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Citation: Chin. Phys. B, 2021, 30 (**6**): 064216. DOI: 10.1088/1674-1056/abcfa0
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Parameter accuracy analysis of weak-value amplification process in the presence of noise*

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(Received 3 November 2020; revised manuscript received 25 November 2020; accepted manuscript online 2 December 2020)

We theoretically introduce the statistical uncertainty of photon number and phase error to discuss the precision of parameters to be measured based on weak measurements. When the photon counting scheme is used, we discuss the relative accuracy of the system in the presence of phase error by using the orthogonal and nonorthogonal pre- and post-selected states, respectively. When using the measurement scheme of pointer shift, we discuss the measurement accuracy in the presence of phase error, pointer resolution, and statistical uncertainty. These results give a guide way to get the smallest relative precision and deepen our understanding about weak measurement.

Keywords: weak measurement, quantum measurement, accuracy analysis

PACS: 42.50.Tx, 03.65.Ta, 42.50.Xa **DOI:** 10.1088/1674-1056/abcfa0

1. introduction

The concept of weak measurement was introduced by Aharonov, Albert, and Vaidman in 1988.[1] Through about 30 years of development, there are many worthy investigations both in fundamental and application. On one side, weak measurement and weak value are interesting phenomenons as they assist us in understanding many counterintuitive quantum results. On the other side, strange characteristics that weak values can even exceed the eigenvalue range of a typical strong or projective measurement and are usually complex, have allowed a wide range of applications in classical and quantum contexts. Especially, a new parameter estimation method based on weak measurement process, called weak-value amplification technique, is rapidly growing, which has shown significantly advantages for precision metrology. The essence of weak-value amplification technique, when the pre- and post-selected states are almost orthogonal, can result in large weak values that can be used to sensitively determine unknown parameters affecting the state evolution, where the parameters are extracted from the coupling pointer shift.

The weak-value amplification technique has been utilized in many scenarios for ultra small physical quantity estimations, such as tiny lateral displacement,[2] light angular deflections,[3,4] angular rotations,[5] longitudinal phase shift,[6] asymmetric spectral response,[7–9] polarization-dependent loss,[10] velocity measurement,[11] high-precision temperature measurement,[12,13] ultra-small angular velocitiy,[14] and so on. Furthermore, there are a series of analyses about its advantageous performance for measurement precision in certain kinds of experimental imperfections, such as detector saturation,[15] temporally correlated noise,[16] decoherence,[17] and more reported in Refs. [18,19].

To completely discuss the technical advantage of the weak-value amplification technique, it is necessary to make clear the effect of noise. In fact, previous research shows that it amplifies the signal while keeping the technical noises caused by the experimental imperfections, *e.g.*, devices resolution,[2,20] unchanged resulting in the improvement of signal to noise ratio (SNR). However, in the ideal case, the statistical noise or called shot noise, cannot be suppressed via the amplification of the signal and is counterbalanced by the small rate of post-selection.[21–23] More than that, the weak-value amplification technique is also deemed suboptimal,[24] even fails to suppress technical noise.[25]

Despite all of these, there are a variety of theoretical[26,27] and experimental efforts to optimize the weak-value amplification technique, such as using technical

*Project supported by the National Key Research and Development Program of China (Grant No. 2017YFA0305200), the National Natural Science Foundation of China (Grant Nos. 11674234 and 11605205), the Natural Science Foundation of Chongqing (Grant Nos. cstc2015jcyjA00021 and cstc2018jcyjAX0656), the Innovation Project of Sichuan University (Grant No. 2018SCUH0021), the Youth Innovation Promotion Association Program of the Chinese Academy of Sciences (CAS) (Grant No. No. 2015317), the Entrepreneurship and Innovation Support Program for Chongqing Overseas Returnees (Grant Nos. cx2017134 and cx2018040), the Fund of CAS Key Laboratory of Microscale Magnetic Resonance, and the Fund of CAS Key Laboratory of Quantum Information.

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noise,[28] entanglement,[29–32] squeezed state,[33] recycled photons,[34–36] and beyond the weak-value amplification, *e.g.* joint weak measurements,[37–39] projective photon-counting measurements.[40] Most of the work mainly concentrate on the average displacement of the pointer to measure tiny physical quantity, meanwhile it is of efficiency to use the post-selected photon number[40–43] to realize the measurement of tiny physical quantities[44–46] and the estimation of quantum states.[47–49]

In fact, on the one hand, we should explore the fundamental significance of the weak-value amplification technique, on the other hand, it is also important to investigate and analyze the optimization of the existing protocol of the weak-value amplification technology. Especially in many specific protocols, the main noise source is relatively clear. This allows us to analyze the noise more pertinently and choose better suppression methods. In this paper, we analyze the precision by using pointer shift and photon number counting in the presence of phase errors and pointer resolution. When the photon counting scheme is used, we discuss the relative accuracy of the system in the presence of phase error by using the orthogonal and nonorthogonal pre- and post-selected states, respectively. For the pointer shift measurement scheme, we discuss the measurement accuracy in the presence of phase error, pointer resolution, and statistical uncertainty. These results give a guide way to get the smallest relative precision and deepen our understanding about weak measurement.

2. The framework for pps measurement

We consider the weak measurement process, which can be described as follows. Firstly, the system and meter are initially prepared in $\rho = |\psi\rangle\langle\psi|$ and $\rho_m = |\psi_m\rangle\langle\psi_m|$ respectively. Then the system and the meter, in the time interval $(t_\mathrm{i}, t_\mathrm{f})$, undergoes the weak interaction, which is given by the Hamiltonian

$$\hat{H} = \gamma \hat{A} \otimes \hat{F}, \tag{1}$$

where $\gamma = \int_{t_\mathrm{i}}^{t_\mathrm{f}} g(t)\,\mathrm{d}t$ is the interaction constant, $g(t)$ is the instantaneous coupling rate, and $\hat{A}$ is the observable of system and $\hat{F}$ is meter's input variable. The joint initial state $\rho \otimes \rho_m$, for the unitary transformation $U = \exp(-\mathrm{i}\gamma \hat{A} \otimes \hat{F})$, becomes $\rho_\mathrm{f} = |\psi_\mathrm{f}\rangle\langle\psi_\mathrm{f}| = \hat{U}\rho \otimes \rho_m \hat{U}^\dagger$. Through this interaction procedure, the meter and the system are entangled and it becomes possible to extract the information about the interaction γ or the system from the meter. If the interaction constant γ is large, this will be corresponding to the standard Von-Neumann measurement scheme, in which the pointer eigenstate of observable $\hat{A}$ can be exactly distinguished. Here we consider the interaction constant γ is weak.

The last step is post-selection. In this process, we have two different treatments for measurement results. One is, we are only concerned about the probability of finding the system in the post-selected state. When post-selecting $\Pi_\phi = |\phi\rangle\langle\phi|$ on the system, we obtain the detected probability of the system state $|\phi\rangle$

$$P = \mathrm{Tr}\,[(\Pi_\phi \otimes I_m)\rho_\mathrm{f}]. \tag{2}$$

The normalized state in meter space is given by

$$|\Psi_p\rangle = \langle\phi|\exp(-\mathrm{i}\gamma \hat{A} \otimes \hat{F})|\psi\rangle|\psi_m\rangle/\sqrt{P}. \tag{3}$$

The above two equations show that after post-selection a single particle to be selected with probability P in Eq. (2) has the probability distribution $|\Psi_p(F)|^2$. When the above process repeats N times, the detecting process can be regarded as a number of N-independent events q, which has the probability P to be one otherwise to be zero, and the detected number n is the binomial distribution $n \sim B(N, P)$. The detected number is $n = \sum_{i=1}^{N} q_i$, of which the detected number concentrates on

$$n = \bar{n} \pm \Delta n, \tag{4}$$

where $\bar{n} = NP$ is the expectation value and $\Delta n = \sqrt{N}\sqrt{P(1-P)}$ is its fluctuation.

The other is selected by the detector of the meter, of which the records are actually the probability distribution $|\Psi_p(R)|^2$, where $\hat{R}$ is the measurement operator of the pointer. The signal is the average shift defined as $r = \sum_{i=1}^{n} r_i/n$. As n increases, the value of r will concentrate on

$$r = \bar{R}_p \pm \Delta R/\sqrt{n}, \tag{5}$$

where $\bar{R}_p = \langle\Psi_p|\hat{R}|\Psi_p\rangle$ is the expectation value and ΔR is the standard deviation of the state $|\Psi_p(R)|^2$. With the linear validity condition[50] $\gamma|A_w|(\Delta F + |\bar{F}|) \ll 1$, equation (3) is derived as $|\Psi_p\rangle \approx \langle\phi|\psi\rangle(1 - \mathrm{i}\gamma A_w \hat{F})|\psi_m\rangle$, where A_w is the weak value defined as $\langle\phi|\hat{A}|\psi\rangle/\langle\phi|\psi\rangle$ and $\bar{F} = \langle\psi_m|\hat{F}|\psi_m\rangle$ is the average value of the coupling variable. The expectation is obtained as $\bar{R}_p \approx \bar{R} + \gamma\,\mathrm{Im}\,A_w(\overline{\{\hat{R}, \hat{F}\}} - 2\bar{R}\bar{F}) + \gamma\,\mathrm{Re}\,A_w\,\mathrm{Im}\,\overline{[\hat{R}, \hat{F}]}$, where the $\langle\Psi_m|\hat{O}|\Psi_m\rangle$ is denoted as $\overline{O}$ for short and $\{\hat{R}, \hat{F}\} = \hat{R}\hat{F} + \hat{F}\hat{R}$ is anticommutator. As $\hat{F} = \hat{R}$, we obtain

$$\widetilde{F} = 2\gamma\,\mathrm{Im}\,A_w(\Delta F)^2, \tag{6}$$

where $\widetilde{F} = \bar{F}_p - \bar{F}$ is the pointer shift as signal.

3. Photon number counting

In this section, we analyze such the measured parameter accuracy, where the parameter is extracted from the probability of finding the system in the post-selected state which is realized by measuring the post-selected photon number. We systematically analyze the precision with the orthogonal and nonorthogonal pre- and post-selected states in the presence of phase error. In realistic experiments, the systematic phase errors due to device imperfections cannot be avoided. For simplicity, we summarize all these in a phase error ε. When we use photon counting method, the pixel or digitization noise

of the meter is ignored. In the following sections, we suppose that $\hat{A} = |0\rangle\langle 0| - |1\rangle\langle 1|$, $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, and $|\phi\rangle = (\exp^{-i\varphi}|0\rangle - \exp^{i\varphi}|1\rangle)/\sqrt{2}$, so $A_{\phi\psi} = \cos\varphi$, $\langle\phi|\psi\rangle = i\sin\varphi$, weak value $A_w = -i\cot\varphi$. For the operator $\hat{A}^2 = \hat{I}$,[51] the unitary operator $\exp(-i\gamma\hat{A}\otimes\hat{F})$ can be described as

$$\hat{U} = \cos(\gamma\hat{F}) - i\hat{A}\sin(\gamma\hat{F}). \tag{7}$$

Then taking Eq. (7) into Eq. (2), without approximations, the post-selected probability is obtained as

$$P = \begin{cases} |\langle\phi|\psi\rangle|^2[1 + \overline{\sin(2\gamma\hat{F})}\,\mathrm{Im}\,A_w + \overline{\sin^2(\gamma\hat{F})}(|A_w|^2 - 1)], & \langle\phi|\psi\rangle \neq 0, \\ |A_{\phi\psi}|^2\overline{\sin^2(\gamma\hat{F})}, & \langle\phi|\psi\rangle = 0. \end{cases} \tag{8}$$

where $A_{\phi\psi} = \langle\phi|\hat{A}|\psi\rangle$. From Eq. (8), there are two ways to extract the information γ.

First, we consider the pre- and post-selection is orthogonal, which has been used already in the measurement of the spin Hall effect of light,[45] and discussed in Refs. [52,53]. Here the weak approximation $\gamma(\bar{F} + \Delta F) \ll 1$ is supposed, the probability is

$$P = \gamma^2(\bar{F}^2 + \Delta F^2), \qquad \langle\phi|\psi\rangle = 0, \tag{9}$$

where $\bar{F}$ is the expected value of the variable F and ΔF is the standard deviation of the meter's initial distribution. From this formula, it shows an ability to extract information about γ through the photon number counting method. If we consider a phase error ε, the value of $A_{\phi\psi}$ becomes $\cos(\varphi + \varepsilon)$. Taking it into Eq. (8), expanding it and neglecting the higher order terms, we obtain the probability with phase error

$$P_\varepsilon \approx (1 - \varepsilon^2)\gamma^2(\bar{F}^2 + \Delta F^2). \tag{10}$$

Taking Eq. (10) into Eq. (4), the detected photon number is expressed as

$$n_\varepsilon = \overline{n_\varepsilon} \pm \Delta n_\varepsilon, \tag{11}$$

where $\overline{n_\varepsilon} = NP_\varepsilon$ is the expectation with pase error and $\Delta n_\varepsilon \approx \sqrt{NP_\varepsilon} \approx \sqrt{NP}$ under the condition that P_ε is extremely low.

Without considering the phase error and statistical uncertainty, an ideal value γ can be obtained from Eqs. (5) and (9),

$$\gamma = \sqrt{\bar{n}/[(\bar{F}^2 + \Delta F^2)N]}. \tag{12}$$

When the phase error and statistical uncertainty are taken into account, $\bar{n}$ in Eq. (12) is replaced with n_ε and the measured parameter γ_ε with errors is given by

$$\gamma_\varepsilon = \sqrt{n_\varepsilon/[(\bar{F}^2 + \Delta F^2)N]}. \tag{13}$$

Under one-order approximation, taking Eq. (11) into Eq. (13) becomes

$$\gamma_\varepsilon = \gamma\left(1 - \frac{\varepsilon^2}{2} \pm \frac{1}{\sqrt{NP}}\right), \tag{14}$$

where the last two terms give the absolute precision

$$\delta_\gamma = \gamma_\varepsilon - \gamma = \gamma\left(\frac{\varepsilon^2}{2} + \frac{1}{\sqrt{NP}}\right). \tag{15}$$

In order to reflect the precision of the measurement more accurately, we introduce the relative precision, which is given by

$$\frac{\delta_\gamma}{\gamma} = \varepsilon^2/2 + \frac{1}{\sqrt{NP}}. \tag{16}$$

From Eq. (16), we can see that the relative precision is dependent on two factors. One is the phase error, which can be reduced by optimizing the experimental system. The other is the input photon numbers that we can increase to improve the relative precision.

Moreover there is also an interesting result. When the pre- and post-selected states are orthogonal corresponding to $\varphi = 0$, the probability $P = \gamma^2(\bar{F}^2 + \Delta F^2)$. Under the condition $P \ll 1$, the standard deviation $\sqrt{NP(1-P)}$ approximates to $\Delta n = \sqrt{N}\gamma\sqrt{\bar{F}^2 + \Delta F^2}$. This shows that the statistical uncertainty also includes the information of the parameter γ. The parameter γ can be extracted from Δn.

Secondly, we consider the case that pre- and post-selected states are nonorthogonal. Under the condition that $\gamma|A_w|(\Delta F + |\bar{F}|) \ll 1$ (the second-order term in γ has been neglected), the post-selected probability in Eq. (8) can be rewritten as

$$P_1 \approx |\langle\phi|\psi\rangle|^2(1 + 2\gamma\bar{F}\,\mathrm{Im}\,A_w), \tag{17}$$

$$P_2 \approx |\langle\phi|\psi\rangle|^2(1 - 2\gamma\bar{F}\,\mathrm{Im}\,A_w), \tag{18}$$

where we take two measurements to get two opposite weak values $\pm|\mathrm{Im}\,A_w|$. Through the difference between P_1 and P_2, the parameter γ can be calculated by

$$\gamma = \frac{P_1 - P_2}{P_1 + P_2}\frac{1}{2\bar{F}\,\mathrm{Im}\,A_w}. \tag{19}$$

Equation (19) can be rewritten as

$$\gamma = \frac{\overline{n_1} - \overline{n_2}}{\overline{n_1} + \overline{n_2}}\frac{1}{2\bar{F}\,\mathrm{Im}\,A_w}, \tag{20}$$

where $\overline{n_i} = NP_i$ $(i = 1, 2)$ are the expectation of detected photon numbers corresponding to the two different measurements.

Replacing φ with $\varphi \pm \varepsilon$ in Eq. (17) and ignoring the second order term in ε, we obtain

$$P_{\varepsilon 1} \approx \varphi^2 - 2\gamma \bar{F}\varphi + 2\varphi\varepsilon, \tag{21}$$

$$P_{\varepsilon 2} \approx \varphi^2 + 2\gamma \bar{F}\varphi + 2\varphi\varepsilon. \tag{22}$$

Considering the statistical uncertainty, the realistic detected numbers are given by

$$n_{\varepsilon 1} \approx N\varphi^2 - N2\gamma \bar{F}\varphi + N2\varphi\varepsilon \pm \sqrt{N}\varphi, \tag{23}$$

$$n_{\varepsilon 2} \approx N\varphi^2 + N2\gamma \bar{F}\varphi + N2\varphi\varepsilon \pm \sqrt{N}\varphi, \tag{24}$$

where $n_{\varepsilon i} = NP_{\varepsilon i}$ $(i = 1, 2)$. Replacing n_1 and n_2 in Eq. (20) with $n_{\varepsilon 1}$ and $n_{\varepsilon 2}$ in Eqs. (23) and (24), the parameter including the error γ_ε is obtained with

$$\gamma_\varepsilon \approx \gamma \pm \left(\frac{\varepsilon}{\bar{F}} + \frac{1}{2\bar{F}\sqrt{N}} \right), \tag{25}$$

and the absolute accuracy is given by

$$\delta_\gamma = \frac{\varepsilon}{\bar{F}} + \frac{1}{2\bar{F}\sqrt{N}}, \tag{26}$$

which shows that in addition to reducing ε and increasing N, we can also increase the expectation value $\bar{F}$ to improve the absolute accuracy for the measurement of γ.

By comparing Eqs. (15) and (26), we can get their absolute precision under specific experimental conditions. According to this absolute accuracy, we can choose a suitable measurement scheme in the situation that the pre- and post-selections are orthogonal or not.

Now we have studied about the way by using photon number counting to realize the ultra-small parameter measurement and obtained its precision under the statistical uncertainty and phase error. Next, we study the accuracy of the parameter to be measured using the pointer shift.

4. Pointer shift

Supposed that the initial state of the pointer is $|\psi_m\rangle = \int \Psi_m(F)\,\mathrm{d}F|F\rangle$. According to Eq. (6) the parameter γ to be measured can be obtained with

$$\gamma = -\widetilde{F}\tan\varphi/(2\Delta F^2). \tag{27}$$

Next, we will discuss the accuracy for the measurement of parameter γ in two cases. One is in the existence of phase error and pointer resolution, the other is in the existence of statistical uncertainty.

First, considering the influence of phase error ε and pointer resolution δ_F on the parameter γ, equation (27) can be rewritten as

$$\gamma_\varepsilon = -(\widetilde{F} \pm \delta_F)\tan(\varphi \pm \varepsilon)/(2\Delta F^2). \tag{28}$$

Expanding it and neglecting the second-order term, we obtain

$$\gamma_\varepsilon \approx -(\widetilde{F}\tan\varphi \pm \delta_F \tan\varphi \pm \widetilde{F}\varepsilon)/(2\Delta F^2). \tag{29}$$

The absolute accuracy is obtained with

$$\delta_\gamma = \gamma_\varepsilon - \gamma = -(\delta_F \tan\varphi + \widetilde{F}\varepsilon)/(2\Delta F^2). \tag{30}$$

The relative precision is given by

$$\frac{\delta_\gamma}{\gamma} = \frac{\delta_F}{\widetilde{F}} + \varepsilon\cot\varphi. \tag{31}$$

Putting Eq. (6) into Eq. (31), we can rewritten the formula with the weak value

$$\frac{\delta_\gamma}{\gamma} = \frac{\delta_F}{2\gamma|A_w|\Delta F^2} + \varepsilon|A_w|. \tag{32}$$

The first item shows that under certain resolution increasing weak value can decrease relative precision. However the second item shows that under certain phase error decreasing the weak value can decrease relative precision. The smallest relative precision condition is given by

$$\frac{\delta_F}{2\gamma|A_w|\Delta F^2} = \varepsilon|A_w| \to |A_w| = \sqrt{\frac{\delta_F}{2\gamma\Delta F^2\varepsilon}}. \tag{33}$$

Combining Eqs. (32) and (33), we obtain the smallest relative precision

$$\frac{\delta_\gamma}{\gamma} = \sqrt{\frac{2\delta_F\varepsilon}{\gamma\Delta F^2}}. \tag{34}$$

Under the condition that δ_F, ε, and γ are determined, we can improve the relative precision by increasing the statistical uncertainty ΔF.

In Ref. [21], by replacing δ_F and ε with $\widetilde{F}$ and φ respectively, the authors have given the absolute accuracy $\varepsilon\delta_F/(2\Delta F^2)$. But in some practical situations, the relative accuracy is also an important criterion.

Second, we consider the influence of the statistical uncertainty $\Delta F/\sqrt{n}$ on the measurement of γ. Equation (27) can be rewritten as

$$\gamma_s = -\left(\widetilde{F} \pm \frac{\Delta F}{\sqrt{n}}\right)\tan\varphi/(2\Delta F^2). \tag{35}$$

The absolute accuracy is

$$\delta_\gamma = \gamma_s - \gamma = -\frac{\tan\varphi}{\sqrt{n}}/(2\Delta F), \tag{36}$$

and the relative precision is

$$\frac{\delta_\gamma}{\gamma} = \frac{\Delta F}{\sqrt{n}\widetilde{F}}. \tag{37}$$

In the usual way, we do not consider the influence of the statistical uncertainty of the output number on the pointer shift. Then $n = \bar{n} \approx N\sin^2\varphi \approx N/|A_w|^2$ and the pointer shift $\widetilde{F} = 2\Delta F^2\,\mathrm{Im}\,A_w$. We can see that n and $\widetilde{F}$ are simultaneously

influenced by the weak value. Putting them into Eq. (36), it can be rewritten as

$$\frac{\delta_\gamma}{\gamma} = \frac{1}{2\sqrt{N}\Delta F}. \tag{38}$$

We find that the relative precision is only related to the initial state of the system and meter, that is, the total number of photons and the uncertainty of the pointer. And it is irrelevant to pre- and post-selected states of the system, *i.e.*, the weak value. Therefore, it is not possible to improve the relative precision by adjusting the weak value under the condition of only considering the statistical uncertainty of the pointer shift.

However, if we consider the statistical uncertainty of the output number n, the realistic detected number is $N\sin^2\varphi \pm \sqrt{N}\sin\varphi$. So the statistical uncertainty of the pointer shift in Eq. (35) has a fluctuation

$$\begin{aligned}\frac{\Delta F}{\sqrt{n}} &= \frac{\Delta F}{\sqrt{N\sin^2\varphi \pm \sqrt{N}\sin\varphi}}, \\ &= \frac{\Delta F}{\sqrt{N/|A_w|^2 \pm \sqrt{N}/|A_w|}}.\end{aligned} \tag{39}$$

In this part, we consider the use of the pointer shift to measure parameters. When the phase error and pointer resolution are taken into account, the minimum relative accuracy is given. In the existence of the statistical uncertainty, we give the relative precision, which is only relevant to the input total number and the standard deviation of the pointer.

5. Conclusion

In this paper, the statistical uncertainty of photon number and phase error have been introduced theoretically, and the parameter measurement accuracy based on weak measurement has been discussed. When using the photon counting scheme, we have investigated the relative accuracy of the system in the case of phase error by using orthogonal and non-orthogonal pre-selected states and post-selected states respectively. The measurement accuracy with phase error, pointer resolution, and statistical certainty have also been investigated. These results provide a guidance for obtaining minimum relative accuracy and deepening understanding of the weak measurement.

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