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Exact solution of the Gaudin model with Dzyaloshinsky–Moriya and Kaplan–Shekhtman–Entin–Wohlman–Aharony interactions*

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We study the exact solution of the Gaudin model with Dzyaloshinsky–Moriya and Kaplan–Shekhtman–Entin–Wohlman–Aharony interactions. The energy and Bethe ansatz equations of the Gaudin model can be obtained via the off-diagonal Bethe ansatz method. Based on the off-diagonal Bethe ansatz solutions, we construct the Bethe states of the inhomogeneous XXX Heisenberg spin chain with the generic open boundaries. By taking a quasi-classical limit, we give explicit closed-form expression of the Bethe states of the Gaudin model. From the numerical simulations for the small-size system, it is shown that some Bethe roots go to infinity when the Gaudin model recovers the $U(1)$ symmetry. Furthermore, it is found that the contribution of those Bethe roots to the Bethe states is a nonzero constant. This fact enables us to recover the Bethe states of the Gaudin model with the $U(1)$ symmetry. These results provide a basis for the further study of the thermodynamic limit, correlation functions, and quantum dynamics of the Gaudin model.

Keywords: integrable models, Gaudin model, Bethe ansatz, Bethe states

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1. Introduction

The Gaudin model is an important integrable system with long-range interaction.[1] It is closely related to the central spin model[2,3] and BCS model,[4–9] which has been widely used in the study of metallic nanoparticles,[10] quantum dots,[11] noisy spins,[12] Rydberg atoms,[13] and so on. Based on the exact solutions of the Gaudin model, its correlation functions,[14] quantum dynamics[15–18] and coherence[19] have always been the focus of research.

The Gaudin Hamiltonian can be constructed by taking the quasi-classical expansion of the transfer matrix of the spin chain.[20–24] For the periodic case, the corresponding Gaudin Hamiltonian includes the external field on the central spin ($\boldsymbol{B}$) and Heisenberg exchange interaction ($\boldsymbol{A}_k$). For the open case, the corresponding Gaudin Hamiltonian also includes the Dzyaloshinsky–Moriya (DM) interaction ($\boldsymbol{D}_k$)[25,26] and the Kaplan–Shekhtman–Entin–Wohlman–Aharony (KSEA) interaction ($\mathcal{M}_k$),[27,28] i.e.,

$$\begin{aligned} H_j = {} & \boldsymbol{B}\cdot\boldsymbol{\sigma}_j + \sum_{k=1,k\neq j}^{N}\left(A_k^x\sigma_j^x\sigma_k^x + A_k^y\sigma_j^y\sigma_k^y + A_k^z\sigma_j^z\sigma_k^z\right) \\ & + \boldsymbol{D}_k\cdot(\boldsymbol{\sigma}_j\times\boldsymbol{\sigma}_k) + \boldsymbol{\sigma}_j\cdot\mathcal{M}_k\cdot\boldsymbol{\sigma}_k, \end{aligned} \tag{1}$$

where N is the number of spins. The constants $\boldsymbol{B}$, $\boldsymbol{A}_k$, $\boldsymbol{D}_k$ and $\mathcal{M}_k$ are defined below and their values rely on the boundary parameters and a set of inhomogeneous parameters $\{\theta_1,\ldots,\theta_N\}$. For the generic open cases, the corresponding Gaudin Hamiltonian has no obvious reference state due to the $U(1)$ symmetry breaking (i.e., the total spin along z-direction is not conserved). This leads to employing the conventional Bethe ansatz methods to approach the eigenvalues and Bethe states of the model very involved. With the help of the off-diagonal Bethe ansatz (ODBA) method,[29,30] the eigenvalues of the XXX model with generic open boundaries were given in Ref. [31]. Recently, the Gaudin model has been revisited many times.[32–34] However, the Bethe states of the Gaudin Hamiltonian (1) are still absent.

In this paper, one of our aims is to construct the Bethe states of the Gaudin model (1). By employing some gauge transformations, we obtain the operators and reference state for constructing the Bethe states, respectively. Taking the quasi-classical expansion of the Bethe states of the spin chain, we obtain the Bethe states of the Gaudin model (1). We also study the degeneration of the Bethe states when the model recovers the $U(1)$ symmetry.

The rest of this paper is organized as follows. In Section 2, we present the ODBA solutions of the Gaudin model. In Section 3, with the help of the ODBA solutions, we construct the Bethe states of the Gaudin model. The Bethe states, eigenvalues and Bethe ansatz equations recovering that of the Gaudin model with the $U(1)$ symmetry are given in Section 4. We summarize our results and give some discussions in Section 5. Some technical proofs are given in Appendix A.

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2. ODBA solutions

The integrability of the Gaudin model is associated with the R-matrix

$$R(u)=\begin{pmatrix} u+\eta & & & \\ & u & \eta & \\ & \eta & u & \\ & & & u+\eta \end{pmatrix}, \tag{2}$$

where u is the spectral parameter and η is the crossing parameter. The R-matrix satisfies the quantum Yang–Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v)=R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{3}$$

Here $R_{21}(u)=P_{12}R_{12}(u)P_{12}$ with P_{12} being the usual permutation operator. In what follows, we adopt the standard notations: for any matrix $O\in \mathrm{End}(\boldsymbol{V})$, O_k is an embedding operator in the tensor space $\boldsymbol{V}\otimes\boldsymbol{V}\otimes\cdots$, which acts as O on the k-th space and as identity on the other factor spaces; $R_{kl}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as an identity on the factor spaces except for the k-th and l-th ones.[35]

The general reflection matrices are given by[31]

$$K^-(u)=\xi+u\boldsymbol{h}_1\cdot\boldsymbol{\sigma},\quad \boldsymbol{h}_1^2=1, \tag{4}$$

which satisfy the reflection equation (RE)

$$\begin{aligned} &R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v)\\ =\;&K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v). \end{aligned} \tag{5}$$

The dual K-matrix $K^+(u)$ is

$$K^+(u)=\bar{\xi}+(u+\eta)\boldsymbol{h}_2\cdot\boldsymbol{\sigma},\quad \boldsymbol{h}_2^2=1, \tag{6}$$

which satisfies the dual RE

$$\begin{aligned} &R_{12}(v-u)K_1^+(u)R_{21}(-u-v-2\eta)K_2^+(v)\\ =\;&K_2^+(v)R_{12}(-u-v-2\eta)K_1^+(u)R_{21}(v-u). \end{aligned} \tag{7}$$

The parameters $\bar{\xi}$ and $\boldsymbol{h}_2$ depend on the crossing parameter η. Expansion of $\bar{\xi}$ and $\boldsymbol{h}_2$ with respect to η is as follows:

$$\begin{aligned} \boldsymbol{h}_2(\eta)&=\boldsymbol{h}_1+\boldsymbol{h}_2^{(1)}\eta+\frac{\boldsymbol{h}_2^{(2)}}{2!}\eta^2+\mathcal{O}(\eta^3),\\ \bar{\xi}(\eta)&=-\xi+\xi^{(1)}\eta+\mathcal{O}(\eta^2). \end{aligned} \tag{8}$$

Using the normalization (6), we have

$$\boldsymbol{h}_1\cdot\boldsymbol{h}_2^{(1)}=0,\quad \boldsymbol{h}_1\cdot\boldsymbol{h}_2^{(2)}=-\boldsymbol{h}_2^{(1)}\cdot\boldsymbol{h}_2^{(1)}. \tag{9}$$

From the above relations, we have $\lim_{\eta\to0}\{K^+(u)K^-(u)\}=(u^2-\xi^2)\times\mathrm{id}$. This identity is essential for the Gaudin model.

Let us introduce the one-row monodromy matrices

$$T_0(u) = R_{0N}(u-\theta_N)R_{0N-1}(u-\theta_{N-1})\cdots R_{01}(u-\theta_1), \tag{10}$$

$$\hat{T}_0(u) = R_{10}(u+\theta_1)R_{20}(u+\theta_2)\cdots R_{N0}(u+\theta_N), \tag{11}$$

and the double-row monodromy matrix

$$\mathcal{U}(u)=T(u)K^-(u)\hat{T}(u)\equiv\begin{pmatrix}\mathcal{A}(u) & \mathcal{B}(u)\\ \mathcal{C}(u) & \mathcal{D}(u)\end{pmatrix}. \tag{12}$$

Then the transfer matrix is given by

$$t(u)=\mathrm{tr}_0(K_0^+(u)\mathcal{U}_0(u)). \tag{13}$$

With the same procedure introduced in Ref. [36], one can obtain the commutativity of the transfer matrix $[t(u),t(v)]=0$, which ensures the integrability of the model. Expanding the transfer matrix $t(u)$ at the point $u=\theta_j$ around $\eta=0$,[36] we obtain

$$\begin{aligned} H_j=\;&\frac{1}{\theta_j\prod_{k=1,k\neq j}^N(\theta_j-\theta_k)\prod_{k=1}^N(\theta_j+\theta_k)}\frac{\partial(t(\theta_j)/\eta)}{\partial\eta}\bigg|_{\eta=0}\\ &-\frac{\xi\xi^{(1)}}{\theta_j}-1+\frac{\xi^2}{\theta_j^2}-\sum_{k=1,k\neq j}^N\frac{\theta_j^2-\xi^2}{\theta_j^2-\theta_k^2}, \end{aligned} \tag{14}$$

where the constants in the Hamiltonian (1) are expressed in terms of the boundary parameters in the corresponding K-matrices given in Eqs. (4) and (6) as follows:

$$\begin{aligned} &\boldsymbol{B}=i\theta_j(\boldsymbol{h}_1\times\boldsymbol{h}_2^{(1)})+(\xi^{(1)}\boldsymbol{h}_1+\xi\boldsymbol{h}_2^{(1)}),\\ &A_k^\alpha=\frac{\theta_j\theta_k-\xi^2}{\theta_j^2-\theta_k^2}+\frac{\theta_j}{\theta_j+\theta_k}(h_1^\alpha)^2,\quad D_k^\alpha=\frac{i\xi}{\theta_j+\theta_k}h_1^\alpha,\\ &\mathcal{M}_k^{\alpha\beta}=\frac{\theta_j}{\theta_j+\theta_k}h_1^\alpha h_1^\beta(1-\delta_{\alpha,\beta}). \end{aligned} \tag{15}$$

It should be remarked that the $U(1)$ symmetry of the Gaudin model (1) is broken for $\boldsymbol{h}_2^{(1)}\neq0$.

With the help of the ODBA method, the eigenvalues $\Lambda(u)$ of the transfer matrix $t(u)$ can be given by an inhomogeneous T–Q relation[31]

$$\begin{aligned} \Lambda(u)=\;&a(u)\frac{Q(u-\eta)}{Q(u)}+d(u)\frac{Q(u+\eta)}{Q(u)}\\ &+c_h u(u+\eta)\frac{F(u)}{Q(u)}, \end{aligned} \tag{16}$$

where the function $Q(u)$ is parameterized by N parameters $\{\lambda_j|j=1,\ldots,N\}$

$$Q(u)=\prod_{l=1}^N(u-\lambda_l)(u+\lambda_l+\eta)=Q(-u-\eta), \tag{17}$$

and the functions $a(u)$, $d(u)$, $F(u)$ and the constant c_h are given by

$$\begin{aligned} a(u)=\;&\frac{2u+2\eta}{2u+\eta}(u+\xi)(u+\bar{\xi})\\ &\times\prod_{l=1}^N(u+\theta_l+\eta)(u-\theta_l+\eta),\\ d(u)=\;&\frac{2u}{2u+\eta}(u-\xi+\eta)(u+\eta-\bar{\xi}) \end{aligned} \tag{18}$$

$$\times\prod_{l=1}^{N}(u+\theta_l)(u-\theta_l), \tag{19}$$

$$F(u) = \prod_{l=1}^{N}(u+\theta_l)(u-\theta_l)(u+\theta_l+\eta)(u-\theta_l+\eta), \tag{20}$$

$$c_h = 2(\boldsymbol{h}_1\cdot\boldsymbol{h}_2-1). \tag{21}$$

By using the relation (14), the energy E_j of the Gaudin Hamiltonian H_j can be obtained,

$$E_j = \xi^{(1)}+(\theta_j^2-\xi^2)\sum_{k=1,k\neq j}^{N}\frac{1}{\theta_j^2-\theta_k^2} -(\theta_j^2-\xi^2)\sum_{k=1}^{N}\frac{2}{\theta_j^2-\lambda_k^2}, \tag{22}$$

and the N Bethe roots $\{\lambda_k|k=1,\ldots,N\}$ should satisfy a set of Bethe ansatz equations (BAEs),

$$\begin{aligned}&\frac{1-\xi^{(1)}}{\lambda_k-\xi}+\frac{1-\xi^{(1)}}{\lambda_k+\xi}-\sum_{m=1}^{N}\left(\frac{1}{\lambda_k+\theta_m}+\frac{1}{\lambda_k-\theta_m}\right)\\&+\sum_{m=1,m\neq k}^{N}\left(\frac{2}{\lambda_k-\lambda_m}+\frac{2}{\lambda_k+\lambda_m}\right)\\&=\frac{\boldsymbol{h}_2^{(1)}\cdot\boldsymbol{h}_2^{(1)}\lambda_k}{2(\lambda_k^2-\xi^2)}\frac{\prod_{l=1}^{N}(\lambda_k+\theta_l)(\lambda_k-\theta_l)}{\prod_{m=1,m\neq k}^{N}(\lambda_k-\lambda_m)(\lambda_k+\lambda_m)},\\&k=1,\ldots,N.\end{aligned} \tag{23}$$

3. Bethe states of the Gaudin model

The boundary matrices $K^-(u)$ and $K^+(u)$ satisfy $[K^-(u),K^+(v)]\neq 0$ due to the parameter $\boldsymbol{h}_2^{(1)}\neq 0$. This fact gives rise to that they cannot be diagonalized simultaneously, and that there is no obvious reference state on which the conventional Bethe ansatz can be performed. For the Gaudin model (1), we generalize the method in Refs. [37,38] to construct its Bethe states.

We first introduce the following gauge matrix:

$$G=\begin{pmatrix}\frac{1}{2}(h_2^z+1) & \frac{1}{2}(h_2^x-\mathrm{i}h_2^y)\\ -\frac{1}{2}(h_2^x+\mathrm{i}h_2^y) & \frac{1}{2}(h_2^z+1)\end{pmatrix}. \tag{24}$$

With the gauge matrix G, we can diagonalize the K^+-matrix

$$\begin{aligned}\tilde{K}^+(u) &= GK^+(u)G^{-1}=\begin{pmatrix}\bar{\xi}+u+\eta & 0\\ 0 & \bar{\xi}-u-\eta\end{pmatrix}\\&\equiv\begin{pmatrix}\tilde{K}_{11}^+(u) & 0\\ 0 & \tilde{K}_{22}^+(u)\end{pmatrix},\end{aligned} \tag{25}$$

and the gauge transformed K^--matrix is

$$\tilde{K}^-(u)=GK^-(u)G^{-1}\equiv\begin{pmatrix}\tilde{K}_{11}^-(u) & \tilde{K}_{12}^-(u)\\ \tilde{K}_{21}^-(u) & \tilde{K}_{22}^-(u)\end{pmatrix}. \tag{26}$$

The elements of the gauged $\tilde{K}^-(u)$ matrix are as follows:

$$\tilde{K}_{11}^-(u)=\xi+\boldsymbol{h}_1\cdot\boldsymbol{h}_2u,\quad \tilde{K}_{22}^-(u)=\xi-\boldsymbol{h}_1\cdot\boldsymbol{h}_2u, \tag{27}$$

$$\tilde{K}_{12}^-(u)=\left(h_1^x-\mathrm{i}h_1^y\right)u-\frac{(h_2^x-\mathrm{i}h_2^y)(\boldsymbol{h}_1\cdot\boldsymbol{h}_2+h_1^z)}{1+h_2^z}u, \tag{28}$$

$$\tilde{K}_{21}^-(u)=\left(h_1^x+\mathrm{i}h_1^y\right)u-\frac{(h_2^x+\mathrm{i}h_2^y)(\boldsymbol{h}_1\cdot\boldsymbol{h}_2+h_1^z)}{1+h_2^z}u. \tag{29}$$

Accordingly, the double-row monodromy matrix can be expressed as

$$\tilde{\mathcal{U}}(u)=G\mathcal{U}(u)G^{-1}\equiv\begin{pmatrix}\tilde{\mathcal{A}}(u) & \tilde{\mathcal{B}}(u)\\ \tilde{\mathcal{C}}(u) & \tilde{\mathcal{D}}(u)\end{pmatrix}. \tag{30}$$

With the above relations, the transfer matrix $t(u)$ can be rewritten as

$$t(u)=\mathrm{tr}_0\left\{\tilde{K}_0^+(u)\tilde{\mathcal{U}}_0(u)\right\}=\tilde{K}_{11}^+(u)\tilde{\mathcal{A}}(u)+\tilde{K}_{22}^+(u)\tilde{\mathcal{D}}(u). \tag{31}$$

Using the gauge transformation matrix G, we obtain the operator $\tilde{\mathcal{B}}(u)$ (or $\tilde{\mathcal{C}}(u)$) for constructing the Bethe states.

We introduce other two gauge transformation matrices

$$\begin{aligned}\bar{G}&=\begin{pmatrix}h_1^z+1 & h_1^x-\mathrm{i}h_1^y\\ h_1^x-\mathrm{i}h_1^y & h_1^z+1\end{pmatrix},\\ \hat{G}&=\begin{pmatrix}h_1^z+1 & -(h_1^x+\mathrm{i}h_1^y)\\ -(h_1^x+\mathrm{i}h_1^y) & h_1^z+1\end{pmatrix}.\end{aligned} \tag{32}$$

Then, the boundary $K^-(u)$ matrix can be transformed into a lower triangular matrix

$$\begin{aligned}\bar{K}^-(u) &= \bar{G}K^-(u)\bar{G}^{-1}=\begin{pmatrix}\xi+u & 0\\ 2uh_1^x & \xi-u\end{pmatrix}\\&\equiv\begin{pmatrix}\bar{K}_{11}^-(u) & 0\\ \bar{K}_{21}^-(u) & \bar{K}_{22}^-(u)\end{pmatrix},\end{aligned} \tag{33}$$

and an upper triangular matrix

$$\hat{K}^-(u)=\hat{G}K^-(u)\hat{G}^{-1}=\begin{pmatrix}\xi+u & 2uh_1^x\\ 0 & \xi-u\end{pmatrix}. \tag{34}$$

Here we use the following notations for the gauge transformed matrices: $\tilde{O}=GOG^{-1}$, $\bar{O}=\bar{G}O\bar{G}^{-1}$ and $\hat{O}=\hat{G}O\hat{G}^{-1}$, where O is an arbitrary matrix in the auxiliary space. With the gauge transformation matrices $\bar{G}$ and $\hat{G}$, we introduce the following new reference states:

$$|\phi_1\rangle=\bigotimes_{n=1}^{N}\bar{G}_n^{-1}|\downarrow\rangle_n,\quad \langle\bar{\phi}_1|=\bigotimes_{n=1}^{N}{}_n\langle\uparrow|\bar{G}_n, \tag{35}$$

$$|\phi_2\rangle=\bigotimes_{n=1}^{N}\hat{G}_n^{-1}|\uparrow\rangle_n,\quad \langle\bar{\phi}_2|=\bigotimes_{n=1}^{N}{}_n\langle\downarrow|\hat{G}_n. \tag{36}$$

Using a similar method in Ref. [37], we construct the left and right Bethe states of the inhomogeneous XXX Heisenberg spin chain with the generic open boundaries

$$\langle\lambda_1,\ldots,\lambda_N| = \langle\bar{\phi}_1|\prod_{l=1}^{N}\tilde{\mathcal{C}}(\lambda_l), \tag{37}$$

$$|\lambda_1,\ldots,\lambda_N\rangle = \prod_{l=1}^{N}\tilde{\mathcal{B}}(\lambda_l)|\phi_2\rangle. \tag{38}$$

The proofs are given in Appendix A.

With the relation (14), the Bethe state of the inhomogeneous XXX Heisenberg spin chain can be expanded as[21]

$$|\lambda_1,\ldots,\lambda_N\rangle=\eta^N|\psi\rangle+\mathcal{O}(\eta^{N+1}). \tag{39}$$

Table 1. Solutions of BAEs (23) for the case of $N=3$, $j=1$, $\{\theta_k\}=\{1.19,2.20,3.58\}$, $\xi=0.8\mathrm{i}$, $\xi^{(1)}=1.2$, $h_1^x=0.3$, $h_1^y=0.5$, $h_2^{(1)x}=0.2\mathrm{i}$, $h_2^{(1)y}=0.7\mathrm{i}$. The symbol n indicates the number of the eigenvalues, $\Delta_j^n=H_j|\psi\rangle-E_j^n|\psi\rangle$, and E_j^n is the corresponding energy. The energy E_j^n calculated from Eq. (22) is the same as that from the exact diagonalization of the Gaudin Hamiltonian (1).

λ_1	λ_2	λ_3	E_j^n	Δ_j^n	n
0.4038	$1.7040+1.4436\mathrm{i}$	$1.7040-1.4436\mathrm{i}$	-3.0624	10^{-15}	1
4.6639	$0.4788+1.0420\mathrm{i}$	$0.4788-1.0420\mathrm{i}$	-2.4126	10^{-15}	2
0.5266	$2.5061+0.4493\mathrm{i}$	$2.5061-0.4493\mathrm{i}$	-1.7616	10^{-14}	3
0.4888i	$4.0676+0.9174\mathrm{i}$	$4.0676-0.9174\mathrm{i}$	-1.6133	10^{-14}	4
5.1762	$3.9424+2.6895\mathrm{i}$	$3.9424-2.6895\mathrm{i}$	0.6952	10^{-14}	5
3.0069	$2.8496+1.4388\mathrm{i}$	$2.8496-1.4388\mathrm{i}$	1.3880	10^{-13}	6
4.1829	$2.1283+0.2312\mathrm{i}$	$2.1283-0.2312\mathrm{i}$	3.1108	10^{-13}	7
1.7049	$2.0069+1.1388\mathrm{i}$	$2.0069-1.1388\mathrm{i}$	3.6559	10^{-14}	8

Here $|\psi\rangle$ is the Bethe state of the Gaudin model (1)

$$|\psi\rangle=\prod_{l=1}^{N}\mathcal{B}_{\mathrm{G}}(\lambda_l)|\phi_2\rangle, \tag{40}$$

where

$$\begin{aligned}\mathcal{B}_{\mathrm{G}}(u)\ =\ &\left(\frac{h_2^{(1)z}}{h_1^z+1}(h_1^x-\mathrm{i}h_1^y)-(h_2^{(1)x}-\mathrm{i}h_2^{(1)y})\right)u\\&+\frac{1}{2}\sum_{k=1}^{N}\left(\frac{u-\xi}{u-\theta_k}-\frac{\xi+u}{u+\theta_k}\right)\\&\times\left[(h_1^x-\mathrm{i}h_1^y)\sigma_k^z+\frac{(h_1^x-\mathrm{i}h_1^y)^2}{h_1^z+1}\sigma_k^+-(1+h_1^z)\sigma_k^-\right].\end{aligned} \tag{41}$$

Considering the normalization (4) and relations (9), the Gaudin model has six free boundary parameters. We choose $\{\xi,\xi^{(1)},h_1^x,h_1^y,h_2^{(1)x},h_2^{(1)y}\}$ as the boundary free parameters. The numerical solutions of BAEs (23) and the corresponding energies (22) and Bethe states (40) of the Gaudin Hamiltonian (1) for $N=3$ are given in Table 1. Those numerical simulations show that the BAEs (23) indeed gives the complete spectrum and eigenstates of the Gaudin model.

4. Degeneration of the Bethe states

For the model with $U(1)$ symmetry,[21] the Bethe states cannot be obtained by the present method, because the eigenstates of the operators $\tilde{C}(u)$ and $\tilde{\mathcal{B}}(u)$ cannot form a SoV basis.[30] However, the Bethe states (40) should reduce to the form in Ref. [21] when the parameter $h_2^{(1)}\to 0$. The numerical solutions for small $h_2^{(1)x}$, $h_2^{(1)y}$ and $h_2^{(1)z}$ are listed in Table 2.

Comparing Table 2 with Table 1, we find that each set of the Bethe roots $\{\lambda_1,\ldots,\lambda_N\}$ split into two sets when $h_2^{(1)}\to 0$: small Bethe roots $\{\lambda_1,\ldots,\lambda_M\}$ and large Bethe roots $\{\tilde{\lambda}_{M+1},\ldots,\tilde{\lambda}_N\}$ with $M\in\{0,1,\ldots,N\}$.[39] In the $U(1)$-symmetric case, the whole Hilbert space can be decomposed into $N+1$ subspaces, i.e., $H_j=\oplus_{M=0}^{N}H_j^{(M)}$.

In order to study the values of the terms $\left(\frac{h_2^{(1)z}}{h_1^z+1}(h_1^x-\mathrm{i}h_1^y)-(h_2^{(1)x}-\mathrm{i}h_2^{(1)y})\right)u$ and $\frac{\xi-u}{u-\theta_k}+\frac{\xi+u}{u+\theta_k}$ for large Bethe roots, without loss of generality, we set $h_2^{(1)y}=h_2^{(1)z}=0$. The values of $h_2^{(1)x}\tilde{\lambda}_l$ versus $h_2^{(1)x}$ are shown in Tables 3 and 4. To show the variation tendency, we plot the real and imaginary parts of $h_2^{(1)x}\tilde{\lambda}$ versus $\left|\log_{10}\left(h_2^{(1)x}\right)\right|$ in Figs. 1 and 2.

Numerical simulations indicate that the values of $h_2^{(1)x}\tilde{\lambda}_l$ go to a nonzero constant as $h_2^{(1)x}\to 0$. Considering the form of the BAEs (23), we have

$$|\tilde{\lambda}_l|\propto\frac{1}{\max\left\{|h_2^{(1)x}|,|h_2^{(1)y}|,|h_2^{(1)z}|\right\}}. \tag{42}$$

Table 2. Solutions of BAEs (23) for the case of $N=3$, $j=1$, $\{\theta_k\}=\{1.19,2.20,3.58\}$, $\xi=0.8\mathrm{i}$, $\xi^{(1)}=1.2$, $h_1^x=0.3$, $h_1^y=0.5$, $h_2^{(1)x}=0.002\mathrm{i}$, $h_2^{(1)y}=0.007\mathrm{i}$. The symbol n indicates the number of the eigenvalues, $\Delta_j^n=H_j|\psi\rangle-E_j^n|\psi\rangle$, and E_j^n is the corresponding energy. The energy E_j^n calculated from Eq. (22) is the same as that from the exact diagonalization of the Gaudin Hamiltonian (1).

λ_1	λ_2	λ_3	E_j^n	Δ_j^n	n
0.6702i	2.9192i	201.9838i	-2.1993	10^{-15}	1
7.9077	4.1051i	0.6465i	-1.9808	10^{-15}	2
0.6290i	$150.8937+112.0518\mathrm{i}$	$150.8937-112.0518\mathrm{i}$	-1.8505	10^{-15}	3
0.5616i	2.8851	201.9208i	-1.3606	10^{-15}	4
414.2882	$319.1713+309.7090\mathrm{i}$	$319.1713-309.7090\mathrm{i}$	0.4192	10^{-14}	5
3.1386	$150.8727+112.0845\mathrm{i}$	$150.8727-112.0845\mathrm{i}$	0.9067	10^{-14}	6
1.7485	$150.8866+112.0628\mathrm{i}$	$150.8866-112.0628\mathrm{i}$	2.9248	10^{-14}	7
$1.4945+0.4144\mathrm{i}$	$1.4945-0.4144\mathrm{i}$	201.9353i	3.1405	10^{-14}	8

Table 3. Values of $h_2^{(1)x}\tilde{\lambda}$ in solutions of BAEs (23) for the case of $N=2$, $j=1$, $\{\theta_k\}=\{1.19,2.20\}$, $\xi=0.8\mathrm{i}$, $\xi^{(1)}=1.2$, $h_1^x=0$, $h_1^y=0.7$, $h_2^{(1)y}=0$. The symbol n indicates the number of the eigenvalues.

$h_2^{(1)x}$	0.1i	0.01i	0.001i	0.0001i
$n=2$	1.0602i	0.8967i	0.8945i	0.8944i
$n=3$	$-1.4794+2.6527\mathrm{i}$	$-1.4817+2.6450\mathrm{i}$	$-1.4818+2.6449\mathrm{i}$	$-1.4818+2.6449\mathrm{i}$
	$1.4794+2.6527\mathrm{i}$	$1.4817+2.6450\mathrm{i}$	$1.4818+2.6449\mathrm{i}$	$1.4818+2.6449\mathrm{i}$
$n=4$	0.9070i	0.8946i	0.8944i	0.8944i

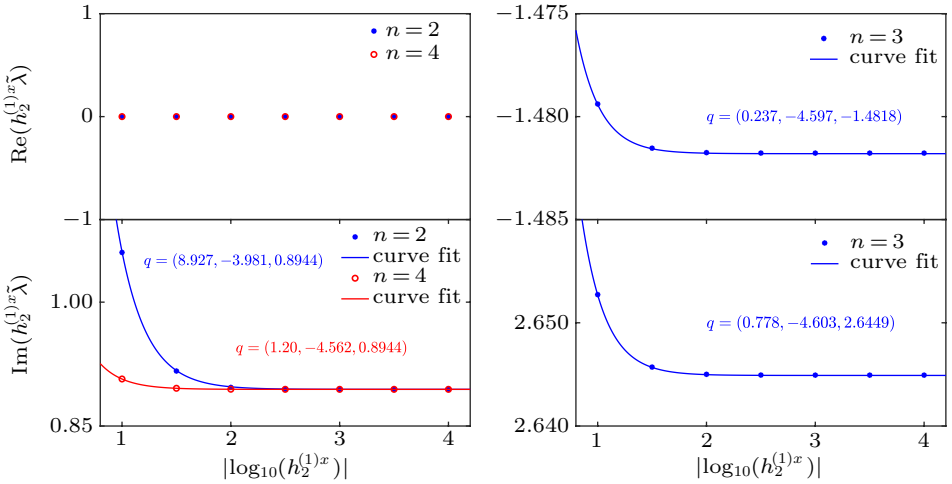


Fig. 1. The real and imaginary parts of $h_2^{(1)x}\tilde{\lambda}$ versus $|\log_{10}(h_2^{(1)x})|$. The data can be fitted as $\mathrm{Re}(h_2^{(1)x}\tilde{\lambda}) = q_1\,\mathrm{e}^{q_2\cdot|\log_{10}(h_2^{(1)x})|}+q_3$ and $\mathrm{Im}(h_2^{(1)x}\tilde{\lambda}) = q_1\,\mathrm{e}^{q_2\cdot|\log_{10}(h_2^{(1)x})|}+q_3$ with $q=(q_1,q_2,q_3)$. Here $N=2$, $j=1$, $\{\theta_k\}=\{1.19,2.20\}$, $\xi=0.8\mathrm{i}$, $\xi^{(1)}=1.2$, $h_1^x=0.3$, $h_1^y=0.5$, $h_2^{(1)x}=0.2\mathrm{i}$, $h_2^{(1)y}=0.7\mathrm{i}$. The symbol n indicates the number of the eigenvalues. For the $n=3$ case, the two sets of data have the same tendency q_2. Without loss of generality, we choose the first set of data to show the variation tendency.

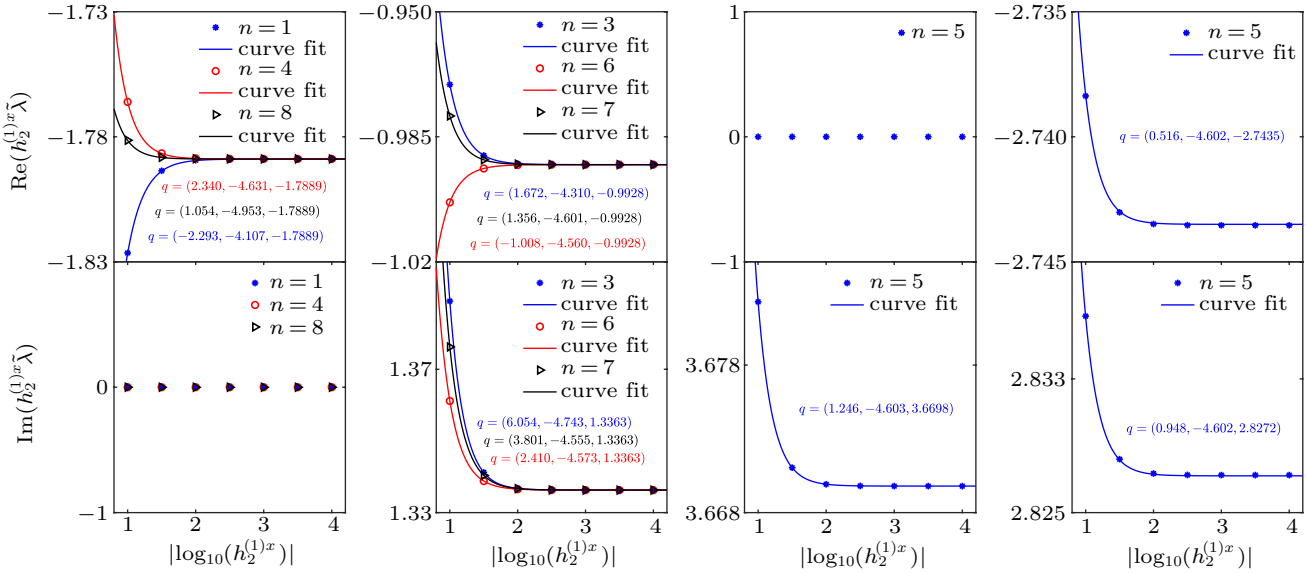


Fig. 2. The real and imaginary parts of $h_2^{(1)x}\tilde{\lambda}$ versus $|\log_{10}(h_2^{(1)x})|$. The data can be fitted as $\mathrm{Re}(h_2^{(1)x}\tilde{\lambda}) = q_1\,\mathrm{e}^{q_2\cdot|\log_{10}(h_2^{(1)x})|}+q_3$ and $\mathrm{Im}(h_2^{(1)x}\tilde{\lambda}) = q_1\,\mathrm{e}^{q_2\cdot|\log_{10}(h_2^{(1)x})|}+q_3$. Here $N=3$, $j=1$, $\{\theta_k\}=\{1.19,2.20,3.58\}$, $\xi=0.8\mathrm{i}$, $\xi^{(1)}=1.2$, $h_1^x=0.3$, $h_1^y=0.5$, $h_2^{(1)x}=0.2\mathrm{i}$, $h_2^{(1)y}=0.7\mathrm{i}$. Similar to the case of $N=2$, we choose the first set of data for $n=3,6,7$, respectively. For $n=5$, we choose the first and second sets of data.

Table 4. Values of $h_2^{(1)x}\tilde{\lambda}$ in solutions of BAEs (23) for the case of $N=3$, $j=1$, $\{\theta_k\}=\{1.19,2.20,3.58\}$, $\xi=0.8\mathrm{i}$, $\xi^{(1)}=1.2$, $h_1^x=0$, $h_1^y=0.7$, $h_2^{(1)y}=0$. The symbol n indicates the number of the eigenvalues.

$h_2^{(1)x}$	0.1i	0.01i	0.001i	0.0001i
$n=1$	-1.8265	-1.7893	-1.7889	-1.7889
$n=3$	$-0.9704+1.3890\mathrm{i}$	$-0.9925+1.3368\mathrm{i}$	$-0.9928+1.3363\mathrm{i}$	$-0.9928+1.3363\mathrm{i}$
	$0.9704+1.3890\mathrm{i}$	$0.9925+1.3368\mathrm{i}$	$0.9928+1.3363\mathrm{i}$	$0.9928+1.3363\mathrm{i}$
$n=4$	-1.7661	-1.7886	-1.7889	-1.7889
	$3.6823\mathrm{i}$	$3.6699\mathrm{i}$	$3.6698\mathrm{i}$	$3.6698\mathrm{i}$
$n=5$	$-2.7384+2.8368\mathrm{i}$	$-2.7435+2.8273\mathrm{i}$	$-2.7435+2.8272\mathrm{i}$	$-2.7435+2.8272\mathrm{i}$
	$2.7384+2.8368\mathrm{i}$	$2.7435+2.8273\mathrm{i}$	$2.7435+2.8272\mathrm{i}$	$2.7435+2.8272\mathrm{i}$
$n=6$	$-1.0033+1.3612\mathrm{i}$	$-0.9929+1.3365\mathrm{i}$	$-0.9928+1.3363\mathrm{i}$	$-0.9928+1.3363\mathrm{i}$
	$1.0033+1.3612\mathrm{i}$	$0.9929+1.3365\mathrm{i}$	$0.9928+1.3363\mathrm{i}$	$0.9928+1.3363\mathrm{i}$
$n=7$	$-0.9792+1.3762\mathrm{i}$	$-0.9927+1.3367\mathrm{i}$	$-0.9928+1.3363\mathrm{i}$	$-0.9928+1.3363\mathrm{i}$
	$0.9792+1.3762\mathrm{i}$	$0.9927+1.3367\mathrm{i}$	$0.9928+1.3363\mathrm{i}$	$0.9928+1.3363\mathrm{i}$
$n=8$	-1.7814	-1.7888	-1.7889	-1.7889

This fact gives rise to

$$\left(\frac{h_2^{(1)z}}{h_1^z+1}(h_1^x-\mathrm{i}h_1^y)-(h_2^{(1)x}-\mathrm{i}h_2^{(1)y})\right)\tilde{\lambda}_l\to\text{const}$$

and $\frac{\xi-\tilde{\lambda}_l}{\tilde{\lambda}_l-\theta_k}+\frac{\xi+\tilde{\lambda}_l}{\tilde{\lambda}_l+\theta_k}\to 0$ for $\boldsymbol{h}_2^{(1)}\to 0$. One can obtain that the contribution of the large Bethe roots to the Bethe states is a nonzero constant when $\boldsymbol{h}_2^{(1)}=0$. This fact enables us to write the Bethe states under the degeneration condition as

$$|\psi\rangle=\prod_{l=1}^{M}\mathcal{B}_G(\lambda_l)|\phi_2\rangle, \tag{43}$$

where

$$\begin{aligned}\mathcal{B}_G(u)=&\frac{1}{2}\sum_{k=1}^{M}\left(\frac{u-\xi}{u-\theta_k}-\frac{\xi+u}{u+\theta_k}\right)\\&\times\left[(h_1^x-\mathrm{i}h_1^y)\sigma_k^z+\frac{(h_1^x-\mathrm{i}h_1^y)^2}{h_1^z+1}\sigma_k^+-(1+h_1^z)\sigma_k^-\right].\end{aligned} \tag{44}$$

Since the large Bethe roots $\{\tilde{\lambda}_{M+1},\ldots,\tilde{\lambda}_N\}$ grow as $1/\max\left\{|h_2^{(1)x}|,|h_2^{(1)y}|,|h_2^{(1)z}|\right\}$, the contribution of the large Bethe roots to the eigenvalues is zero when $\boldsymbol{h}_2^{(1)}=0$. Then the eigenvalue E_j reads

$$\begin{aligned}E_j=&\xi^{(1)}+(\theta_j^2-\xi^2)\sum_{k=1,k\neq j}^{N}\frac{1}{\theta_j^2-\theta_k^2}-(\theta_j^2-\xi^2)\\&\times\sum_{k=1}^{M}\frac{2}{\theta_j^2-\lambda_k^2},\end{aligned} \tag{45}$$

and the M Bethe roots $\{\lambda_k|k=1,\ldots,M\}$ should satisfy a set of BAEs

$$\begin{aligned}&\frac{1-\xi^{(1)}}{\lambda_k-\xi}+\frac{1-\xi^{(1)}}{\lambda_k+\xi}-\sum_{m=1}^{N}\left(\frac{1}{\lambda_k+\theta_m}+\frac{1}{\lambda_k-\theta_m}\right)\\&+\sum_{m\neq k}\left(\frac{2}{\lambda_k-\lambda_m}+\frac{2}{\lambda_k+\lambda_m}\right)=0,\quad k=1,2,\ldots,M.\end{aligned} \tag{46}$$

The numerical solutions of BAEs (46) and the corresponding eigenvalues (45) and Bethe states (43) of the Gaudin Hamiltonian (1) for $N=3$ are listed in Table 5.

Equations (45) and (46) indicate that the parameter $\boldsymbol{h}_1$ does not contribute to the energy. Because the boundary matrices $K^-(u)$ and $K^+(u)$ can be diagonalized simultaneously by the gauge transformation G for $\boldsymbol{h}_2^{(1)}=0$. We can set $h_1^x=h_1^y=0$, and the Bethe states (43) can be written as

$$|\psi\rangle=\prod_{l=1}^{M}\left[\sum_{k=1}^{N}\left(\frac{\xi+\lambda_l}{\lambda_l+\theta_k}-\frac{\lambda_l-\xi}{\lambda_l-\theta_k}\right)\sigma_k^-\right]|0\rangle. \tag{47}$$

The Bethe states (47), eigenvalues (45) and BAEs (46) coincide with the results obtained by the algebraic Bethe ansatz method.[21] This indicates that the Bethe states of the model with $U(1)$ symmetry (47) can be obtained by taking the limit $\boldsymbol{h}_2^{(1)}\to 0$. Up to now, we construct the Bethe states of the Gaudin model with or without $U(1)$ symmetry based on the off-diagonal Bethe ansatz method solutions.

Table 5. Solutions of BAEs (46) for the case of $N=3$, $j=1$, $\{\theta_k\}=\{1.19,2.20,3.58\}$, $\xi=0.8\mathrm{i}$, $\xi^{(1)}=1.2$, $h_1^x=0.4$, $h_1^y=0.7$. The symbol n indicates the number of the eigenvalues, $\Delta_j^n=H_j|\psi\rangle-E_j^n|\psi\rangle$, and E_j^n is the corresponding energy. The energy E_j^n calculated from Eq. (45) is the same as that from the exact diagonalization of the Gaudin Hamiltonian (1).

λ_1	λ_2	λ_3	E_j^n	Δ_j^n	n
2.9201i	0.6702i	–	−2.1990	10^{-15}	1
7.9106	4.1044i	0.6465i	−1.9809	10^{-15}	2
0.6290i	–	–	−1.8506	10^{-15}	3
0.5616i	2.8851	–	−1.3605	10^{-15}	4
–	–	–	0.4191	10^{-15}	5
3.1386	–	–	0.9067	10^{-14}	6
1.7485	–	–	2.9248	10^{-15}	7
1.4947 + 0.4145i	1.4947 − 0.4145i	–	3.1404	10^{-14}	8

5. Conclusion

In summary, we have obtained the explicit closed-form expression of the Bethe states of the Gaudin model with DM and KSEA interactions. Taking the Gaudin model as a concrete example, we have studied the degeneration of the Bethe states (40). This method provide some insights into the degeneration of the Bethe states for other integral models without $U(1)$ symmetry. In the near future, we will study the correlation functions and quantum dynamics of the Gaudin model (1) based on the results in this paper.

Appendix A: Proof of Eqs. (37) and (38)

In this appendix, we will construct the left Bethe state (37) and right Bethe state (38) of the inhomogeneous XXX Heisenberg spin chain.

A.1. Separation-of-variables (SoV) basis

With the help of the gauge transformation G, we introduce two local states of site n

$$\begin{aligned}&|1\rangle_n=G_n^{-1}|\uparrow\rangle_n,\quad |2\rangle_n=G_n^{-1}|\downarrow\rangle_n,\\&{}_n\langle 1|={}_n\langle\uparrow|G_n,\quad {}_n\langle 2|={}_n\langle\downarrow|G_n.\end{aligned} \tag{A1}$$

It is easy to check that these states satisfy the orthogonal relations ${}_j\langle a|b\rangle_k=\delta_{a,b}\delta_{j,k}$, where $a,b=1,2$ and $j,k=1,\ldots,N$. Using the above local states, we define

$$|\Omega\rangle=\otimes_{j=1}^{N}|1\rangle_j,\quad \langle\bar{\Omega}|=\otimes_{j=1}^{N}{}_j\langle 2|. \tag{A2}$$

The SoV states can be constructed as[37]

$$|\theta_{p_1},\ldots,\theta_{p_n}\rangle=\tilde{\mathcal{A}}(\theta_{p_1})\cdots\tilde{\mathcal{A}}(\theta_{p_n})|\Omega\rangle, \tag{A3}$$

$$\langle\theta_{q_1},\ldots,\theta_{q_n}|=\langle\bar{\Omega}|\tilde{\mathcal{D}}(-\theta_{q_1})\cdots\tilde{\mathcal{D}}(-\theta_{q_n}). \tag{A4}$$

It is easy to derive

$$\tilde{\mathcal{C}}(u)|\theta_{p_1},\ldots,\theta_{p_n}\rangle=c(u,\{\theta_{p_1},\ldots,\theta_{p_n}\})|\theta_{p_1},\ldots,\theta_{p_n}\rangle, \tag{A5}$$

$$\langle\theta_{q_1},\ldots,\theta_{q_n}|\tilde{\mathcal{C}}(u)=\bar{c}(u,\{\theta_{q_1},\ldots,\theta_{q_n}\})\langle\theta_{q_1},\ldots,\theta_{q_n}|, \quad \text{(A6)}$$

where

$$c(u,\{\theta_{p_1},\ldots,\theta_{p_n}\})=(-1)^N\tilde{K}_{21}^{-}(u)\tilde{d}(u)\tilde{d}(-u-\eta)\prod_{j=1}^{n}\frac{(u+\theta_{p_j})(u-\theta_{p_j}+\eta)}{(u-\theta_{p_j})(u+\theta_{p_j}+\eta)}, \quad \text{(A7)}$$

$$\bar{c}(u,\{\theta_{p_1},\ldots,\theta_{p_n}\})=(-1)^N\tilde{K}_{21}^{-}(u)\tilde{a}(u)\tilde{a}(-u-\eta)\prod_{j=1}^{n}\frac{(u-\theta_{p_j})(u+\theta_{p_j}+\eta)}{(u+\theta_{p_j})(u-\theta_{p_j}+\eta)}. \quad \text{(A8)}$$

Here $\tilde{d}(u)=\tilde{a}(u-\eta)=\prod_{j=1}^{N}(u-\theta_j)$. These results show that the right SoV states and left SoV states are the eigenstates of the operator $\tilde{\mathcal{C}}(u)$. The right states $\{|\theta_{p_1},\ldots,\theta_{p_n}\rangle\}$ (or $\{\langle\theta_{q_1},\ldots,\theta_{q_n}|\}$) form a right (or left) basis of Hilbert space.[30]

The left SoV states and right SoV states satisfy the orthogonal relation

$$\langle\theta_{q_1},\ldots,\theta_{q_m}|\theta_{p_1},\ldots,\theta_{p_n}\rangle=f_n(\theta_{p_1},\ldots,\theta_{p_n})\delta_{m+n,N}\delta_{\{q\},\{p\}}, \quad \text{(A9)}$$

where $p_j,q_j\in\{1,\ldots,N\}$, $p_1<p_2<\ldots p_n$, $q_1<q_2<\ldots q_n$ and $\delta_{\{q\},\{p\}}$ is defined as

$$\delta_{\{q\},\{p\}}=\begin{cases}1 & \text{if } \{q_1,\ldots,q_m,p_1,\ldots,p_n\}=\{1,\ldots,N\},\\ 0 & \text{otherwise,}\end{cases} \quad \text{(A10)}$$

and $f_n(\theta_{p_1},\ldots,\theta_{p_n})$ is defined by

$$f_n(\theta_{p_1},\ldots,\theta_{p_n})=\langle\theta_{p_{n+1}},\ldots,\theta_{p_N}|\theta_{p_1},\ldots,\theta_{p_n}\rangle, \quad \text{(A11)}$$

with the convention $p_{n+1}<\cdots<p_N$.[37]

A.2. Eigenstates of the inhomogeneous *XXX* Heisenberg spin chain

In Ref. [37], the key to construct the eigenstates $\langle\Psi|$ of the transfer matrix $t(u)$ is to calculate the scalar product $\langle\Psi|\theta_{p_1},\ldots,\theta_{p_n}\rangle$ with the SoV basis and the inhomogeneous T–Q relation (A16). For convenience, we denote $\langle\Psi|\theta_{p_1},\ldots,\theta_{p_n}\rangle$ as

$$F_n(\theta_{p_1},\ldots,\theta_{p_n})=\langle\Psi|\theta_{p_1},\ldots,\theta_{p_n}\rangle. \quad \text{(A12)}$$

Obviously, $F_0=\langle\Psi|\Omega\rangle$. Following the method used in Refs. [30,37], we have

$$F_n(\theta_{p_1},\ldots,\theta_{p_n})=\prod_{j=1}^{n}(\theta_{p_j}+\xi)(-1)^N\tilde{d}(-\theta_{p_j}-\eta)\tilde{a}(\theta_{p_j})\frac{Q(\theta_{p_j}-\eta)}{Q(\theta_{p_j})}F_0. \quad \text{(A13)}$$

The eigenstates $\langle\Psi|$ can be decomposed as a unique linear combination of the basis

$$\langle\Psi|=\sum_{n=0}^{N}\sum_{\{p_j\}}\frac{F_n(\theta_{p_1},\ldots,\theta_{p_n})}{f_n(\theta_{p_1},\ldots,\theta_{p_n})}\langle\theta_{p_{n+1}},\ldots,\theta_{p_N}|, \quad \text{(A14)}$$

where $p_1<\cdots<p_n$ and $p_{n+1}<\cdots<p_N$.

Based on the SoV basis and the inhomogeneous T–Q relation (A16), we construct the eigenstates of the transfer matrix $t(u)$ (or the Hamiltonian $H_{XXX}=\partial\ln t(u)/\partial u|_{u=0}$). The benefit of this approach is that an reference states is not needed. Next, we construct the Bethe states of the inhomogeneous *XXX* Heisenberg spin chain using the SoV basis and the scalar product in Eq. (A13).

A.3. Inner product $\langle\bar{\phi}_1|\theta_{p_1},\ldots,\theta_{p_n}\rangle$

With the help of the gauge transformation matrix $\bar{G}$, we introduce

$$\bar{T}(u)=\bar{G}T(u)\bar{G}^{-1}\equiv\begin{pmatrix}\bar{A}(u) & \bar{B}(u)\\ \bar{C}(u) & \bar{D}(u)\end{pmatrix}, \quad \text{(A15)}$$

$$\bar{\mathcal{U}}(u)=\bar{G}\mathcal{U}(u)\bar{G}^{-1}\equiv\begin{pmatrix}\bar{\mathcal{A}}(u) & \bar{\mathcal{B}}(u)\\ \bar{\mathcal{C}}(u) & \bar{\mathcal{D}}(u)\end{pmatrix}. \quad \text{(A16)}$$

The reference states in Eq. (35) have the following properties:

$$\bar{A}(u)|\phi_1\rangle=\tilde{d}(u)|\phi_1\rangle,\quad \bar{D}(u)|\phi_1\rangle=\tilde{a}(u)|\phi_1\rangle,\quad \bar{C}(u)|\phi_1\rangle\neq 0,\quad \bar{B}(u)|\phi_1\rangle=0, \quad \text{(A17)}$$

$$\langle\bar{\phi}_1|\bar{A}(u)=\langle\bar{\phi}_1|\tilde{a}(u),\quad \langle\bar{\phi}_1|\bar{D}(u)=\langle\bar{\phi}_1|\tilde{d}(u),\quad \langle\bar{\phi}_1|\bar{C}(u)\neq 0,\quad \langle\bar{\phi}_1|\bar{B}(u)=0. \quad \text{(A18)}$$

With the above properties, we obtain

$$\langle\bar{\phi}_1|\bar{\mathcal{B}}(u)=0, \quad \text{(A19)}$$

$$\langle\bar{\phi}_1|\bar{\mathcal{A}}(u)=(-1)^N\bar{K}_{11}^{-}(u)\tilde{a}(u)\tilde{d}(-u-\eta)\langle\bar{\phi}_1|, \quad \text{(A20)}$$

$$\langle\bar{\phi}_1|\bar{\mathcal{D}}(u)=(-1)^N\left\{\frac{(2u+\eta)\bar{K}_{22}^{-}(u)-\eta K_{11}^{-}(u)}{2u+\eta}\tilde{d}(u)\tilde{a}(-u-\eta)+\frac{\eta\bar{K}_{11}^{-}(u)}{2u+\eta}\tilde{a}(u)\tilde{d}(-u-\eta)\right\}\langle\bar{\phi}_1|. \quad \text{(A21)}$$

Now we calculate the scaler product $\langle\bar{\phi}_1|\theta_{p_1},\ldots,\theta_{p_n}\rangle$, which is very important to construct the Bethe states. From the definition of the gauge transformed matrices, we find

$$\tilde{\mathcal{U}}(u)=G\mathcal{U}(u)G^{-1}=G\bar{G}^{-1}\bar{\mathcal{U}}(u)\bar{G}G^{-1}. \quad \text{(A22)}$$

For convenience, we define

$$\bar{G}G^{-1}=\begin{pmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\end{pmatrix}. \quad \text{(A23)}$$

Then, we obtain the following relations

$$\tilde{\mathcal{A}}(u)=\frac{1}{y_{11}y_{22}-y_{12}y_{21}}\{y_{11}y_{22}\bar{\mathcal{A}}(u)+y_{21}y_{22}\bar{\mathcal{B}}(u)-y_{11}y_{12}\bar{\mathcal{C}}(u)-y_{12}y_{21}\bar{\mathcal{D}}(u)\}, \quad \text{(A24)}$$

$$\tilde{\mathcal{C}}(u)=\frac{1}{y_{11}y_{22}-y_{12}y_{21}}\{-y_{11}y_{21}\bar{\mathcal{A}}(u)-y_{21}^2\bar{\mathcal{B}}(u)+y_{11}^2\bar{\mathcal{C}}(u)+y_{11}y_{21}\bar{\mathcal{D}}(u)\},\tag{A25}$$

$$\tilde{\mathcal{A}}(u)=-\frac{y_{22}}{y_{21}}\tilde{\mathcal{C}}(u)+\frac{y_{11}}{y_{21}}\bar{\mathcal{C}}(u)+\bar{\mathcal{D}}(u).\tag{A26}$$

Using the identity $\tilde{\mathcal{C}}(\theta_{p_n})|\theta_{p_1},\ldots,\theta_{p_{n-1}}\rangle=0$ and Eqs. (A19), (A20), (A26), one can prove

$$\begin{aligned}&\langle\bar{\phi}_1|\theta_{p_1},\ldots,\theta_{p_n}\rangle\\&=\langle\bar{\phi}_1|\tilde{\mathcal{A}}(\theta_{p_n})|\theta_{p_1},\ldots,\theta_{p_{n-1}}\rangle\\&=\prod_{j=1}^{n}(-1)^N(\theta_{p_j}+\xi)\tilde{a}(\theta_{p_j})\tilde{d}(-\theta_{p_j}-\eta)\langle\bar{\phi}_1|\Omega\rangle.\end{aligned}\tag{A27}$$

A.4. Bethe states of the inhomogeneous *XXX* Heisenberg spin chain

We introduce the left Bethe state

$$\langle\lambda_1,\ldots,\lambda_N|=\langle\bar{\phi}_1|\prod_{l=1}^{N}\tilde{\mathcal{C}}(\lambda_l).\tag{A}$$

Using the SoV basis and the relations (A5) and (A27), we obtain

$$\begin{aligned}&\frac{\langle\lambda_1,\ldots,\lambda_N|\theta_{p_1},\ldots,\theta_{p_n}\rangle}{\prod_{j=1}^{N}(-1)^N\tilde{K}_{21}^{-}(\lambda_j)\tilde{d}(\lambda_j)\tilde{d}(-\lambda_j-\eta)}\\&=\prod_{k=1}^{n}(-1)^N(\theta_{p_k}+\xi)\tilde{a}(\theta_{p_k})\tilde{d}(-\theta_{p_k}-\eta)\frac{Q(\theta_{p_k}-\eta)}{Q(\theta_{p_k})}\langle\bar{\phi}_1|\Omega\rangle.\end{aligned}\tag{A29}$$

Comparing Eq. (A29) with Eq. (A13), we find that the left Bethe state $\langle\lambda_1,\ldots,\lambda_N|$ is the eigenstates of the inhomogeneous XXX Heisenberg spin chain when $\{\lambda_k|k=1,\ldots,N\}$ satisfy the BAEs (A23).[37] With the same procedure, the right Bethe state can be constructed as

$$|\lambda_1,\ldots,\lambda_N\rangle=\prod_{l=1}^{N}\tilde{\mathcal{B}}(\lambda_l)|\phi_2\rangle.\tag{A30}$$

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