2r}(\hat{x}\hat{p}+\hat{p}\hat{x})=-\frac{\dot{\omega}(t)}{4\omega(t)}(\hat{x}\hat{p}+\hat{p}\hat{x}), \tag{15}$$

which is firstly obtained by Berry's formula.[16] Clearly, this method provides a way to design the adiabatic driving of Eq. (4) without resorting to the instantaneous states of $\hat{H}_0(t)$. The result indicates that the higher the speed of the state evolution, the larger the intensity of the required auxiliary field needed to recover the adiabatic evolution. The above result neatly proves that the evolution generated by Hamiltonian $\hat{H}(t)$ exactly leads to the adiabatic state $\Psi(t)=\sum_n c_n|\psi_n(t)\rangle=\sum_n c_n|n(t)\rangle\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_0^t\varepsilon_n(t')\,\mathrm{d}t'}$ from an initial state described by a linear superposition $\Psi(0)=\sum_n c_n|n(0)\rangle$.

However, the control field $\hat{H}_1(t)$ obtained by the above squeezing transformation is still unpractical and hard to be realized because it is non-local. An alternative control field that is local and experimentally realizable can be established by a further rotation transformation

$$\hat{U}_+(t)=\mathrm{e}^{\mathrm{i}\theta_+(t)\hat{J}_+},\quad \theta_+(t)=-\frac{m\dot{r}(t)}{r(t)}, \tag{16}$$

which continues to transform Hamiltonian $\hat{H}(t)$ into

$$\begin{aligned}\hat{H}'(t)&=\hat{U}_+(t)\hat{H}(t)\hat{U}_+^\dagger(t)+\mathrm{i}\hbar\dot{\hat{U}}_+(t)\hat{U}_+^\dagger(t)\\&=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\Omega^2(t)\hat{x}^2,\end{aligned} \tag{17}$$

where the effective frequency

$$\Omega^2(t)=\omega^2(t)-\frac{3}{4}\left[\frac{\dot{\omega}(t)}{\omega(t)}\right]^2+\frac{1}{2}\frac{\ddot{\omega}(t)}{\omega(t)}, \tag{18}$$

which can be easily realized in the practical control.[17] The new Hamiltonian $\hat{H}'(t)$ has the same structure as Hamiltonian $\hat{H}_0(t)$ but with different time-dependent frequencies, which can be conducted by magnetic and optical fields in the cold atom experiments for a time-varying trapping.

Under the above successive transformations, the time evolution of the initial state $|n(0)\rangle$ is mapped to

$$|\Phi_n(t)\rangle=\hat{U}_+(t)|\psi_n(t)\rangle=\mathrm{e}^{-\mathrm{i}\frac{m\dot{\omega}(t)}{4\hbar\omega(t)}x^2}|\psi_n(t)\rangle, \tag{19}$$

which generates a similar final state of an adiabatic process evolving slowly from $\hat{H}_0(0)$ to $\hat{H}_0(\tau)$ if the control frequency $\Omega(t)$ meets the boundary conditions

$$\Omega(0)=\omega(0)\equiv\omega_0,\quad \Omega(\tau)=\omega(\tau)\equiv\omega_\tau. \tag{20}$$

The above conditions lead to

$$\dot{\omega}(0)=\dot{\omega}(\tau)=0,\quad \ddot{\omega}(0)=\ddot{\omega}(\tau)=0. \tag{21}$$

The zero of the first time-derivative means $|\Phi_n(0,\tau)\rangle = |\psi_n(0,\tau)\rangle$ and the zero of the second one implies $\hat{H}'(0,\tau) = \hat{H}_0(0,\tau)$.

In order to design the frequency $\Omega(t)$ to meet Eq. (21), a polynomial ansatz of time function for $\omega(t)$ is often utilized,[11]

$$\omega(t) = \omega_0 + 10\delta s^3 - 15\delta s^4 + 6\delta s^5, \tag{22}$$

where $\delta = \omega_\tau - \omega_0$ and $s = t/\tau$. The trajectories with different control times are displayed in Fig. 1.

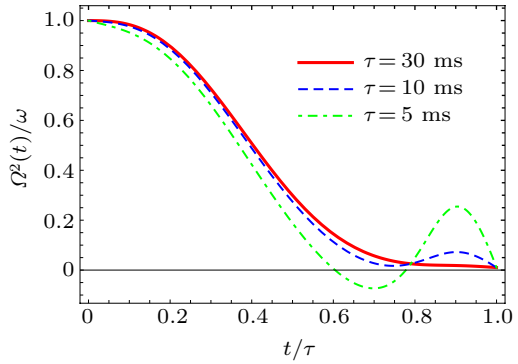


Fig. 1. The adiabatic drivings of the trap frequency $\Omega^2(t)$ changing from $\omega_0 = 2\pi \times 250$ Hz to $\omega_\tau = 2\pi \times 25$ Hz, displayed for three different control times $\tau = 30$ ms (red solid line), $\tau = 10$ ms (blue dashed line) and $\tau = 5$ ms (green dot-dashed line).

We can see that $\Omega^2(t)$ may become negative during some time intervals if the control process is very fast (green dot-dashed line), driving the potential to an expulsive parabola. These state-independent shortcut controls are ideal for maximal robustness owing to no excitation in the final state but allowing for excitations in the intermediate. With Lie transformations, it is convenient to verify that the above STA design is equivalent to the Lewis–Riesenfeld (LR) invariant method.[11] If we set the total transformation $\hat{U}_I(t) \equiv \hat{U}_+(t)\hat{U}_0(t)$, then the new adiabatic Hamiltonian $\hat{H}'(t)$ is obtained by

$$\hat{H}'(t) = r^2\hat{I}(t) + \mathrm{i}\hbar\dot{\hat{U}}_I(t)\hat{U}_I^\dagger(t), \tag{23}$$

where the LR invariant is defined by

$$\hat{I}(t) = \hat{U}_I(t)\hat{H}_0(0)\hat{U}_I^\dagger(t). \tag{24}$$

Naturally, we can see $\hat{I}(0) = \hat{H}_0(0)$ and it satisfies

$$\frac{\mathrm{d}\hat{I}(t)}{\mathrm{d}t} = \frac{\partial\hat{I}(t)}{\partial t} - \frac{1}{\mathrm{i}\hbar}[\hat{H}'(t),\hat{I}(t)] = 0. \tag{25}$$

From Eq. (26), we can find that the eigenvectors of $\hat{I}(t)$ are

$$|\phi_n(t)\rangle = \hat{U}_I(t)|n(0)\rangle, \tag{26}$$

which satisfy

$$\hat{I}(t)|\phi_n(t)\rangle = \lambda_n|\phi_n(t)\rangle, \tag{27}$$

where $\lambda_n = (n+\frac{1}{2})\hbar\omega_0 = \varepsilon_n(0)$. According to the invariant-based inverse engineering, the orthonormal eigenvectors of the invariant $\hat{I}(t)$ and the instantaneous eigenstates of system $\hat{H}_0(t)$ coincide at the initial and final times, then we naturally have

$$\hat{I}(t) = \sum_n |\phi_n(t)\rangle\lambda_n\langle\phi_n(t)| = \sum_n |n(t)\rangle\lambda_n\langle n(t)|, \tag{28}$$

which leads to

$$\hat{I}(t) = r^{-2}\hat{H}_0(t) = \frac{\omega_0}{\omega(t)}\left[\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(t)\hat{x}^2\right]. \tag{29}$$

As the new Hamiltonian of Eq. (23) must be equivalent to Hamiltonian of the system $\hat{H}_0(t)$ at the initial and final times for STA controls, the boundary conditions will be

$$\dot{\hat{U}}_0(t=0,\tau) = 0, \quad \dot{\hat{U}}_+(t=0,\tau) = 0. \tag{30}$$

These new constraints can naturally recover Eq. (21) and make $\hat{H}_1(t)$ vanish at the initial and final times. It should be noted that the transformed Hamiltonian $\hat{H}'(t)$ we found here is just the Hamiltonian for LR dynamical invariant $\hat{I}(t)$. This is the essence of invariant-based inverse engineering and our method avoids the difficulty for constructing dynamical invariants $\hat{I}(t)$ for a given system.[18]

We can conclude that, referring to the previous approaches about harmonic oscillator expansions, the Lie transformation method proposed here provides a solid algebraic framework for STA controls, which consistently introduces auxiliary counterdiabatic Hamiltonian without resorting to classical analog of dissipationless driving.[19]

3. Power-law trap

In the Lie algebraic framework, a generation to other trapping potentials is straightforward. We apply our trick on a free particle inside an expanding one-dimensional box (quantum piston) described by[20]

$$\hat{H}_0(t) = \frac{\hat{p}^2}{2m} + V_{\mathrm{box}}(x,t), \tag{31}$$

where the potential is

$$V_{\mathrm{box}}(x,t) = \begin{cases} 0, & 0 < x < L(t), \\ \infty, & \text{otherwise}, \end{cases} \tag{32}$$

which defines an infinite square well with a moving boundary at $x = L(t)$. If the well expansion is slow enough to follow adiabatic theorem, the exact solution can be expressed as

$$\begin{aligned}\psi_n(x,t) &= \sqrt{\frac{2}{L(t)}}\sin\left[\frac{n\pi x}{L(t)}\right]\exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t'\frac{n^2\pi^2\hbar^2}{2mL^2(t)}\right) \\ &\equiv \varphi_n\left[\frac{x}{L(t)}\right]\exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t \varepsilon_n(t')\mathrm{d}t'\right),\end{aligned} \tag{33}$$

where the instantaneous states satisfy

$$\hat{H}_0(t)\varphi_n\left[\frac{x}{L(t)}\right] = \varepsilon_n(t)\varphi_n\left[\frac{x}{L(t)}\right]. \tag{34}$$

Since the width of box depends on time, a squeezing transformation can be introduced

$$\hat{U}_0(t) = \mathrm{e}^{2\ln\frac{L(0)}{L(t)}\hat{f}_0}, \tag{35}$$

with an initial width $L(0) \equiv L_0$. In the coordinate representation, the squeezing means increasing the wavelength of the free particle and reducing its momentum, which gives

$$\hat{U}_0(t)\varphi_n\left(\frac{x}{L_0}\right) = \varphi_n\left[\frac{x}{L(t)}\right]. \tag{36}$$

Using the same STA design method, we have

$$\begin{aligned}\hat{H}(t) &= \sum_n \hat{U}_0\left|\varphi_n\left(\frac{x}{L_0}\right)\right\rangle\varepsilon_n(t)\left\langle\varphi_n\left(\frac{x}{L_0}\right)\right|\hat{U}_0^\dagger \\ &\quad + \mathrm{i}\hbar\sum_n \dot{\hat{U}}_0\left|\varphi_n\left(\frac{x}{L_0}\right)\right\rangle\left\langle\varphi_n\left(\frac{x}{L_0}\right)\right|\hat{U}_0^\dagger \\ &= \frac{\hat{p}^2}{2m} + \frac{\dot{L}(t)}{2L(t)}(\hat{x}\hat{p}+\hat{p}\hat{x}).\end{aligned} \tag{37}$$

The above Hamiltonian can be used to control quantum gas of non-interacting particles confined in an expanding box with the initial populations sampled microcanonically. The gas will remain a uniform distribution for an arbitrary expanding boundary $L(t)$ because the auxiliary term in Eq. (37) suppresses shock waves (excitations) by uniformly expanding the gas.[21]

This transformation technique can be easily generalized to all even-power-law potentials:[22]

$$\hat{V}(rx_1, rx_2, \ldots, rx_n) = r^k\hat{V}(x_1, x_2, \ldots, x_n), \tag{38}$$

with $k = 2, 4, 6\ldots$ due to the trapping properties of the potentials. Clearly, the case $k = 2$ is the harmonic trapping we have considered in Section 2 and $k \to \infty$ corresponds to the quantum box. The virial theorem shows that the average kinetic energy of the particle in the above potential is $\overline{T}_n = \frac{k}{k+2}\bar{\varepsilon}_n \equiv \mu\bar{\varepsilon}_n$.[23]

For convenience, we set $L_0 = 1$ and the squeezing operator becomes $\hat{U}_0(t) = \mathrm{e}^{-\mu\ln[L(t)]\hat{f}_0}$, which leads to

$$\hat{U}_0(t)\varphi_n(x) = \varphi_n\left[\frac{x}{L^\mu(t)}\right], \tag{39}$$

where $\varphi_n(x)$ are the wavefunctions of Hamiltonian with a one-dimensional potential of Eq. (38). Substituting Eq. (39) into Eq. (4), we can find the controlling adiabatic Hamiltonian

$$\hat{H}(t) = \hat{H}_0(t) + \frac{k}{k+2}\frac{\dot{L}(t)}{2L(t)}(\hat{x}\hat{p}+\hat{p}\hat{x}). \tag{40}$$

It shows that squeezing transformation greatly facilitates the STA design for expansion or compression of the homogeneous trapping potentials. Still, the auxiliary Hamiltonian is difficult to implement in the laboratory, we conduct another transformation $\hat{U}_+(t) = \exp\left[\frac{\mathrm{i}m\dot{\xi}}{2\hbar\xi}\hat{x}^2\right]$ with $\xi = L^\mu(t)$ on Eq. (40) and arrive at

$$\hat{H}'(t) = \hat{H}_0(t) - \frac{1}{2}m\frac{k}{k+2}\frac{\ddot{L}(t)}{L(t)}x^2. \tag{41}$$

The above new Hamiltonian introduces an auxiliary potential

$$U_{\mathrm{aux}}(x,t) = -\frac{1}{2}m\frac{k}{k+2}\frac{\ddot{L}(t)}{L(t)}x^2 \tag{42}$$

to control the system to a final adiabatic state. Now the auxiliary potential is a realistic harmonic potential which can be implemented in the laboratory with well-established technology and have been applied to perform a fast adiabatic squeezing in Bose–Einstein condensate.[24] In this case, the LR invariant in the inverse-engineering approach will be defined by $\hat{I}(t) = \hat{U}_I(t)\hat{H}_0(0)\hat{U}_I^\dagger(t)$. Here, the effective frequency of the auxiliary potential, $\Omega^2(t) = -\frac{k}{k+2}\frac{\ddot{L}(t)}{L(t)}$, is the key to engineer the STA. According to Eq. (30), we impose the boundary conditions $\dot{L}(0,\tau) = \ddot{L}(0,\tau) = 0$. A polynomial ansatz to meet the boundary condition can be designed by

$$L(t) = 1 + [L(\tau) - 1][10s^3 + 3s^4(2s-5)]. \tag{43}$$

The driving frequencies $\Omega^2(t)$ for different power-law potentials are displayed in Fig. 2. It shows a symmetric control process and $U_{\mathrm{aux}}(x,t)$ performs a repulsive harmonic potential in an early stage ($t < \frac{\tau}{2}$), achieving a speed-up STA in an arbitrary finite time τ. In the subsequent stage ($t > \frac{\tau}{2}$), $\Omega^2(t)$ changes its sign to positive and $U_{\mathrm{aux}}(x,t)$ becomes a trapping potential, slowing down the expanding mode. In other words, when the intermediate state deviates from the target state, the auxiliary potential gives a negative feedback, which reproduces the adiabatic result at the end of the evolution. Figure 2 also reveals that a stronger trapping potential means a higher auxiliary control energy. This conclusion has been considered in the expansion stroke of a quantum piston,[25] where an auxiliary harmonic trap is introduced by an electromotive circuit which dissipates a mass of energy to preform the speed-up adiabatic Otto circle.

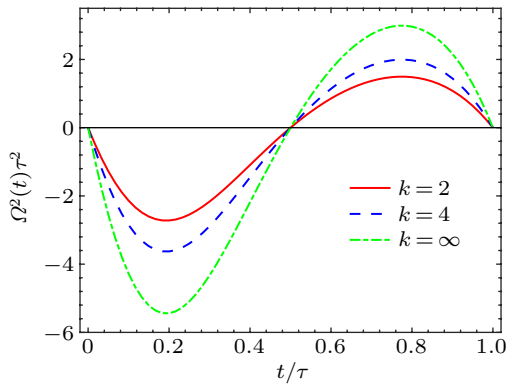


Fig. 2. Shortcuts to adiabaticity by adiabatic driving, for an expansion factor $L(\tau) = 2$ with different power-law traps. The exponent values of auxiliary potentials correspond to harmonic oscillator (red solid line), a quartic trap potential (blue dashed line), and an expanding quantum piston (green dot-dashed line).

4. Conclusion

A unified algebraic framework is developed to design shortcuts to adiabaticity in both transitionless quantum driving and invariant-based inverse engineering method. We demonstrate that the two approaches are mathematically equivalent and share a common physical ground with Lie transformations, and provides a universal method to design STA for the general time-dependent parametric driving systems. It removes the demanding requirement to diagonalize the Hamiltonian of the system in the transitionless driving method, and the difficulty to find the invariant quantities in the invariant-based inverse engineering technique. The Lie transformation method presents a powerful tool to design STA control in the homogenous power-law trapping system and can neatly combine the invariant-based method with the transitionless driving method by quickly diagonalizing the reference Hamiltonian. In summary, this method provides a deeper insight into STA techniques that can be easily applied to a variety of systems in quantum control problems.

Appendix A: Derivations of squeezing transformation

The squeezing transformation is a unitary operator and it is easy to prove

$$\begin{aligned}\hat{U}_0(t)\hat{a}_0\hat{U}_0^{\dagger}(t) &= \hat{a}_0\cosh(-\ln r) - \hat{a}_0^{\dagger}\sinh(-\ln r)\\ &= \sqrt{\frac{m\omega_0}{2\hbar}}\left[\hat{x}+\frac{\mathrm{i}}{m\omega_0}\hat{p}\right]\frac{1+r^2}{2r}\\ &\quad - \sqrt{\frac{m\omega_0}{2\hbar}}\left[\hat{x}-\frac{\mathrm{i}}{m\omega_0}\hat{p}\right]\frac{1-r^2}{2r}\\ &= \sqrt{\frac{m\omega(t)}{2\hbar}}\left[\hat{x}+\frac{\mathrm{i}}{m\omega(t)}\hat{p}\right] \equiv \hat{a}_t, \end{aligned} \tag{A1}$$

where $\hat{a}_0 \equiv \hat{a}(0)$ and $\hat{a}_t \equiv \hat{a}(t)$. Similarly, its conjugate transformation gives

$$\hat{U}_0(t)\hat{a}_0^{\dagger}\hat{U}_0^{\dagger}(t) = \sqrt{\frac{m\omega(t)}{2\hbar}}\left[\hat{x}-\frac{\mathrm{i}}{m\omega(t)}\hat{p}\right] \equiv \hat{a}_t^{\dagger}. \tag{A2}$$

Therefore, we have

$$\begin{aligned}\hat{U}_0(t)|n(0)\rangle &= \hat{U}_0(t)\frac{1}{\sqrt{n!}}(\hat{a}_0^{\dagger})^n|0_0\rangle\\ &= \frac{1}{\sqrt{n!}}\hat{U}_0(t)(\hat{a}_0^{\dagger})^n\hat{U}_0^{\dagger}(t)\hat{U}_0(t)|0_0\rangle\\ &= \frac{1}{\sqrt{n!}}\left[\hat{U}_0(t)\hat{a}_0^{\dagger}\hat{U}_0^{\dagger}(t)\right]^n\hat{U}_0(t)|0_0\rangle\\ &\equiv \frac{1}{\sqrt{n!}}(\hat{a}_t^{\dagger})^n|0_t\rangle = |n(t)\rangle, \end{aligned} \tag{A3}$$

where the vacuum state at time t is defined by $|0_t\rangle \equiv \hat{U}_0(t)|0_0\rangle$.

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