

Wave–activity relation containing wave–basic flow interaction based on decomposition of general potential vorticity

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Wave–activity relation containing wave–basic flow interaction based on decomposition of general potential vorticity*

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On the basis of the general potential vorticity theorem (GPV), a new wave–activity relation is derived in an ageostrophic and nonhydrostatic dynamic framework. When the Reynolds average is taken, the wave–activity relation shares an exchange term with the equation of the basic-state GPV. Thus, the two equations are capable of presenting the dynamic process of the wave–basic flow interaction. Unlike the E–P flux theory which can only be used in large-scale atmosphere, the corresponding derivation provides a useful tool to analyze the feedback of waves to basic states and the forcing of basic states to waves simultaneously, and it can be used in mesoscale systems, such as heavy rainfall processes. The theory was initially applied to the landfalling Typhoon Mujigae (2015) by assigning the scalar φ to the generalized potential temperature (GPT). The results showed that the newly-derived wave–activity density is able to describe the wave activities associated with strong precipitation in Typhoon Mujigae (2015), including the eyewall and spiral rainbands. However, the interaction between the basic-state vortex and mesoscale waves denoted by the exchange term between basic-state GPV and wave–activity density mainly occurs in the eyewall in Typhoon Mujigae (2015). A comparison of the exchange term with other forcing terms in the newly-derived wave–activity relation indicates that the basic state–wave interaction plays a significant role in enhancing wave activities in the high-precipitation eyewall. By a magnitude analysis of the interaction term, it is found that the strong interaction between basic-state vortex and mesoscale waves is mainly attributed to two factors: the vertical vorticity intensity of the basic-state vortex and the averaged perturbation advection of perturbation GPT which is an exchange between the basic-state GPT and perturbation GPT.

Keywords: wave–activity density, basic state–wave interaction, wave–activity relation

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1. Introduction

The wave–activity relation is a useful tool for illustrating disturbance developments and evaluating the interaction between waves and basic flow.[1,2] Generally, the relation takes the following flux form:

$$\frac{\partial A}{\partial t} + \nabla \cdot \boldsymbol{F} = S, \tag{1}$$

where A and $\boldsymbol{F}$ are the wave–activity density and the wave–activity flux, respectively, and S is the source or sink of the wave–activity density. The wave–activity density and wave–activity flux are quadratic or higher-order in the disturbance amplitude. In recent decades, many types of wave–activity relations have been constructed for various purposes.[3–12] However, most of these wave–activity relations were based on a quasi-geostrophic approximation or built by using entropy coordinates, which are not suitable to mesoscale motions.

Ran and Gao[13] derived a three-dimensional (3D) wave–activity relation for the pseudomomentum in Cartesian coordinates by the momentum–Casimir method. Because the relation was constructed in an ageostrophic and nonhydrostatic dynamic framework, it is applicable to mesoscale phenomena. Nevertheless, the wave–activity relation constructed by the Casimir method requires the basic-state velocity and basic-state Casimir function to satisfy a restriction relation, which greatly limits its applications. Realizing this, Gao and Ran[14] constructed the general wave–activity relations associated with vertical velocity perturbation on the basis of potential vorticity (PV) theorem.[15] These wave–activity relations are easy to calculate and flexible to use. An application of them to a heavy-rainfall event demonstrated that the general wave–activity densities were consistent with the rain rate simulated by numerical models.

However, the wave–basic flow interaction essentially contains two dynamic processes: the forcing of the basic states to waves and the feedback of waves to basic states. The former is represented by the "wave–activity (conservation) relation", while the latter is generally described by the "Eliassen–Palm (E–P) flux theory". A common problem in the previous investigations of wave–activity relations (also seen in Eq. (1)) is that

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the two processes were separated. When the wave–activity relation was mainly considered, the feedback of waves to basic states was not taken into account.

In this paper, we derive a new wave–activity relation, in which the wave–activity density is the general PV (GPV) perturbation quadratic in disturbance amplitude. The GPV perturbation is defined as the scalar product of the curl of the horizontal velocity perturbation and the spatial gradient of an arbitrary scalar perturbation based on the work of Haynes and McIntyre[15] and Gao and Ran.[15] An important feature of this new wave–activity relation that differs from previous wave–activity relation is that it shares an exchange term with the basic-state GPV equation. Thus, the two equations connect the two processes in the wave–basic flow interaction. The theory is initially applied to a landfalling typhoon. The characteristics of the interaction between the wave–activity density and basic-state GPV during the event and their relations to heavy rainfall in the typhoon are examined.

The rest of this paper is arranged as follows. The theory is detailed in Section 2. The application of the newly-derived wave–activity relation to a landfall-typhoon heavy-rainfall event is discussed and analyzed in Section 3. Concluding remarks are presented in Section 4.

2. Formulae

The general form of potential vorticity (GPV)[15] is defined as

$$q = (\nabla \times \boldsymbol{v}_{\rm h}) \cdot \nabla \varphi, \tag{2}$$

where $\boldsymbol{v}_{\rm h} = (u,\ v,\ 0)$ is the horizontal velocity vector, and $\varphi = \varphi(x,y,z,t)$ is an arbitrary scalar function and satisfies the tendency equation,

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{v} \cdot \nabla \varphi = S_\varphi. \tag{3}$$

Note that φ can be any scalar, *e.g.*, potential temperature, moist potential temperature or even potential vorticity, as long as it satisfies Eq. (3). S_φ is the source or sink of φ.

All the variables in the primitive equations of atmosphere[16] and Eq. (3), *e.g.*, u, v, φ, are assumed to be able to be decomposed into a basic state and a perturbation,

$$F = \overline{F} + F'. \tag{4}$$

Here, F represents the variables, the superscript "–" denotes the basic state, and the superscript "′" refers to the perturbation from the corresponding basic state.

Substituting decompositions of the variables (such as Eq. (4)) into Eq. (2), GPV can be decomposed into,

$$q = \bar{q} + q' + A, \tag{5}$$

where

$$\bar{q} = (\nabla \times \bar{\boldsymbol{v}}_{\rm h}) \cdot \nabla \bar{\varphi}, \tag{6}$$

$$q' = \left(\nabla \times \boldsymbol{v}'_{\rm h}\right) \cdot \nabla \bar{\varphi} + (\nabla \times \bar{\boldsymbol{v}}_{\rm h}) \cdot \nabla \varphi', \tag{7}$$

$$A = \left(\nabla \times \boldsymbol{v}'_{\rm h}\right) \cdot \nabla \varphi' \tag{8}$$

are respectively the basic-state, first-order, and quadratic-order GPVs, $\bar{\boldsymbol{v}}_{\rm h} = (\bar{u},\ \bar{v},\ 0)$ is the basic-state horizontal velocity vector, $\boldsymbol{v}'_{\rm h} = (u',\ v',\ 0)$ is the horizontal velocity vector perturbation, $\bar{\varphi}$ and φ' are respectively the basic state and perturbations of the scalar φ.

In Eq. (5), q is decomposed into three portions and these three portions have different physical meanings. $\bar{q}$ is the basic-state GPV which is composed of basic-state variables describing large-scale motions. It represents the projection of the basic-state vorticity onto the gradient of the basic-state scalar $\bar{\varphi}$, and q' is the first-order GPV, which contains the first-order disturbance. It denotes the sum of the projection of the perturbation vorticity onto the gradient of the basic-state scalar $\bar{\varphi}$ and the projection of the basic-state vorticity onto the gradient of the scalar perturbation. A is the second-order GPV and represents the projection of the perturbation vorticity onto the gradient of the scalar perturbation.

Since A is quadratic in perturbation fields, it is in essence a type of wave energy and is capable of representing mesoscale perturbation imposed on a basic flow. Therefore, A is the so-called wave–activity density in this paper. What kind of physical information is contained in the wave–activity density A is determined by the decomposition of the basic state and the perturbation. Here, the Reynolds average is used to yield the basic state and to derive the wave–activity relation for A. It will be found that through the Reynolds average, the basic state–wave interaction is contained in the wave–activity relation.

Substituting the decompositions like Eq. (4) into the primitive equations[16] and Eq. (3) and applying the Reynolds average, one can obtain the basic-state tendency equations as follows:

$$\frac{\partial \bar{u}}{\partial t} = -\bar{\boldsymbol{v}} \cdot \nabla \bar{u} + f \bar{v}_a - \overline{\boldsymbol{v}' \cdot \nabla u'}, \tag{9}$$

$$\frac{\partial \bar{v}}{\partial t} = -\bar{\boldsymbol{v}} \cdot \nabla \bar{v} - f \bar{u}_a - \overline{\boldsymbol{v}' \cdot \nabla v'}, \tag{10}$$

$$\frac{\partial \bar{\varphi}}{\partial t} = -\bar{\boldsymbol{v}} \cdot \nabla \bar{\varphi} - \overline{\boldsymbol{v}' \cdot \nabla \varphi'} + \bar{S}_\varphi, \tag{11}$$

where $\bar{\boldsymbol{v}} = (\bar{u},\ \bar{v},\ \bar{w})$ is the three-dimensional basic-state velocity vector, $\boldsymbol{v}' = \left(u',\ v',\ w'\right)$ is the three-dimensional perturbation velocity vector, $\bar{u}_{\rm a}$ and $\bar{v}_{\rm a}$ are respectively basic-state zonal and meridional ageostrophic velocities, $\bar{S}_\varphi$ is the basic-state S_φ, meaning that the source or sink of φ is also decomposed into a basic state and a perturbation. The Reynolds conditions which are used in the above equations can be found in the paper by Liu and Liu.[16] The terms ($\overline{\boldsymbol{v}' \cdot \nabla u'}$, $\overline{\boldsymbol{v}' \cdot \nabla v'}$, and $\overline{\boldsymbol{v}' \cdot \nabla \varphi'}$) that are quadratic in disturbance amplitude resulting from the Reynolds average are kept on the right-hand

sides of the above basic-state equations. From Eqs. (9)–(11) it is found that those terms serve as the stress forces to dissipate the basic-state fields.

Subtracting Eqs. (9)–(11) from the primitive equations[16] and Eq. (3) respectively yields the perturbation equations as follows:

$$\frac{\partial u'}{\partial t} = -\boldsymbol{v}'\cdot\nabla\bar{u} - \bar{\boldsymbol{v}}\cdot\nabla u' + fv'_{\rm a} - \boldsymbol{v}'\cdot\nabla u' + \overline{\boldsymbol{v}'\cdot\nabla u'}, \quad (12)$$

$$\frac{\partial v'}{\partial t} = -\boldsymbol{v}'\cdot\nabla\bar{v} - \bar{\boldsymbol{v}}\cdot\nabla v' - fu'_{\rm a} - \boldsymbol{v}'\cdot\nabla v' + \overline{\boldsymbol{v}'\cdot\nabla v'}, \quad (13)$$

$$\frac{\partial \varphi'}{\partial t} = -\boldsymbol{v}'\cdot\nabla\bar{\varphi} - \bar{\boldsymbol{v}}\cdot\nabla\varphi' - \boldsymbol{v}'\cdot\nabla\varphi' + \overline{\boldsymbol{v}'\cdot\nabla\varphi'} + S'_{\varphi}, \quad (14)$$

where $u'_{\rm a}$ and $v'_{\rm a}$ are respectively perturbation zonal and meridional ageostrophic velicities, and S'_{φ} is the perturbation of the source or sink of φ which is retained in Eq. (14).

Comparing Eqs. (12)–(14) with Eqs. (9)–(11) respectively, the averaged quadratic terms ($\overline{\boldsymbol{v}'\cdot\nabla u'}$, $\overline{\boldsymbol{v}'\cdot\nabla v'}$, and $\overline{\boldsymbol{v}'\cdot\nabla\varphi'}$) in the basic-state equations are also involved in the above perturbation equations, but with opposite signs. They thus represent the transformation between the basic state and perturbation. These terms can partially describe the interaction between the wave and basic flow. However, the dynamic and thermodynamic interactions are separated in Eqs. (9)–(14), thus unable to describe the wave–flow interaction comprehensively. To exactly describe the interaction, an alternative set of basic-state and perturbation equations that share the same term and synthetically describe the dynamic and thermodynamical characteristics of the atmosphere is needed.

By applying $\partial/\partial t$ to Eqs. (6)–(8) separately and using the basic state and perturbation equations given above, the tendency equations of the three GPV components A, q', and, $\bar{q}$ can be obtained as follows:

$$\begin{aligned}\frac{\partial A}{\partial t} &= \nabla\cdot\boldsymbol{F}_{A1} + \nabla\cdot\boldsymbol{F}_{A2} + \nabla\cdot\boldsymbol{F}_{A3} + \nabla\cdot\boldsymbol{F}_{A4} + \nabla\cdot\boldsymbol{F}_{\rm Aex} \\ &\quad + \nabla\cdot\left(\boldsymbol{v}'_{\rm h}\times\nabla S'_{\varphi}\right), \end{aligned} \quad (15)$$

$$\begin{aligned}\frac{\partial q'}{\partial t} &= \nabla\cdot\boldsymbol{F}_{q'1} + \nabla\cdot\boldsymbol{F}_{q'2} + \nabla\cdot\boldsymbol{F}_{q'3} + \nabla\cdot\boldsymbol{F}_{q'4} + \nabla\cdot\boldsymbol{F}_{q'5} \\ &\quad - \nabla\cdot\boldsymbol{F}_{\rm Aex} + \nabla\cdot\left(\boldsymbol{v}'_{\rm h}\times\nabla\bar{S}_{\varphi} + \bar{\boldsymbol{v}}_{\rm h}\times\nabla S'\right), \end{aligned} \quad (16)$$

$$\begin{aligned}\frac{\partial \bar{q}}{\partial t} &= \nabla\cdot\boldsymbol{F}_{\bar{q}1} + \nabla\cdot\boldsymbol{F}_{\bar{q}2} + \nabla\cdot\bar{\boldsymbol{F}}_{q'3} - \nabla\cdot\bar{F}_{\rm Aex} \\ &\quad + \nabla\cdot\left(\bar{\boldsymbol{v}}_{\rm h}\times\nabla\bar{S}_{\varphi}\right). \end{aligned} \quad (17)$$

Specific forms of the forcing terms on the right-hand side of Eqs. (15)–(17) are given in Appendix A. Here, we only see the term

$$\begin{aligned}F_{\rm Aex} = \boldsymbol{i}&\left[\left(v'\frac{\partial v'}{\partial y} + w'\frac{\partial v'}{\partial z}\right)\frac{\partial\bar{\varphi}}{\partial z} - \frac{\partial\bar{v}}{\partial z}\left(v'\frac{\partial\varphi'}{\partial y} + w'\frac{\partial\varphi'}{\partial z}\right)\right] + \boldsymbol{j}\left[-\left(u'\frac{\partial u'}{\partial x} + w'\frac{\partial u'}{\partial z}\right)\frac{\partial\bar{\varphi}}{\partial z} + \frac{\partial\bar{u}}{\partial z}\left(u'\frac{\partial\varphi'}{\partial x} + w'\frac{\partial\varphi'}{\partial z}\right)\right] \\ &+ \boldsymbol{k}\left[-\left(u'\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y}\right)\frac{\partial\bar{\varphi}}{\partial x} + \left(u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y}\right)\frac{\partial\bar{\varphi}}{\partial y} + \left(\frac{\partial\bar{v}}{\partial x} - \frac{\partial\bar{u}}{\partial y}\right)\left(u'\frac{\partial\varphi'}{\partial x} + v'\frac{\partial\varphi'}{\partial y}\right)\right], \end{aligned} \quad (18)$$

for the reason that it is related to the interactions between the basic state and wave.

As in the above equations, the term $\nabla\cdot\bar{\boldsymbol{F}}_{\rm Aex}$ in Eq. (15) is also involved in Eq. (17) when the Reynolds average is taken into consideration in Eqs. (15)–(17), but its sign is opposite (note that when the Reynolds average is taken into account in Eq. (16), the terms on its both sides become zero). This means that the term $\nabla\cdot\bar{\boldsymbol{F}}_{\rm Aex}$ is an exchange term between the averaged wave–activity diversity and the basic-state GPV, denoting the transformation of the momentum and heat between waves and basic states..

In the case of the Reynolds average, when $\nabla\cdot\bar{\boldsymbol{F}}_{\rm Aex} > 0$, the transfer of the basic-state GPV to the wave–activity density is prone to reducing $\bar{q}$ (namely, $\partial\bar{q}/\partial t < 0$) and increasing $\bar{A}$ (namely, $\partial\bar{A}/\partial t > 0$). The convergence of $\bar{\boldsymbol{F}}_{\rm Aex}$ (namely, $\nabla\cdot\bar{\boldsymbol{F}}_{\rm Aex} < 0$), representing the transfer of wave–activity density to basic-state GPV, is prone to promoting $\bar{q}$ (namely, $\partial\bar{q}/\partial t > 0$) and suppressing $\bar{A}$ (namely, $\partial\bar{A}/\partial t < 0$).

In the classical theory of interaction between the wave and basic flow, the wave–activity density and wave–activity flux divergence satisfy an important dynamic relation:

$$\rho_0\bar{u}_t = \nabla_{y,z}\bar{\boldsymbol{F}}_{E-P} = \nabla_{y,z}\bar{\boldsymbol{F}} = -\frac{\partial\bar{A}}{\partial t}, \quad (19)$$

where $\bar{\boldsymbol{F}}_{\rm E-P}$ is the E–P flux. It is noted that "–" denotes a zonal average here. The E–P flux and wave–activity density satisfy a group velocity relation, namely,

$$\boldsymbol{F} = \boldsymbol{c}_{\rm g}A, \quad (20)$$

where $c_{\rm g}$ is the group velocity. Equation (20) is a concise statement of the wave–mean flow interaction. Nevertheless, the relation holds only in some special cases such as quasi-geostrophic Rossby waves and two-dimensional or three-dimensional pure gravity waves (excluding the effects of the Earth rotation). It is not applicable for all types of wave–activity relations and cannot be extended to general circumstances. In contrast, the wave–activity relation (Eq. (15)) derived in this paper is not limited to a specific type of wave. Any perturbation which may be composed of various-scale waves can be described by Eq. (15). Moreover, when the term $\nabla\cdot\bar{\boldsymbol{F}}_{\rm Aex}$ in Eq. (15) is eliminated by the same term in Eq. (17)

with the Reynolds average, one obtains

$$\frac{\partial \bar{q}}{\partial t} = -\nabla \cdot \bar{\boldsymbol{F}}_{\mathrm{Aex}} + \cdots = -\frac{\partial \bar{A}}{\partial t} + \cdots, \tag{21}$$

which states the interaction between the wave–activity density and basic-state GPV, and thus may be used to study the wave–mean flow interaction in mesoscale process.

3. Application

As a first application, the above newly-derived wave–activity density and wave–activity relation are calculated and analyzed in a landfalling typhoon in China–Mujigae (2015). The purpose is to preliminarily see the characteristics of the interactions between waves and basic states diagnosed from Eq. (23), and also the relation of the interaction to strong precipitation.

3.1. Determination of φ

Before application, an important task is to determine the arbitrary scalar φ. Here, considering the characteristics of the rainfall atmosphere, a moist potential temperature called generalized potential temperature (θ^*, GPT) is used, namely, $\varphi = \theta^*$. The GPT proposed by Gao *et al.*[17] is used to describe a non-uniformly saturated characteristic of the precipitation atmosphere. It is defined as

$$\theta^* = \theta \exp\left[\frac{L_v q_{vs}}{c_p T_{\mathrm{c}}}\left(\frac{q_v}{q_{vs}}\right)^k\right], \tag{22}$$

where L_v is he latent heat, q_v is the specific humidity, q_{vs} is the saturated specific humidity, T_{c} is the temperature at the condensation lift level, and k is a constant, θ^* decreases to θ for dry air, and is equal to the equivalent potential temperature $\theta_{\mathrm{se}} = \theta \exp(L_v q_{vs}/c_p T_{\mathrm{c}})$ for a saturated moist atmosphere. The GPT may describe the thermodynamical situation of a moist atmosphere more realistically than θ and θ_{se}.

3.2. Data

The data for the analysis come from a high-resolution simulation of Mujigae with the weather research and forecasting model (WRF)[18] performed by Jiao *et al.*[19,20] The simulation began at 1200 UTC 3 October 2015 and ran for 30 h with a horizontal grid spacing of 2 km. Specific configurations of the experiment and also verifications of the simulation can be found in Jiao *et al.*[19] Therefore, here we directly use the model output to perform the analyses. The data, which have 1-h time interval, are interpolated into a local z coordinate for calculation in this paper. In the vertical includes 40 layers with a fixed vertical grid spacing of 500 m.

As indicated above, the whole theory described in Section 2 needs the data to be decomposed into basic-state fields and perturbations (or waves) deviated from the basic states through the Reynolds average, to keep the interactive term. However, Reynolds average conditions are difficult to satisfy only if the zonal average is used. In that case, meso- or smaller-scale process that are related to heavy precipitation in typhoon will not be separated from the basic states. Therefore, in this paper, we use the Barnes filter technique to replace the zonal average, which is seen to approximately satisfy the Reynolds average conditions here. The Barnes filter technique is conducted with a horizontal scan radius of approximated 20 km. The response function larger than 0.8 corresponds to the theoretical wave length approximately larger than 250 km, which referred to as the basic state. The fields deviated from the basic state are explained as the perturbation field with most of wave lengths being less than 250 km.

3.3. Results

3.3.1. Wave activities and precipitation in typhoon

Typhoon Mujigae, which was the 22nd typhoon in 2015, formed in the Luzon Island of Philippine on 1 October 2015 and underwent landfall at Zhanjiang in Guangdong Province at approximately 0600 UTC on 4 October 2015. Figure 1 presents an overview of distributions of the wave–activity density $\langle|A|\rangle$ and precipitation in Mujigae at 0300 UTC and 0600 UTC on 4 October 2015. Here, the operator $\langle|\ |\rangle$ denotes the average of absolute values of A over 0.5 km–5 km levels, which may more thoroughly reflect the characteristics of the perturbation in the troposphere than a single level. As shown, a main feature of Figs. 1(a)–1(b) is that the wave activities (shaded areas) is mostly collocated to the strong precipitation (contours) in Typhoon Mujigae. The simulated Mujigae's precipitation is characterized by evident asymmetric distribution. The eye (approximately at 21° N, 111.2° E) is surrounded by high-precipitation eyewall, with precipitation centered on the northwest part of the eyewall. Out of the eyewall, three arc-shaped spiral rainbands (denoted as "Band 1", "Band 2", and "Band 3" in the figures) were located on the northeast and east part of the typhoon, which produced the heavy precipitation in Guangdong Province. These simulated 1-h precipitation regions are covered by the anomalous-value regions of $\langle|A|\rangle$, including the eyewall and three spiral rainbands. The collocation between $\langle|A|\rangle$ and precipitation suggests that there are evident wave activities over the precipitation region of Typhoon Mujigae. This is consistent with the results by Jiao *et al.*[20] that the vortex Rossby waves (wavenumbers 1 and 2) are associated with the high reflectivities (indicating strong precipitation) in Mujigae. However, it is noted that A does not just refer to the activity of vortex Rossby waves although they are always the most dominant in typhoon. Instead, it denotes a mixed perturbation of physical variables (vorticity and GPT) caused by the activities of multi-scale waves (the wave length less than 250 km according to the Barnes filter). Figures 1(a) and 1(b)

indicate that these multi-scale waves are closely related to the high precipitation in typhoon.

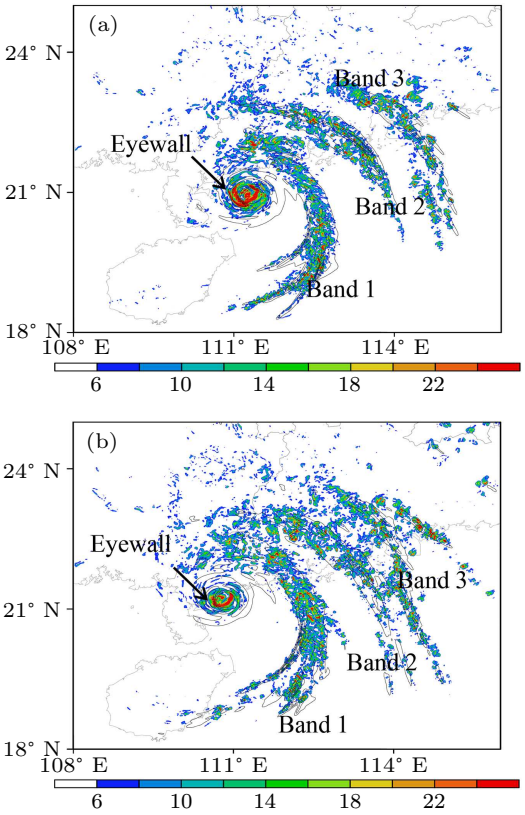


Fig. 1. Horizontal distributions of $\langle|A|\rangle$ (average of absolute values of $\bar{A}$ over 0.5 km–5 km levels, color shaded, unit: 10^{-6} K·s^{-1}·m^{-2}) superposed by 1-h simulated precipitation (10, 30, 50, 70, 100, 150 black contours, unit: mm) at (a) 0300 UTC and (b) 0600 UTC on 4 October 2015.

3.3.2. Interaction between wave–activity density and basic-state GPV

As indicated in Section 2, a main feature of the newly-derived wave–activity density A is that its average $\bar{A}$ shares an exchange term $(\nabla\cdot\bar{\boldsymbol{F}}_{\rm Aex})$ with the basic-state GPV $\bar{q}$. Here, with the exchange term $\nabla\cdot\bar{\boldsymbol{F}}_{\rm Aex}$, characteristics of the interaction between waves and basic states are analyzed in Typhoon Mujigae. Figure 2 presents vertical distributions of $\bar{A}$ and $\bar{q}$ along 21° N (approximately though the eye) at 0300 UTC on 4 Oct 2015. Note that $\bar{A}$ is obtained by applying the Barnces filter to A, which thus denotes an averaged wave activity over a limited area (approximately 40-km wide). From Fig. 2(a) it follows that vertical distribution of $\bar{A}$ is similar to A by corresponding to the high-precipitation areas–eyewall and spiral rainbands in Mujigae. Both the intensity and vertical extent of $\bar{A}$ within the eyewall is larger than those in the spiral rainbands, indicating stronger wave activities in the eyewall than in the spiral rainbands. In addition, after being averaged, $\bar{A}$ also presents strong anomalies in the eye region (approximately 111.2° N) in the boundary and lower troposphere below 3-km height, and strong negative values of $\bar{A}$ are superposed by strong positive values there. In Fig. 2(b), basic-state GPV $\bar{q}$ presents strong negative- and positive-value column in the eye and eyewall region. The negative areas extend from the surface to approximately 3-km height while the positive areas extend from 3-km to 10-km height in the mid troposphere and extends from 15 km to 18 km height in the upper troposphere. In contrast, no evident $\bar{q}$ is found in the spiral rainbands compared with the eye and eyewall region. This actually implies that the interaction between averaged wave–activity densities and basic-state GPV $(\nabla\cdot\bar{\boldsymbol{F}}_{\rm Aex})$ is the most evident in the eye and eyewall region, which can be seen in Fig. 3.

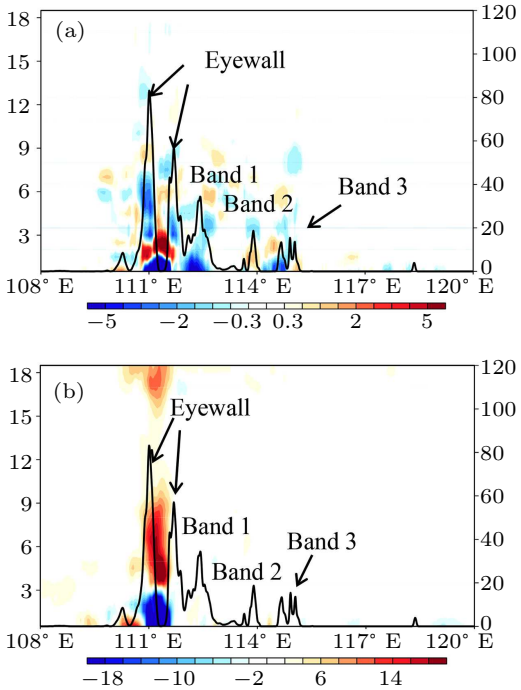


Fig. 2. Vertical distributions of (a) $\bar{A}$ (color shaded, unit: 10^{-9} K·s^{-1}·m^{-1}) and (b) $\bar{q}$ (color shaded, unit: 10^{-9} K·s^{-1}·m^{-1}) along 21° N at 0300 UTC on 4 October 2015, with black line being 1-h simulation precipitation, left y axis being height (unit: km) and the right y axis being 1-h precipitation (unit: mm).

To evaluate the significance of $\nabla\cdot\bar{\boldsymbol{F}}_{\rm Aex}$ compared with other forcing terms relating to the change of $\bar{A}$ (averaged Eq. (15)), figure 3 compares the magnitudes of these forcing terms in the vertical. These terms have been weighted by $\bar{A}/|\bar{A}|$ to exclude the influence of the sign of $\bar{A}$. For example, a positive value of the forcing $\nabla\cdot\bar{\boldsymbol{F}}$ may increase the wave activities denoted by positive $\bar{A}$, but reduce the wave activities denoted by negative $\bar{A}$, which makes the discussion on the role of $\nabla\cdot\bar{\boldsymbol{F}}$ very complicated. After being weighted by $\bar{A}/|\bar{A}|$, the positive value of $(\bar{A}/|\bar{A}|)\nabla\cdot\bar{\boldsymbol{F}}$ will increase the intensity of wave activities while negative values will reduce them. After being weighted, the forcing term is noted by $[\nabla\cdot\bar{\boldsymbol{F}}]$.

In Fig. 3(a), similar to $\bar{A}$, anomalies of $[\nabla\cdot\bar{F}]$ in the eyewall regions are evidently larger than the spiral rainbands regions. Thus, we focus on the analysis in the eyewall where wave activities and their forcing are both strong. As in Fig. 3(a), positive and negative values of $[\nabla\cdot\bar{\boldsymbol{F}}]$ are alternatively distributed in the vertical in the eyewall. An important fact is that the boundary layers of the eyewall have the most

significant positive $[\nabla\cdot\bar{\boldsymbol{F}}]$, indicating that the eyewall boundary layer is a main region to produce wave activities associated with the precipitation in the eyewall (Fig. 1). The importance of the interactions between $\bar{A}$ and $\bar{q}$ for this process can be seen by comparing Fig. 3(f) and Figs. 3(a)–3(e). In Fig. 3(f), in the layers below 3-km height, $[\nabla\cdot\bar{\boldsymbol{F}}_{\rm Aex}]$ possesses evident positive values in the eyewall, exhibiting a major contribution to the strong boundary-layer positive $[\nabla\cdot\bar{\boldsymbol{F}}]$ in Fig. 3(a) compared with the other two contributors $[\nabla\cdot\bar{\boldsymbol{F}}_1]$ in Fig. 3(b) and $[\nabla\cdot\bar{\boldsymbol{F}}_4]$ in Fig. 3(e). According to the physical meaning of $\nabla\cdot\bar{\boldsymbol{F}}_{\rm Aex}$ which denotes the transformation between basic-state GPV and wave–activity density, the distribution of $[\nabla\cdot\bar{F}_{\rm Aex}]$ in Fig. 3(e) indicates that the transformation between basic-state GPV and wave–activity density is a major process to increase the precipitation-related wave activities in the eyewall (Fig. 1). However, it is noted that the transformation direction needs to be ascertained with the aid of the signs of both $\bar{q}$ and $\bar{A}$.

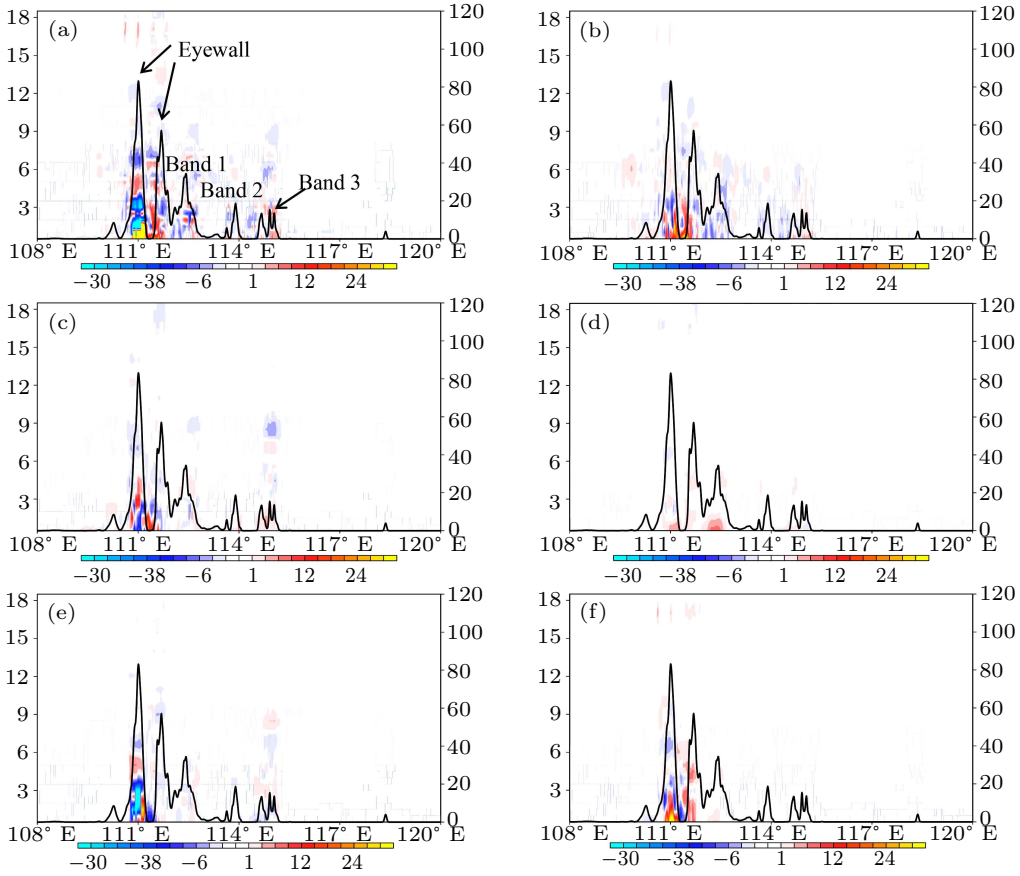


Fig. 3. Vertical distributions of (a) $[\nabla\cdot\bar{\boldsymbol{F}}]$ (unit: 10^{-9} K·s^{-2}·m^{-1}), (b) $[\nabla\cdot\bar{\boldsymbol{F}}_1]$ (unit: 10^{-9} K·s^{-2}·m^{-1}), (c) $[\nabla\cdot\bar{\boldsymbol{F}}_2]$ (unit: 10^{-9} K·s^{-2}·m^{-1}), (d) $[\nabla\cdot\bar{\boldsymbol{F}}_3]$ (unit: 10^{-10} K·s^{-2}·m^{-1}), (e) $[\nabla\cdot\bar{\boldsymbol{F}}_4]$ (unit: 10^{-9} K·s^{-2}·m^{-1}), and (f) $[\nabla\cdot\bar{\boldsymbol{F}}_{\rm ex}]$ (unit: 10^{-9} K·s^{-2}·m^{-1}) along 21° N at 0300 UTC on 4 October 2015, with [ ] denoting variable weighted by $\bar{A}/|\bar{A}|$, black line being 1-h simulation precipitation, left y axis being height (unit: km) and right y axis being 1-h precipitation (unit: mm).

Further, to reveal the main process in $\nabla\cdot\bar{\boldsymbol{F}}_{\rm Aex}$ to contribute to the transformation of $\bar{q}$ and $\bar{A}$ and thus the strong wave–activity densities in the eyewall, $\nabla\cdot\bar{\boldsymbol{F}}_{\rm Aex}$ is decomposed into 14 components by applying $\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k}$ to Eq. (18). Finally, it is found that the vertical gradient of the coupling of basic-state vertical vorticity to averaged perturbation advection of perturbation GPT, that is,

$$\nabla\cdot\overline{\boldsymbol{F}}_{\rm Aex}\approx\frac{\partial}{\partial z}\left[\left(\frac{\partial\bar{v}}{\partial x}-\frac{\partial\bar{u}}{\partial y}\right)\overline{\left(u'\frac{\partial\theta^*}{\partial x}+v'\frac{\partial\theta^*}{\partial y}\right)}\right]=\frac{\partial}{\partial z}\left(\bar{\zeta}_z\overline{\boldsymbol{v}'\cdot\nabla\theta^{*\prime}}\right)\tag{23}$$

plays a major part in $\nabla\cdot\overline{\boldsymbol{F}}_{\rm Aex}$, which can be seen in Figs. 4(a) and 4(b). Basic-state vertical vorticity reflects the intensity of the basic vortex of Typhoon Mujigae as shown in Fig. 4(c). The strong vertical gradients of the basic-state vertical vorticity in the eyewall, especially the western part of the eyewall, are one factor that is favorable for large $\nabla\cdot\overline{\boldsymbol{F}}_{\rm Aex}$ in the boundary-layer eyewall. The $\overline{\boldsymbol{v}'\cdot\nabla\theta^{*\prime}}$ is an exchange term between the basic-state GPT and perturbation GPT according to Eqs. (11) and (14). It mainly possesses large values in large-precipitation areas in typhoon, including the eyewall and spiral rainbands, with the negative values superposing on the positive values (Fig. 4(d)). In the eyewall, positive $\overline{\boldsymbol{v}'\cdot\nabla\theta^{*\prime}}$ which denotes a transformation of the basic-state GPT to perturbation GPT is confined below 3-km height while negative $\overline{\boldsymbol{v}'\cdot\nabla\theta^{*\prime}}$ that represents the transformation of perturbation

GPT to basic-state GPT is in the 3 km–13 km height. This is consistent with the characteristic of the precipitating atmosphere that in the lower atmosphere a large amount of moisture is transported into the inner region of typhoon to produce a high $\overline{\boldsymbol{v}' \cdot \nabla \theta^{*'}}$ area, driving the basic-state GPT to transform into GPT perturbations. In the mid- to high-troposphere, a large amount of latent heat is released due to the condensation of moisture which could warm the basic vortex, inducing the transformation of perturbation GPT into basic-state GPT. The difference of $\overline{\boldsymbol{v}' \cdot \nabla \theta^{*'}}$ in the vertical in the eyewall is the other factor that is favorable for large $\nabla \cdot \overline{\boldsymbol{F}}_{\mathrm{Aex}}$ in the boundary-layer eyewall.

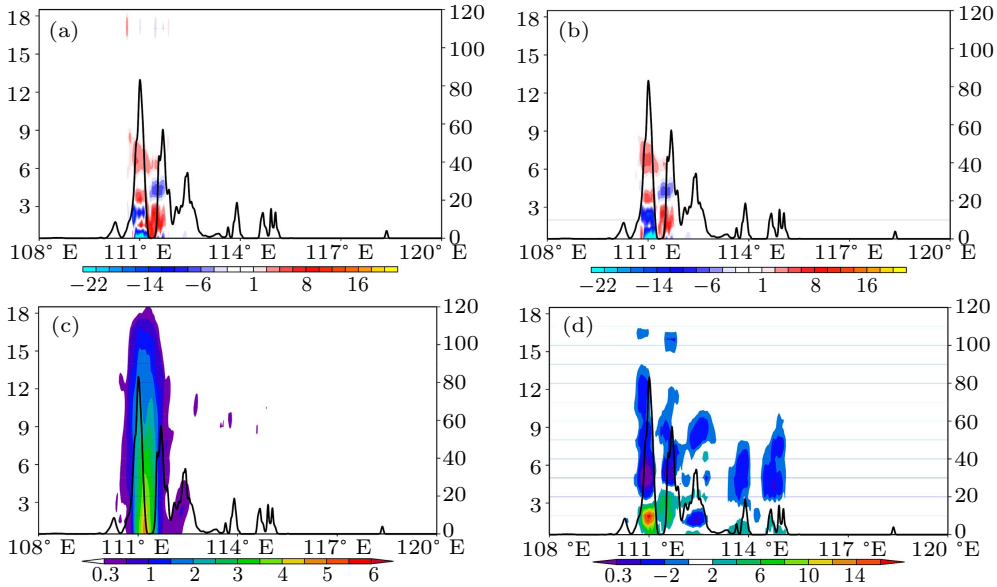


Fig. 4. Vertical distributions of (a) $\nabla \cdot \bar{\boldsymbol{F}}_{\mathrm{Aex}}$ (in units: 10^{-9} K·s^{-2}·m^{-1}), (b) $(\partial/\partial z)\left(\bar{\zeta}_z \overline{\boldsymbol{v}' \cdot \nabla \theta^{*'}}\right)$ (in units: 10^{-9} K·s^{-2}·m^{-1}), (c) $\bar{\zeta}_z$ (in unit: 10^{-3} s^{-1}), and (d) $\overline{\boldsymbol{v}' \cdot \nabla \theta^{*}}$ (in units: 10^{-3} K·s^{-1}) along 21° N at 0300 UTC on 4 October 2015, with black line being 1-h simulation precipitation, left y axis being height (in unit: km), and right y axis being 1-h precipitation (in unit: mm).

4. Conclusions

In this study, a new wave–activity relation is derived in an ageostrophic and nonhydrostatic dynamic framework. The perturbation portion of the general potential vorticity (GPV), defined by the horizontal velocity vector perturbation and arbitrary scalar perturbation, is referred to as the wave–activity density, which is quadratic in the disturbance amplitude. A basic-state GPV equation and its corresponding wave–activity relation are derived. When the Reynolds average is determined, the wave–activity relation shares an exchange term with the equation of the basic-state GPV. Thus, the two equations are capable of describing the two dynamic processes of interaction between waves and the basic flow, namely, the feedback of waves to basic states and the forcing of basic states to waves.

With the landfalling Typhoon Mujigae (2015), the above theory is used to analyze the relation between the newly-derived wave–activity density and high precipitation, the characteristics of the interaction between the basic-state vortex and mesoscale waves, and also the significance of the interaction for the wave activities in typhoon. To make the derived wave–activity relation conform to the characteristics of the precipitation atmosphere, the scalar φ is assigned to the generalized potential temperature, which is capable of describing the nonuniform saturation of the atmosphere.

The results indicate that the wave–activity density is consistent with the precipitation region in both the eyewall and spiral rainbands, indicating its ability to describe the mesoscale wave activities associated with precipitation in typhoon. The calculation of the interaction term $\nabla \cdot \bar{\boldsymbol{F}}_{\mathrm{ex}}$ between basic-state GPV and wave–activity density shows that the $\nabla \cdot \bar{\boldsymbol{F}}_{\mathrm{ex}}$ performs large values in the eyewall, indicating that the interaction between basic-state vortex and mesoscale wave mainly occurs in the eyewall region. Moreover, a comparison of $\nabla \cdot \bar{\boldsymbol{F}}_{\mathrm{ex}}$ with other forcing terms in the newly-derived wave–activity relation shows that the interaction between basic-state vortex and mesoscale waves exhibits a major contribution to the increase of the precipitation-related wave activities in the eyewall. Further, a magnitude analysis of the interaction term $\nabla \cdot \bar{\boldsymbol{F}}_{\mathrm{ex}}$ reveals that the main process in $\nabla \cdot \bar{\boldsymbol{F}}_{\mathrm{ex}}$ is responsible for the transformation between $\bar{q}$ and $\bar{A}$ and thus the strong wave–activity density in the eyewall is the vertical gradient of the coupling of basic-state vertical vorticity to averaged perturbation advection of perturbation GPT

As a theoretical study, this paper provides a very preliminary application and analysis of the newly-derived wave–activity relation and the interaction between the wave and basic state. In the future, more detailed studies will be done, such as how the basic state interacts with wave (by analyzing their

interaction term), to what extent the strongest interaction can reach, and what effect this interaction has on the development of precipitating system.

Appendix A: Derivations of equations of A, q', and $\bar{q}$

The equations of A, q', and $\bar{q}$ are derived in detail in this section. Taking the partial derivatives of Eqs. (6)–(8) with respective to time respectively, the following equations are obtained:

$$\frac{\partial A}{\partial t} = \nabla\cdot\left[\left(\nabla\times\boldsymbol{v}'_{\rm h}\right)\frac{\partial\varphi'}{\partial t} - \nabla\varphi'\times\frac{\partial\boldsymbol{v}'_{\rm h}}{\partial t}\right], \tag{A1}$$

$$\frac{\partial q'}{\partial t} = \nabla\cdot\left[\left(\nabla\times\boldsymbol{v}'_{\rm h}\right)\frac{\partial\bar{\varphi}}{\partial t} + \left(\nabla\times\bar{\boldsymbol{v}}_{\rm h}\right)\frac{\partial\varphi'}{\partial t} - \left(\nabla\bar{\varphi}\times\frac{\partial\boldsymbol{v}'_{\rm h}}{\partial t}\right) - \left(\nabla\varphi'\times\frac{\partial\bar{\boldsymbol{v}}_{\rm h}}{\partial t}\right)\right], \tag{A2}$$

$$\frac{\partial\bar{q}}{\partial t} = \nabla\cdot\left[\left(\nabla\times\bar{\boldsymbol{v}}_{\rm h}\right)\frac{\partial\bar{\varphi}}{\partial t} - \nabla\bar{\varphi}\times\frac{\partial\bar{\boldsymbol{v}}_{\rm h}}{\partial t}\right]. \tag{A3}$$

The term on the right-hand side of Eq. (A1) can be expressed by using Eqs. (12)–(14):

$$\begin{aligned}&\nabla\cdot\left[\left(\nabla\times\boldsymbol{v}'_{\rm h}\right)\frac{\partial\varphi'}{\partial t} - \nabla\varphi'\times\frac{\partial\boldsymbol{v}'_{\rm h}}{\partial t}\right]\\ &= \nabla\cdot\left(\boldsymbol{T}_A\boldsymbol{1}+\boldsymbol{T}_A\boldsymbol{2}\right)+\nabla\cdot\boldsymbol{F}_{A3}+\nabla\cdot\boldsymbol{F}_{A4}\\ &\quad+\nabla\cdot\left(\boldsymbol{v}'_{\rm h}\times\nabla S'_{\varphi}\right),\end{aligned} \tag{A4}$$

where

$$\begin{aligned}\boldsymbol{T}_A\boldsymbol{1} = {}& \boldsymbol{i}\left[\bar{u}\left(\frac{\partial v'}{\partial z}\frac{\partial\varphi'}{\partial x}-\frac{\partial v'}{\partial x}\frac{\partial\varphi'}{\partial z}\right)+u'\left(\frac{\partial v'}{\partial z}\frac{\partial\bar{\varphi}}{\partial x}-\frac{\partial\bar{v}}{\partial x}\frac{\partial\varphi'}{\partial z}\right)+w'\left(\frac{\partial v'}{\partial z}\frac{\partial\bar{\varphi}}{\partial z}-\frac{\partial\bar{v}}{\partial z}\frac{\partial\varphi'}{\partial z}\right)\right]\\ &+\boldsymbol{j}\left[\bar{v}\left(-\frac{\partial u'}{\partial z}\frac{\partial\varphi'}{\partial y}+\frac{\partial u'}{\partial y}\frac{\partial\varphi'}{\partial z}\right)+v'\left(-\frac{\partial u'}{\partial z}\frac{\partial\bar{\varphi}}{\partial y}+\frac{\partial\bar{u}}{\partial y}\frac{\partial\varphi'}{\partial z}\right)+w'\left(-\frac{\partial u'}{\partial z}\frac{\partial\bar{\varphi}}{\partial z}+\frac{\partial\bar{u}}{\partial z}\frac{\partial\varphi'}{\partial z}\right)\right]\\ &+\boldsymbol{k}\left\{\bar{w}\left[\frac{\partial v'}{\partial z}\frac{\partial\varphi'}{\partial x}-\frac{\partial u'}{\partial z}\frac{\partial\varphi'}{\partial y}-\left(\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}\right)\frac{\partial\varphi'}{\partial z}\right]+w'\left[\frac{\partial\bar{v}}{\partial z}\frac{\partial\varphi'}{\partial x}-\frac{\partial\bar{u}}{\partial z}\frac{\partial\varphi'}{\partial y}-\left(\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}\right)\frac{\partial\bar{\varphi}}{\partial z}\right]\right.\\ &\left.+v'\left(\frac{\partial\bar{\varphi}}{\partial y}\frac{\partial u'}{\partial y}-\frac{\partial\bar{u}}{\partial y}\frac{\partial\varphi'}{\partial y}\right)-u'\left(\frac{\partial\bar{\varphi}}{\partial x}\frac{\partial v'}{\partial x}-\frac{\partial\bar{v}}{\partial x}\frac{\partial\varphi'}{\partial x}\right)\right\},\end{aligned} \tag{A5}$$

$$\begin{aligned}\boldsymbol{T}_A\boldsymbol{2} = {}& \boldsymbol{i}\left[\frac{\partial v'}{\partial z}\left(\bar{v}\frac{\partial\varphi'}{\partial y}+v'\frac{\partial\bar{\varphi}}{\partial y}\right)-\left(\bar{v}\frac{\partial v'}{\partial y}+v'\frac{\partial\bar{v}}{\partial y}\right)\frac{\partial\varphi'}{\partial z}\right]+\boldsymbol{j}\left[-\frac{\partial u'}{\partial z}\left(\bar{u}\frac{\partial\varphi'}{\partial x}+u'\frac{\partial\bar{\varphi}}{\partial x}\right)+\left(\bar{u}\frac{\partial u'}{\partial x}+u'\frac{\partial\bar{u}}{\partial x}\right)\frac{\partial\varphi'}{\partial z}\right]\\ &+\boldsymbol{k}\left[\left(\bar{v}\frac{\partial v'}{\partial y}+v'\frac{\partial\bar{v}}{\partial y}\right)\frac{\partial\varphi'}{\partial x}-\left(u'\frac{\partial\bar{u}}{\partial x}+\bar{u}\frac{\partial u'}{\partial x}\right)\frac{\partial\varphi'}{\partial y}+\left(\bar{u}\frac{\partial\varphi'}{\partial x}+u'\frac{\partial\bar{\varphi}}{\partial x}\right)\frac{\partial u'}{\partial y}-\left(v'\frac{\partial\bar{\varphi}}{\partial y}+\bar{v}\frac{\partial\varphi'}{\partial y}\right)\frac{\partial v'}{\partial x}\right],\end{aligned} \tag{A6}$$

$$\boldsymbol{F}_{A3} = \left(f\boldsymbol{v}'_{\rm a}\times\boldsymbol{k}\right)\times\nabla\varphi', \tag{A7}$$

$$\begin{aligned}\boldsymbol{F}_{A4} = {}& \boldsymbol{i}\left[\frac{\partial\boldsymbol{v}'}{\partial z}\left(\boldsymbol{v}'\cdot\nabla\varphi'-\overline{\boldsymbol{v}'\cdot\nabla\varphi'}\right)-\frac{\partial\varphi'}{\partial z}\left(\boldsymbol{v}'\cdot\nabla\boldsymbol{v}'-\overline{\boldsymbol{v}'\cdot\nabla\boldsymbol{v}'}\right)\right]+\boldsymbol{j}\left[-\frac{\partial u'}{\partial z}\left(\boldsymbol{v}'\cdot\nabla\varphi'-\overline{\boldsymbol{v}'\cdot\nabla\varphi'}\right)+\frac{\partial\varphi'}{\partial z}\left(\boldsymbol{v}'\cdot\nabla u'-\overline{\boldsymbol{v}'\cdot\nabla u'}\right)\right]\\ &+\boldsymbol{k}\left[-\left(\frac{\partial\boldsymbol{v}'}{\partial x}-\frac{\partial u'}{\partial y}\right)\left(\boldsymbol{v}'\cdot\nabla\varphi'-\overline{\boldsymbol{v}'\cdot\nabla\varphi'}\right)+\frac{\partial\varphi'}{\partial x}\left(\boldsymbol{v}'\cdot\nabla\boldsymbol{v}'-\overline{\boldsymbol{v}'\cdot\nabla\boldsymbol{v}'}\right)-\frac{\partial\varphi'}{\partial y}\left(\boldsymbol{v}'\cdot\nabla u'-\overline{\boldsymbol{v}'\cdot\nabla u'}\right)\right].\end{aligned} \tag{A8}$$

In addition,

$$\nabla\cdot\boldsymbol{T}_A\boldsymbol{2} = \nabla\cdot\boldsymbol{T}_A\boldsymbol{3}, \tag{A9}$$

where

$$\begin{aligned}\boldsymbol{T}_A\boldsymbol{3} = {}& \boldsymbol{i}\left[\bar{u}\left(-\frac{\partial u'}{\partial z}\frac{\partial\varphi'}{\partial y}+\frac{\partial u'}{\partial y}\frac{\partial\varphi'}{\partial z}\right)+u'\left(-\frac{\partial\bar{u}}{\partial z}\frac{\partial\varphi'}{\partial y}+\frac{\partial\bar{u}}{\partial y}\frac{\partial\varphi'}{\partial z}-\frac{\partial u'}{\partial z}\frac{\partial\bar{\varphi}}{\partial y}+\frac{\partial u'}{\partial y}\frac{\partial\bar{\varphi}}{\partial z}\right)-v'\left(\frac{\partial\bar{v}}{\partial z}\frac{\partial\varphi'}{\partial y}-\frac{\partial v'}{\partial y}\frac{\partial\bar{\varphi}}{\partial z}\right)\right]\\ &+\boldsymbol{j}\left[\bar{v}\left(\frac{\partial v'}{\partial z}\frac{\partial\varphi'}{\partial x}-\frac{\partial v'}{\partial x}\frac{\partial\varphi'}{\partial z}\right)+v'\left(\frac{\partial\bar{v}}{\partial z}\frac{\partial\varphi'}{\partial x}-\frac{\partial\bar{v}}{\partial x}\frac{\partial\varphi'}{\partial z}+\frac{\partial v'}{\partial z}\frac{\partial\bar{\varphi}}{\partial x}-\frac{\partial v'}{\partial x}\frac{\partial\bar{\varphi}}{\partial z}\right)+u'\left(\frac{\partial\bar{u}}{\partial z}\frac{\partial\varphi'}{\partial x}-\frac{\partial u'}{\partial x}\frac{\partial\bar{\varphi}}{\partial z}\right)\right]\\ &+\boldsymbol{k}\left[v'\left(\frac{\partial\varphi'}{\partial y}\frac{\partial\bar{v}}{\partial x}-\frac{\partial v'}{\partial y}\frac{\partial\bar{\varphi}}{\partial x}\right)+u'\left(\frac{\partial u'}{\partial x}\frac{\partial\bar{\varphi}}{\partial y}-\frac{\partial\varphi'}{\partial x}\frac{\partial\bar{u}}{\partial y}\right)\right].\end{aligned} \tag{A10}$$

Thus, one has

$$\nabla\cdot\left(\boldsymbol{T}_A\boldsymbol{1}+\boldsymbol{T}_A\boldsymbol{2}\right) = \nabla\cdot\left(\boldsymbol{T}_A\boldsymbol{1}+\boldsymbol{T}_A\boldsymbol{3}\right) = \nabla\cdot\boldsymbol{F}_{A1}+\nabla\cdot\boldsymbol{F}_{A2}+\nabla\cdot\boldsymbol{F}_{\rm Aex}, \tag{A11}$$

where

$$\boldsymbol{F}_{A1} = -\bar{\boldsymbol{v}}A, \tag{A12}$$

$$\begin{aligned}\boldsymbol{F}_{A2} = &-u'\left[-\frac{\partial v'}{\partial z}\frac{\partial \bar{\varphi}}{\partial x}+\frac{\partial u'}{\partial z}\frac{\partial \bar{\varphi}}{\partial y}-\frac{\partial u'}{\partial y}\frac{\partial \bar{\varphi}}{\partial z}+\frac{\partial \bar{u}}{\partial z}\frac{\partial \varphi'}{\partial y}+\left(\frac{\partial \bar{v}}{\partial x}-\frac{\partial \bar{u}}{\partial y}\right)\frac{\partial \varphi'}{\partial z}\right]\boldsymbol{i}\\ &-v'\left[-\frac{\partial v'}{\partial z}\frac{\partial \bar{\varphi}}{\partial x}+\frac{\partial u'}{\partial z}\frac{\partial \bar{\varphi}}{\partial y}+\frac{\partial v'}{\partial x}\frac{\partial \bar{\varphi}}{\partial z}-\frac{\partial \bar{v}}{\partial z}\frac{\partial \varphi'}{\partial x}+\left(\frac{\partial \bar{v}}{\partial x}-\frac{\partial \bar{u}}{\partial y}\right)\frac{\partial \varphi'}{\partial z}\right]\boldsymbol{j}\\ &-w'\left[\left(\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}\right)\frac{\partial \bar{\varphi}}{\partial z}-\frac{\partial \bar{v}}{\partial z}\frac{\partial \varphi'}{\partial x}+\frac{\partial \bar{u}}{\partial z}\frac{\partial \varphi'}{\partial y}\right]\boldsymbol{k}, \end{aligned} \tag{A13}$$

$$\begin{aligned}\boldsymbol{F}_{\mathrm{Aex}} = &\boldsymbol{i}\left[\left(v'\frac{\partial v'}{\partial y}+w'\frac{\partial v'}{\partial z}\right)\frac{\partial \bar{\varphi}}{\partial z}-\frac{\partial \bar{v}}{\partial z}\left(v'\frac{\partial \varphi'}{\partial y}+w'\frac{\partial \varphi'}{\partial z}\right)\right]+\boldsymbol{j}\left[-\left(u'\frac{\partial u'}{\partial x}+w'\frac{\partial u'}{\partial z}\right)\frac{\partial \bar{\varphi}}{\partial z}+\frac{\partial \bar{u}}{\partial z}\left(u'\frac{\partial \varphi'}{\partial x}+w'\frac{\partial \varphi'}{\partial z}\right)\right]\\ &+\boldsymbol{k}\left[-\left(u'\frac{\partial v'}{\partial x}+v'\frac{\partial v'}{\partial y}\right)\frac{\partial \bar{\varphi}}{\partial x}+\left(u'\frac{\partial u'}{\partial x}+v'\frac{\partial u'}{\partial y}\right)\frac{\partial \bar{\varphi}}{\partial y}+\left(\frac{\partial \bar{v}}{\partial x}-\frac{\partial \bar{u}}{\partial y}\right)\left(u'\frac{\partial \varphi'}{\partial x}+v'\frac{\partial \varphi'}{\partial y}\right)\right]. \end{aligned} \tag{A14}$$

After substituting Eq. (A11) into Eq. (A4), one has

$$\frac{\partial A}{\partial t} = \nabla\cdot\boldsymbol{F}_{A1}+\nabla\cdot\boldsymbol{F}_{A2}+\nabla\cdot\boldsymbol{F}_{A3}+\nabla\cdot\boldsymbol{F}_{A4}+\nabla\cdot\boldsymbol{F}_{\mathrm{Aex}}+\nabla\cdot\left(\boldsymbol{v}'_{\mathrm{h}}\times\nabla S'_{\varphi}\right). \tag{A15}$$

The substitution of Eqs. (9)–(14) into the term on the right-hand side of Eq. (A2) yields

$$\begin{aligned}\nabla\cdot\left[\left(\nabla\times\boldsymbol{v}'_{\mathrm{h}}\right)\frac{\partial \bar{\varphi}}{\partial t}-\left(\nabla\bar{\varphi}\times\frac{\partial \boldsymbol{v}'_{\mathrm{h}}}{\partial t}\right)+\left(\nabla\times\bar{\boldsymbol{v}}_{\mathrm{h}}\right)\frac{\partial \varphi'}{\partial t}-\left(\nabla\varphi'\times\frac{\partial \bar{\boldsymbol{v}}_{\mathrm{h}}}{\partial t}\right)\right] = &\nabla\cdot\left(\boldsymbol{T}'1+\boldsymbol{T}'2\right)+\nabla\cdot\boldsymbol{F}_{q'3}+\nabla\cdot\boldsymbol{F}_{q'4}+\nabla\cdot\boldsymbol{F}_{q'5}\\ &-\nabla\cdot\boldsymbol{F}_{\mathrm{Aex}}+\nabla\cdot\left(\boldsymbol{v}'_{\mathrm{h}}\times\nabla\bar{S}_{\varphi}\right)+\nabla\cdot\left(\bar{\boldsymbol{v}}_{\mathrm{h}}\times\nabla S'_{\varphi}\right), \end{aligned} \tag{A16}$$

where

$$\begin{aligned}\boldsymbol{T}'\boldsymbol{1} = &\boldsymbol{i}\left[\bar{u}\left(\frac{\partial \bar{v}}{\partial z}\frac{\partial \varphi'}{\partial x}-\frac{\partial \bar{v}}{\partial x}\frac{\partial \varphi'}{\partial z}+\frac{\partial v'}{\partial z}\frac{\partial \bar{\varphi}}{\partial x}-\frac{\partial v'}{\partial x}\frac{\partial \bar{\varphi}}{\partial z}\right)+u'\left(\frac{\partial \bar{v}}{\partial z}\frac{\partial \bar{\varphi}}{\partial x}-\frac{\partial \bar{v}}{\partial x}\frac{\partial \bar{\varphi}}{\partial z}\right)\right]\\ &+\boldsymbol{j}\left[\bar{v}\left(-\frac{\partial u'}{\partial z}\frac{\partial \bar{\varphi}}{\partial y}+\frac{\partial u'}{\partial y}\frac{\partial \bar{\varphi}}{\partial z}-\frac{\partial \bar{u}}{\partial z}\frac{\partial \varphi'}{\partial y}+\frac{\partial \bar{u}}{\partial y}\frac{\partial \varphi'}{\partial z}\right)+v'\left(-\frac{\partial \bar{u}}{\partial z}\frac{\partial \bar{\varphi}}{\partial y}+\frac{\partial \bar{u}}{\partial y}\frac{\partial \bar{\varphi}}{\partial z}\right)\right]\\ &+\boldsymbol{k}\left\{\bar{w}\left[\frac{\partial v'}{\partial z}\frac{\partial \bar{\varphi}}{\partial x}-\frac{\partial u'}{\partial z}\frac{\partial \bar{\varphi}}{\partial y}-\left(\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}\right)\frac{\partial \bar{\varphi}}{\partial z}+\frac{\partial \bar{v}}{\partial z}\frac{\partial \varphi'}{\partial x}-\frac{\partial \bar{u}}{\partial z}\frac{\partial \varphi'}{\partial y}-\left(\frac{\partial \bar{v}}{\partial x}-\frac{\partial \bar{u}}{\partial y}\right)\frac{\partial \varphi'}{\partial z}\right]\right.\\ &\left.\cdot w'\left[\frac{\partial \bar{v}}{\partial z}\frac{\partial \bar{\varphi}}{\partial x}-\frac{\partial \bar{u}}{\partial z}\frac{\partial \bar{\varphi}}{\partial y}-\left(\frac{\partial \bar{v}}{\partial x}-\frac{\partial \bar{u}}{\partial y}\right)\frac{\partial \bar{\varphi}}{\partial z}\right]\right\}, \end{aligned} \tag{A17}$$

$$\begin{aligned}\boldsymbol{T}'\boldsymbol{2} = &\boldsymbol{i}\left[\bar{v}\left(\frac{\partial \bar{v}}{\partial z}\frac{\partial \varphi'}{\partial y}-\frac{\partial \bar{v}}{\partial y}\frac{\partial \varphi'}{\partial z}+\frac{\partial v'}{\partial z}\frac{\partial \bar{\varphi}}{\partial y}-\frac{\partial v'}{\partial y}\frac{\partial \bar{\varphi}}{\partial z}\right)+v'\left(\frac{\partial \bar{v}}{\partial z}\frac{\partial \bar{\varphi}}{\partial y}-\frac{\partial \bar{v}}{\partial y}\frac{\partial \bar{\varphi}}{\partial z}\right)\right]\\ &+\boldsymbol{j}\left[\bar{u}\left(-\frac{\partial u'}{\partial z}\frac{\partial \bar{\varphi}}{\partial x}+\frac{\partial u'}{\partial x}\frac{\partial \bar{\varphi}}{\partial z}-\frac{\partial \bar{u}}{\partial z}\frac{\partial \varphi'}{\partial x}+\frac{\partial \bar{u}}{\partial x}\frac{\partial \varphi'}{\partial z}\right)+u'\left(-\frac{\partial \bar{u}}{\partial z}\frac{\partial \bar{\varphi}}{\partial x}+\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{\varphi}}{\partial z}\right)\right]\\ &+\boldsymbol{k}\left\{\bar{v}\left[-\frac{\partial \bar{v}}{\partial x}\frac{\partial \varphi'}{\partial y}+\frac{\partial \bar{v}}{\partial y}\frac{\partial \varphi'}{\partial x}+\frac{\partial v'}{\partial y}\frac{\partial \bar{\varphi}}{\partial x}-\frac{\partial v'}{\partial x}\frac{\partial \bar{\varphi}}{\partial y}\right]+v'\left(\frac{\partial \bar{v}}{\partial y}\frac{\partial \bar{\varphi}}{\partial x}-\frac{\partial \bar{v}}{\partial x}\frac{\partial \bar{\varphi}}{\partial y}\right)\right.\\ &\left.+\bar{u}\left[-\frac{\partial \bar{u}}{\partial x}\frac{\partial \varphi'}{\partial y}+\frac{\partial \bar{u}}{\partial y}\frac{\partial \varphi'}{\partial x}+\frac{\partial u'}{\partial y}\frac{\partial \bar{\varphi}}{\partial x}-\frac{\partial u'}{\partial x}\frac{\partial \bar{\varphi}}{\partial y}\right]+u'\left(\frac{\partial \bar{u}}{\partial y}\frac{\partial \bar{\varphi}}{\partial x}-\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{\varphi}}{\partial y}\right)\right\}, \end{aligned} \tag{A18}$$

$$\boldsymbol{F}_{q'3} = u'\left(\frac{\partial \varphi'}{\partial x}\frac{\partial \bar{v}}{\partial z}-\frac{\partial v'}{\partial x}\frac{\partial \bar{\varphi}}{\partial z}\right)\boldsymbol{i}+v'\left(-\frac{\partial \varphi'}{\partial y}\frac{\partial \bar{u}}{\partial z}+\frac{\partial u'}{\partial y}\frac{\partial \bar{\varphi}}{\partial z}\right)\boldsymbol{j}+w'\left[-\frac{\partial \varphi'}{\partial z}\left(\frac{\partial \bar{v}}{\partial x}-\frac{\partial \bar{u}}{\partial y}\right)+\frac{\partial v'}{\partial z}\frac{\partial \bar{\varphi}}{\partial x}-\frac{\partial u'}{\partial z}\frac{\partial \bar{\varphi}}{\partial y}\right]\boldsymbol{k}, \tag{A19}$$

$$\boldsymbol{F}_{q'4} = \left(f\bar{\boldsymbol{v}}_{\mathrm{a}}\times\boldsymbol{k}\right)\times\nabla\varphi'+\left(f\boldsymbol{v}'_{\mathrm{a}}\times\boldsymbol{k}\right)\times\nabla\bar{\varphi}, \tag{A20}$$

$$\begin{aligned}\boldsymbol{F}_{q'5} = &\boldsymbol{i}\left[-\left(\frac{\partial \bar{v}}{\partial z}-\frac{\partial v'}{\partial z}\right)\overline{\boldsymbol{v}'\cdot\nabla\varphi'}+\left(\frac{\partial \bar{\varphi}}{\partial z}-\frac{\partial \varphi'}{\partial z}\right)\overline{\boldsymbol{v}'\cdot\nabla v'}\right]+\boldsymbol{j}\left[\left(\frac{\partial \bar{u}}{\partial z}-\frac{\partial u'}{\partial z}\right)\overline{\boldsymbol{v}'\cdot\nabla\varphi'}-\left(\frac{\partial \bar{\varphi}}{\partial z}-\frac{\partial \varphi'}{\partial z}\right)\overline{\boldsymbol{v}'\cdot\nabla u'}\right]\\ &+\boldsymbol{k}\left\{\left[\left(\frac{\partial \bar{v}}{\partial x}-\frac{\partial \bar{u}}{\partial y}\right)-\left(\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}\right)\right]\overline{\boldsymbol{v}'\cdot\nabla\varphi'}+\left(\frac{\partial \bar{\varphi}}{\partial y}-\frac{\partial \varphi'}{\partial y}\right)\overline{\boldsymbol{v}'\cdot\nabla u'}-\left(\frac{\partial \bar{\varphi}}{\partial x}-\frac{\partial \varphi'}{\partial x}\right)\overline{\boldsymbol{v}'\cdot\nabla v'}\right\}. \end{aligned} \tag{A21}$$

It can be proved that

$$\nabla\cdot\boldsymbol{T}'\boldsymbol{2} = \nabla\cdot\boldsymbol{T}'\boldsymbol{3}, \tag{A22}$$

where

$$\boldsymbol{T}'\boldsymbol{3} = \boldsymbol{i}\left[\bar{u}\left(-\frac{\partial u'}{\partial z}\frac{\partial \bar{\varphi}}{\partial y}+\frac{\partial \bar{u}}{\partial y}\frac{\partial \varphi'}{\partial z}-\frac{\partial \bar{u}}{\partial z}\frac{\partial \varphi'}{\partial y}+\frac{\partial u'}{\partial y}\frac{\partial \bar{\varphi}}{\partial z}\right)+u'\left(-\frac{\partial \bar{u}}{\partial z}\frac{\partial \bar{\varphi}}{\partial y}+\frac{\partial \bar{u}}{\partial y}\frac{\partial \bar{\varphi}}{\partial z}\right)\right]$$

$$+\boldsymbol{j}\left[\bar{v}\left(\frac{\partial\bar{v}}{\partial z}\frac{\partial\varphi'}{\partial x}-\frac{\partial v'}{\partial x}\frac{\partial\bar{\varphi}}{\partial z}+\frac{\partial v'}{\partial z}\frac{\partial\bar{\varphi}}{\partial x}-\frac{\partial\bar{v}}{\partial x}\frac{\partial\varphi'}{\partial z}\right)+v'\left(\frac{\partial\bar{v}}{\partial z}\frac{\partial\bar{\varphi}}{\partial x}-\frac{\partial\bar{v}}{\partial x}\frac{\partial\bar{\varphi}}{\partial z}\right)\right]. \tag{A23}$$

After rearrangement, one has

$$\begin{aligned}\nabla\cdot\left(\boldsymbol{T'1}+\boldsymbol{T'2}\right)&=\nabla\cdot\left(\boldsymbol{T'1}+\boldsymbol{T'3}\right)\\&=\nabla\cdot\boldsymbol{F}_{q'1}+\nabla\cdot\boldsymbol{F}_{q'2},\end{aligned} \tag{A24}$$

where

$$\boldsymbol{F}_{q'1}=-\bar{v}q', \tag{A25}$$

$$\boldsymbol{F}_{q'2}=-\boldsymbol{v}'\bar{q}. \tag{A26}$$

The substitution of Eq. (A24) into Eq. (A16) yields

$$\begin{aligned}\frac{\partial q'}{\partial t}&=\nabla\cdot\boldsymbol{F}_{q'1}+\nabla\cdot\boldsymbol{F}_{q'2}+\nabla\cdot\boldsymbol{F}_{q'3}+\nabla\cdot\boldsymbol{F}_{q'4}+\nabla\cdot F_{q'5}\\&\quad-\nabla\cdot\boldsymbol{F}_{\mathrm{Aex}}+\nabla\cdot\left(\boldsymbol{v}'_{\mathrm{h}}\times\nabla\bar{S}_{\varphi}+\bar{\boldsymbol{v}}_{\mathrm{h}}\times\nabla S'\right).\end{aligned} \tag{A27}$$

Eliminating the local-change terms on the right-hand side of Eq. (A3) by using Eqs. (9)–(11) yields

$$\nabla\cdot\left[\left(\nabla\times\bar{\boldsymbol{v}}_{\mathrm{h}}\right)\frac{\partial\bar{\varphi}}{\partial t}-\nabla\bar{\varphi}\times\frac{\partial\bar{\boldsymbol{v}}_{\mathrm{h}}}{\partial t}\right]=\nabla\cdot\left(\bar{\boldsymbol{T}}\boldsymbol{1}+\bar{\boldsymbol{T}}\boldsymbol{2}\right)+\nabla\cdot\boldsymbol{F}_{\bar{q}2}-\bar{\boldsymbol{F}}_{\mathrm{Aex}}+\nabla\cdot\left(\bar{\boldsymbol{v}}_{\mathrm{h}}\times\nabla\bar{S}_{\varphi}\right), \tag{A28}$$

where

$$\begin{aligned}\bar{\boldsymbol{T}}\boldsymbol{1}&=\boldsymbol{i}\left[\bar{u}\left(\frac{\partial\bar{v}}{\partial z}\frac{\partial\bar{\varphi}}{\partial x}-\frac{\partial\bar{v}}{\partial x}\frac{\partial\bar{\varphi}}{\partial z}\right)+\overline{u'\left(\frac{\partial\bar{v}}{\partial z}\frac{\partial\varphi'}{\partial x}-\frac{\partial v'}{\partial x}\frac{\partial\bar{\varphi}}{\partial z}\right)}\right]+\boldsymbol{j}\left[\bar{v}\left(\frac{\partial\bar{u}}{\partial y}\frac{\partial\bar{\varphi}}{\partial z}-\frac{\partial\bar{u}}{\partial z}\frac{\partial\bar{\varphi}}{\partial y}\right)+\overline{v'\left(\frac{\partial u'}{\partial y}\frac{\partial\bar{\varphi}}{\partial z}-\frac{\partial\bar{u}}{\partial z}\frac{\partial\varphi'}{\partial y}\right)}\right]\\&\quad+\boldsymbol{k}\left\{\bar{w}\left[\frac{\partial\bar{v}}{\partial z}\frac{\partial\bar{\varphi}}{\partial x}-\frac{\partial\bar{u}}{\partial z}\frac{\partial\bar{\varphi}}{\partial y}+\left(\frac{\partial\bar{u}}{\partial y}-\frac{\partial\bar{v}}{\partial x}\right)\frac{\partial\bar{\varphi}}{\partial z}\right]+\overline{w'\left[\frac{\partial v'}{\partial z}\frac{\partial\bar{\varphi}}{\partial x}-\frac{\partial u'}{\partial z}\frac{\partial\bar{\varphi}}{\partial y}-\left(\frac{\partial\bar{v}}{\partial x}-\frac{\partial\bar{u}}{\partial y}\right)\frac{\partial\varphi'}{\partial z}\right]}\right\},\end{aligned} \tag{A29}$$

$$\bar{\boldsymbol{T}}\boldsymbol{2}=\bar{v}\left(\frac{\partial\bar{\varphi}}{\partial y}\frac{\partial\bar{v}}{\partial z}-\frac{\partial\bar{v}}{\partial y}\frac{\partial\bar{\varphi}}{\partial z}\right)\boldsymbol{i}-\bar{u}\left(\frac{\partial\bar{\varphi}}{\partial x}\frac{\partial\bar{u}}{\partial z}-\frac{\partial\bar{u}}{\partial x}\frac{\partial\bar{\varphi}}{\partial z}\right)\boldsymbol{j}+\left[\bar{u}\left(\frac{\partial\bar{\varphi}}{\partial x}\frac{\partial\bar{u}}{\partial y}-\frac{\partial\bar{u}}{\partial x}\frac{\partial\bar{\varphi}}{\partial y}\right)-\bar{v}\left(\frac{\partial\bar{\varphi}}{\partial y}\frac{\partial\bar{v}}{\partial x}-\frac{\partial\bar{v}}{\partial y}\frac{\partial\bar{\varphi}}{\partial x}\right)\right]\boldsymbol{k}, \tag{A30}$$

$$\boldsymbol{F}_{\bar{q}2}=\left(f\bar{\boldsymbol{v}}_{\mathrm{a}}\times\boldsymbol{k}\right)\times\nabla\bar{\varphi}. \tag{A31}$$

It is clear that

$$\nabla\cdot\bar{\boldsymbol{T}}\boldsymbol{2}=\nabla\cdot\bar{\boldsymbol{T}}\boldsymbol{3}, \tag{A32}$$

where

$$\begin{aligned}\bar{\boldsymbol{T}}\boldsymbol{3}&=\bar{u}\left(-\frac{\partial\bar{u}}{\partial z}\frac{\partial\bar{\varphi}}{\partial y}+\frac{\partial\bar{u}}{\partial y}\frac{\partial\bar{\varphi}}{\partial z}\right)\boldsymbol{i}\\&\quad+\bar{v}\left(-\frac{\partial\bar{v}}{\partial x}\frac{\partial\bar{\varphi}}{\partial z}+\frac{\partial\bar{v}}{\partial z}\frac{\partial\bar{\varphi}}{\partial x}\right)\boldsymbol{j}.\end{aligned} \tag{A33}$$

Thus, after the substitution of Eq. (A32), one has

$$\begin{aligned}\nabla\cdot\left(\bar{\boldsymbol{T}}\boldsymbol{1}+\bar{\boldsymbol{T}}\boldsymbol{2}\right)&=\nabla\cdot\left(\bar{\boldsymbol{T}}\boldsymbol{1}+\bar{\boldsymbol{T}}\boldsymbol{3}\right)\\&=\nabla\cdot\boldsymbol{F}_{\bar{q}1}+\nabla\cdot\bar{\boldsymbol{F}}_{q'3},\end{aligned} \tag{A34}$$

where

$$\boldsymbol{F}_{\bar{q}1}=-\bar{v}\bar{q}. \tag{A35}$$

Substituting Eq. (A34) into Eq. (A28) and then substituting the result into Eq. (A3), we have

$$\begin{aligned}\frac{\partial\bar{q}}{\partial t}&=\nabla\cdot\boldsymbol{F}_{\bar{q}1}+\nabla\cdot\boldsymbol{F}_{\bar{q}2}+\nabla\cdot\bar{\boldsymbol{F}}_{q'3}-\nabla\cdot\bar{F}_{\mathrm{Aex}}\\&\quad+\nabla\cdot\left(\bar{\boldsymbol{v}}_{\mathrm{h}}\times\nabla\bar{S}_{\varphi}\right).\end{aligned} \tag{A36}$$

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