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Citation: Chin. Phys. B, 2021, 30 (**3**): 030201. DOI: 10.1088/1674-1056/abc2ba
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Soliton interactions and asymptotic state analysis in a discrete nonlocal nonlinear self-dual network equation of reverse-space type*

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(Received 17 September 2020; revised manuscript received 3 October 2020; accepted manuscript online 20 October 2020)

We propose a reverse-space nonlocal nonlinear self-dual network equation under special symmetry reduction, which may have potential applications in electric circuits. Nonlocal infinitely many conservation laws are constructed based on its Lax pair. Nonlocal discrete generalized $(m,N-m)$-fold Darboux transformation is extended and applied to solve this system. As an application of the method, we obtain multi-soliton solutions in zero seed background via the nonlocal discrete N-fold Darboux transformation and rational solutions from nonzero-seed background via the nonlocal discrete generalized $(1,N-1)$-fold Darboux transformation, respectively. By using the asymptotic and graphic analysis, structures of one-, two-, three- and four-soliton solutions are shown and discussed graphically. We find that single component field in this nonlocal system displays unstable soliton structure whereas the combined potential terms exhibit stable soliton structures. It is shown that the soliton structures are quite different between discrete local and nonlocal systems. Results given in this paper may be helpful for understanding the electrical signals propagation.

Keywords: reverse-space nonlocal nonlinear self-dual network equation, nonlocal discrete generalized $(m,N-m)$-fold Darboux transformation, multi-soliton solutions, rational solutions

PACS: 02.30.Ik, 05.45.Yv, 04.60.Nc **DOI:** 10.1088/1674-1056/abc2ba

1. Introduction

Nonlinear lattice equations (NLEs) can be regarded as spatially discrete analog of nonlinear partial differential equations. As the models of some physical phenomena, NLEs have received widespread attention and may be used to describe many certain physical situations such as nonlinear optics, nonlinear lattice dynamics, ladder-type electric circuits, feader-eater process in biology, Langmuir waves in plasma physics (see Refs. [1–7] and references therein). For example, Wadati[1] proposed a ladder-type nonlinear LC circuit diagram, which can be governed by the NLE as follows:

$$\begin{cases} q_{n,t} = I_n - I_{n+1}, \\ \Phi_{n,t} = V_{n-1} - V_n, \end{cases} \tag{1}$$

where $q_n = C(V_n)V_n$ and $\Phi_n = L(I_n)I_n$ represent the electric charge in the n-th capacitance and magnetic flux across n-th inductance, respectively. The voltage dependent capacitance $C(V_n)$ and the current dependent inductance $L(I_n)$ can result in nonlinearities of Eq. (1). When taking $C(V_n) = \arctan(V_n)/V_n$ and $L(I_n) = \arctan(I_n)/I_n$, Eq. (1) reduces to the famous nonlinear self-dual network equation which was first introduced by Hirota to describe the propagation of electrical signals in a nonlinear, lumped, self-dual ladder-type electric circuit.[2] Since nonlinear self-dual network equation was proposed, it has attracted a great deal of attention due to its special physical background.[1,3,8–14] In Ref. [8], Ablowitz and Ladik proposed a coupled discrete nonlinear integrable equation with four potentials:

$$\begin{cases} Q_{n,t} = (1 - Q_n R_n)(gS_n - hS_{n-1}), \\ R_{n,t} = (1 - Q_n R_n)(hT_n - gT_{n+1}), \\ S_{n,t} = (1 - S_n T_n)(gQ_{n+1} - hQ_n), \\ T_{n,t} = (1 - S_n T_n)(hR_{n+1} - gR_n), \end{cases} \tag{2}$$

where $Q_n = Q(n,t)$, $R_n = R(n,t)$, $S_n = S(n,t)$, $T_n = T(n,t)$ are the functions of discrete variable n and time variable t, $Q_{n,t} = \frac{\mathrm{d}Q_n}{\mathrm{d}t}$, $R_{n,t} = \frac{\mathrm{d}R_n}{\mathrm{d}t}$, $S_{n,t} = \frac{\mathrm{d}S_n}{\mathrm{d}t}$, $T_{n,t} = \frac{\mathrm{d}T_n}{\mathrm{d}t}$, and g,h are arbitrary constants. Equation (6) was first construed using the Ablowitz–Kaup–Newell–Segur (AKNS) technique and its some exact soliton solutions have been derived by inverse scattering transformation method.[8] Due to the denominator in Lax pair in Ref. [8], it is difficult to construct its Darboux transformation (DT). Therefore in Ref. [9], Geng constructed the new Lax pair of Eq. (6) to obtain its multi-soliton solutions in zero seed background using the DT method. In Ref. [10], a discrete integrable coupled Ablowitz–Ladik lattice hierarchy covering Eq. (6) has been proposed and some related integrable properties have been studied. When we choose $R_n = -\sigma Q_n = I_n$, $S_n = -\sigma T_n = V_n$, $g = h = 1$, $\sigma = \pm 1$, Eq. (6) reduces to a nonlinear self-dual network equation only taking $\sigma = 1$.

In 2013, Ablowitz and Musslimani first proposed the continuous PT-symmetry nonlocal nonlinear Schrödinger (NNLS) equation from a new symmetry reduction of the well-known AKNS system, which is an integrable system admitting the Lax pair and an infinite number of conservation laws, and has

*Project supported by the National Natural Science Foundation of China (Grant Nos. 12071042 and 61471406), the Beijing Natural Science Foundation, China (Grant No. 1202006), and Qin Xin Talents Cultivation Program of Beijing Information Science and Technology University (QXTCP-B201704).

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been solved by the inverse scattering transform method.[15] The PT-symmetric NNLS equation, which can keep lossless propagation due to the balanced gain and loss,[16] admits remarkable new characteristics that are not observed by comparison with its classical standard NLS equation and has received great attention, see Refs. [15–21]. Moreover, in Refs. [22,23], Ablowitz and Musslimani proposed many different nonlocal counterparts of known continuous and discrete nonlinear integrable equations. Subsequently, many other nonlocal nonlinear integrable equations have also been proposed and studied, such as the nonlocal modified Korteweg–de Vries equation,[24] (2+1)-dimensional nonlocal NLS equation,[25] nonlocal Manakov system,[26] nonlocal Burgers equation,[27] nonlocal two-wave interaction system,[28] nonlocal short pulse equation,[29] nonlocal Gross-pitaevskii equation,[30,31] PT-symmetric coupled NNLS equations.[32] Like their local versions, nonlocal integrable systems admit Lax pair, conservation laws, soliton solutions and some integrable properties,[15,17] and they also admit certain new features and cannot be realized in contrast with their local counterparts that further make the subject attractive. However, it is easy to find that most research results of nonlocal equations are the study of continuous nonlocal integrable equations, while the revelent research of discrete NLEs is not sufficient.[9,33–37] Especially, the study of nonlocal discrete equations is not systematic,[38–41] so focusing on discrete nonlocal NLEs motivates us.

In this paper, inspired by the idea of Ablowitz and Musslimani in Refs. [15,22,23] and based on the previous work in Refs. [9,10], for Eq. (6), if we take a simple reduction $R_n = I_n$, $S_n = V_n$, $Q_n = \sigma I_{-n}$, $T_n = \sigma V_{-n}$ and $h = g = 1$, then Eq. (6) can reduce to the following new discrete reverse-space nonlocal NLE:

$$\begin{cases} I_{n,t} = (\sigma - I_n I_{-n})(V_{-n} - V_{-n-1}), \\ V_{n,t} = (\sigma - V_n V_{-n})(I_{-n+1} - I_{-n}), \end{cases} \tag{3}$$

where $I_n = I(n,t)$ and $V_n = V(n,t)$ are real-valued scalar functions of the discrete variable n and time variable t, $I_{-n} = I(-n,t)$, $V_{-n} = V(-n,t)$, $I_{n,t} = \frac{\mathrm{d}I_n}{\mathrm{d}t}$, $I_{-n,t} = \frac{\mathrm{d}I_{-n}}{\mathrm{d}t}$, $V_{n,t} = \frac{\mathrm{d}V_n}{\mathrm{d}t}$, $V_{-n,t} = \frac{\mathrm{d}V_{-n}}{\mathrm{d}t}$, and $\sigma = \pm 1$ correspond to the self-focusing and defocusing cases. It is noted that Eq. (3) is a real reverse-space discrete nonlinear integrable equation which may also have potential applications for illustrating some physical phenomena in electricity. Here we call Eq. (3) a reverse-space nonlocal nonlinear self-dual network equation compared to the classical nonlinear self-dual network equation. According to Ref. [9], the Lax pair of Eq. (3) is given as follows:

$$\begin{aligned} &E\varphi_n = U_n \varphi_n, \\ &U_n = \begin{pmatrix} \lambda + I_n V_n & \lambda^{-1} V_n + \sigma I_{-n} \\ \sigma\lambda V_{-n} + I_n & \lambda^{-1} + I_{-n} V_{-n} \end{pmatrix}, \\ &\varphi_{n,t} = W_n \varphi_n, \end{aligned} \tag{4}$$

$$W_n = \begin{pmatrix} \lambda - I_{-n} V_{-n-1} & \sigma I_{-n} + \lambda^{-1} V_{n-1} \\ \sigma\lambda V_{-n-1} + I_n & \lambda^{-1} - I_n V_{n-1} \end{pmatrix}, \tag{5}$$

where $\varphi_n = (\phi_{1,n}, \psi_{1,n})^{\mathrm{T}}$, λ means the spectral parameter independent of time t, and in this nonlocal system we define the shift operator E by $Ef(\pm n,t) = f(\pm n + 1,t) = f_{\pm n+1}, E^{-1} f(\pm n,t) = f(\pm n - 1,t) = f_{\pm n-1}$. The compatibility condition $U_{n,t} = (EW_n)U_n - U_n W_n$ between Lax pair (4) and (5) just leads to Eq. (3), which is PT-symmetric for the reason that the potential terms $G(n,t) = \sigma I(n,t) I(-n,t)$ and $H(n,t) = \sigma V(n,t) V(-n,t)$ obey the PT-symmetry, i.e., $G(n,t) = G(-n,t)$, $H(n,t) = H(-n,t)$. PT-symmetry invariance is particular because the local solution state at location n is directly coupled to the nonlocal solution at remote location $-n$, which is very similar to quantum entanglement between a pair of particles.[29] Due to the physical background of nonlinear self-dual network equation, physicists and engineers may design a PT-symmetry nonlocal electronic circuit diagram by utilizing Eq. (3), so Eq. (3) may possess potential application value in electric circuits.

Some methods for solving local equations have been extended to nonlocal equations such as the inverse scattering transform method,[15,24,30] Hirota's bilinear method,[25,26,31,32,42–44] the function expansion method,[27] the Riemann–Hilbert method,[28] and the DT method.[9,12,16,29,33,35–41,45–51] Among them, DT is a simple algebraic technique to generate a family of infinite nontrivial solutions of a linear equation from a known trivial solution which is widely used in soliton theory.[9,45,46] It needs to be noted that the N-fold DT technique is a very effective method to derive N-soliton solutions compared to iterative DT because there is no iterative process. Recently, a generalized $(m, N-m)$-fold DT method, taken as a generalization of the N-fold DT, was proposed by one of the authors of this paper, and it not only expresses soliton solutions but also gives some rational solutions and mixed interaction solutions of rational solutions and usual soliton solutions. This technique was originally used in continuous local models, and then gradually extended to continuous nonlocal models, discrete models and discrete nonlocal models. However, for the discrete nonlocal NLEs, the relevant research and theory are still not systematic and sufficient, so it is a meaningful research work to continue to develop and improve this technique. To the best of our knowledge, the discrete nonlocal multi-soliton solutions and relevant dynamical behaviors of the new reverse-space nonlocal Eq. (3) have not been reported yet in literature. Therefore, in this paper, we further investigate Eq. (3) via the generalized $(m, N-m)$-fold DT technique.

In addition, before discussing Eq. (3), according to Ref. [10], we also propose a new high-order reverse-space nonlocal nonlinear self-dual network equation

$$\begin{cases} I_{n,t} = (\sigma - I_n I_{-n})[I_n V_{-n-1} V_n - I_{-n} V_{-n}^2 - I_n V_{-n} V_{-n-1} + I_{n-1} V_{n-1} V_{-n-1} + I_{-n} V_{-n-1}^2 - I_{n+1} V_n V_{-n} - \sigma(I_{n-1} - I_{n+1})], \\ V_{n,t} = (\sigma - V_n V_{-n})[I_n I_{-n} V_{n-1} - I_n I_{-n+1} V_n + I_{-n}^2 V_{-n} + I_{-n} I_{n+1} V_n - I_{-n+1}^2 V_{-n} - I_{n+1} I_{-n+1} V_{n+1} - \sigma(V_{n-1} - V_{n+1})], \end{cases} \tag{6}$$

and we here give its new lax pair different form one in Ref. [10], the space part of Lax pair for Eq. (6) is same as Eq. (3) (i.e., Eq. (4)), while the time part of its Lax pair is given in the following form:

$$\varphi_{n,t} = W_n^{(2)} \varphi_n, \quad W_n^{(2)} = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \tag{7}$$

with

$$\begin{aligned} A_n &= -\frac{1}{2\lambda^2} + I_n V_{n-1} \frac{1}{\lambda} + I_n V_n I_{-n} V_{-n-1} \\ &\quad + I_{-n}^2 V_{-n-1}^2 + I_{-n} I_{n-1} V_{-n-1} V_{n-1} \\ &\quad - \sigma(I_{-n} I_{n-1} + V_n V_{-n-1}) \\ &\quad - I_{-n} V_{-n-1} \lambda + \frac{1}{2}\lambda^2, \\ B_n &= \sigma I_{-n} \lambda - \sigma I_n I_{-n} V_n - \sigma I_{-n}^2 V_{-n-1} + V_n + (-I_n V_{n-1}^2 \\ &\quad - I_{-n-1} V_{-n-1} V_{n-1} + \sigma I_{-n-1}) \frac{1}{\lambda} + V_{n-1} \frac{1}{\lambda^2}, \\ C_n &= I_n \frac{1}{\lambda} - I_n I_{-n} V_{-n} - I_n^2 V_{n-1} + \sigma V_{-n} \\ &\quad + (-\sigma I_{-n} V_{-n-1}^2 - \sigma I_{n-1} V_{-n-1} V_{n-1} + I_{n-1}) \lambda \\ &\quad + \sigma V_{-n-1} \lambda^2, \\ D_n &= \frac{1}{2\lambda^2} - I_n V_{n-1} \frac{1}{\lambda} + I_{-n} V_{-n} I_n V_{n-1} \\ &\quad + I_n^2 V_{n-1}^2 + I_n I_{-n-1} V_{-n-1} V_{n-1} \\ &\quad - \sigma(I_n I_{-n-1} + V_{-n} V_{n-1}) + I_{-n} V_{-n-1} \lambda - \frac{1}{2}\lambda^2. \end{aligned}$$

The compatibility condition $U_{n,t} = (EW_n^{(2)})U_n - U_n W_n^{(2)}$ between Eqs. (4) and (7) just yields Eq. (6). Theoretically, Eq. (6) possesses more nonlinear terms which is possible to be used to design more complex nonlocal electronic circuit diagrams, which may have the potential application in describing electric signal propagation in complex electric circuits.

The rest contents of this paper are the followings. In Section 2, we construct nonlocal infinitely many conservation laws of Eq. (3). In Section 3, the nonlocal version of $(m, N-m)$-fold DT for Eq. (3) is constructed based on its Lax representation. As applications, the discrete nonlocal multi-soliton solutions and rational solutions are calculated via the nonlocal discrete N-fold DT technique (discrete generalized $(m, N-m)$-fold DT with $m = N$) from vanishing background and the nonlocal discrete $(1, N-1)$-fold DT (discrete generalized $(m, N-m)$-fold DT with $m = 1$) from nonzero seed background, respectively. Moreover, the soliton propagation properties and asymptotic state analysis are analyzed and discussed graphically, and a few wave features of rational solutions are summed up as well. Last section will be devoted to some conclusions.

2. Conservation laws of Eq. (3)

Conservation laws are important to justify the integrability for integrable system.[1,52] In this section, we shall derive the infinitely many conservation laws for Eq. (3). Firstly, we can get from Eqs. (4) and (5),

$$\begin{aligned} \varphi_{1,n+1} &= (\lambda + I_n V_n)\varphi_{1,n} + \left(\frac{V_n}{\lambda} + \sigma I_{-n}\right)\varphi_{2,n}, \\ \varphi_{2,n+1} &= (I_n + \sigma\lambda V_{-n})\varphi_{1,n} + \left(\frac{1}{\lambda} + I_{-n} V_{-n}\right)\varphi_{2,n}. \end{aligned} \tag{8}$$

Setting $\theta_n = \varphi_{2,n}/\varphi_{1,n}$, we obtain

$$\begin{aligned} \frac{\varphi_{1,n+1}}{\varphi_{1,n}} &= \lambda + I_n V_n + \left(\frac{V_n}{\lambda} + \sigma I_{-n}\right)\theta_n, \\ \frac{\varphi_{2,n+1}}{\varphi_{2,n}} &= \frac{I_n + \sigma\lambda V_{-n}}{\theta_n} + \frac{1}{\lambda} + I_{-n} V_{-n}, \end{aligned} \tag{9}$$

from which we have

$$\begin{aligned} &\left(\frac{V_n}{\lambda} + \sigma I_{-n}\right)\theta_n\theta_{n+1} + (\lambda + I_n V_n)\theta_{n+1} \\ &\quad - \left(\frac{1}{\lambda} + I_{-n} V_{-n}\right)\theta_n - I_n - \sigma\lambda V_{-n} = 0. \end{aligned} \tag{10}$$

Substituting $\theta_n = \sum_{j=0}^{\infty} \frac{\theta_n^{(j)}}{\lambda^j}$ into Eq. (10) yields the following recursion relations:

$$\begin{aligned} \theta_n^{(0)} &= \sigma V_{-n-1}, \\ \theta_n^{(1)} &= I_{n-1}(1 - \sigma V_{n-1} V_{-n-1}), \\ \theta_n^{(2)} &= (\sigma V_{n-1} V_{-n-1} - 1)(I_{n-1} I_{-n-1} V_{-n-2} \\ &\quad + I_{n-1}^2 V_{n-1} - \sigma V_{-n-2}), \\ \theta_n^{(3)} &= (1 - \sigma V_{-n-1} V_{n-1})(2I_{n-1}^2 I_{-n-1} V_{n-1} V_{-n-2} \\ &\quad + I_{n-1}^3 V_{n-1}^2 + (V_{-n-2}^2 I_{-n-1}^2 \\ &\quad + I_{n-2}(V_{-n-2} V_{n-2} - \sigma) I_{-n-1} - 2\sigma V_{-n-2} V_{n-1}) I_{n-1} \\ &\quad - \sigma I_{-n-1} V_{-n-2}^2 - \sigma I_{n-2} V_{-n-2} V_{n-2} + I_{n-2}), \\ &\quad \cdots \end{aligned} \tag{11}$$

By means of Eqs. (5) and (9), we have

$$\begin{aligned} &\left[\ln(\lambda + I_n V_n + \left(\frac{V_n}{\lambda} + \sigma I_{-n}\right)\theta_n)\right]_t \\ &= (E-1)\left(\lambda - I_{-n} V_{-n-1} + \left(\sigma I_{-n} + \frac{V_{n-1}}{\lambda}\right)\theta_n\right). \end{aligned} \tag{12}$$

Taking Eq. (11) into Eq. (12) and equating the same powers of λ in Eq. (12) generate an infinite number of conservation laws to Eq. (3), and the first two conservation laws are listed as follows:

$$(T_k)_t = (E-1)(X_k), \quad (k = 1, 2) \tag{13}$$

with

$$\begin{aligned} T_1 &= I_nV_n + I_{-n}V_{-n-1}, \\ X_1 &= (-V_{-n-1}V_{n-1}+\sigma)I_{-n}I_{n-1} + \sigma V_{-n-1}V_{n-1}, \\ T_2 &= \sigma V_nV_{-n-1} - I_{n-1}(V_{-n-1}V_{n-1}-\sigma)I_{-n} \\ &\quad -\frac{1}{2}(I_nV_n + I_{-n}V_{-n-1})^2, \\ X_2 &= (V_{-n-1}V_{n-1}-\sigma)(I_{-n}I_{n-1}^2V_{n-1} \\ &\quad +(I_{-n}I_{-n-1}V_{-n-2} - \sigma V_{n-1})I_{n-1} - \sigma I_{-n}V_{-n-2}), \end{aligned}$$

where T_k and X_k are the conserved densities and associated fluxes, respectively. The expressions $-\sum_{n=-\infty}^{\infty} T_1$ and $\sum_{n=-\infty}^{\infty} T_2$ represent the total momentum and total energy of lattice, respectively.

3. Discrete generalized $(m, N-m)$-fold DT

In this section, we construct the generalized $(m,N-m)$-fold DT of Eq. (3) on the basis of Lax pair (4) and (5). First of all, we assume that $\varphi_n(\lambda_i) = (\phi_n(\lambda_i), \psi_n(\lambda_i))^{\mathrm{T}}$ are m distinct solutions to Eqs. (4) and (5) for $\lambda = \lambda_i$ $(i = 1,2,\ldots,m)$ and the initial solutions I_n, V_n of Eq. (3). We introduce the following special gauge transformation:

$$\widetilde{\varphi}_n = T_n\varphi_n = \begin{pmatrix} \lambda^N + \sum\limits_{j=0}^{N-1} a_n^{(j)}\lambda^j & \sum\limits_{j=0}^{N-1} b_n^{(j)}\lambda^j \\ \sigma\sum\limits_{j=0}^{N-1} b_{-n}^{(j)}\lambda^{N-j} & 1+\sum\limits_{j=0}^{N-1} a_{-n}^{(j)}\lambda^{N-j} \end{pmatrix}\varphi_n, \quad (14)$$

where N is a positive integer, $a_n^{(j)}, b_n^{(j)}$ are the functions of variables n,t to be determined later, while $a_{-n}^{(j)}, b_{-n}^{(j)}$ are derived through $a_n^{(j)}, b_n^{(j)}$ by replacing n with $-n$. From the knowledge of DT, Eq. (14) ensures that $\widetilde{\varphi}_n$ satisfies $E\widetilde{\varphi}_n = \widetilde{U}_n\widetilde{\varphi}_n$ and $\widetilde{\varphi}_{n,t} = \widetilde{W}_n\widetilde{\varphi}_n$, in which $\widetilde{U}_n$ and $\widetilde{W}_n$ have the same forms with U_n and W_n under transformations $\widetilde{U}_n = T_{n+1}U_nT_n^{-1}$ and $\widetilde{W}_n = (T_{n,t} + T_nW_n)T_n^{-1}$, except replacing the old potentials I_n, V_n with new ones $\widetilde{I}_n$, $\widetilde{V}_n$. To construct the generalized $(m,N-m)$-fold DT, $a_n^{(j)}$ and $b_n^{(j)}$ are determined by the following linear algebraic system with $2N$ equations ($N = m + \sum_{i=1}^{m} M_i$, $i = 1,2,\ldots,m$):

$$\begin{cases} T^{(0)}(\lambda_i)\varphi^{(0)}(\lambda_i) = 0, \\ T^{(0)}(\lambda_i)\varphi^{(1)}(\lambda_i) + T^{(1)}(\lambda_i)\varphi^{(0)}(\lambda_i) = 0, \\ T^{(0)}(\lambda_i)\varphi^{(2)}(\lambda_i) + T^{(1)}(\lambda_i)\varphi^{(1)}(\lambda_i) \\ \qquad\qquad + T^{(2)}(\lambda_i)\varphi^{(0)}(\lambda_i) = 0, \\ \cdots\cdots \\ \sum\limits_{j=0}^{M_i} T^{(j)}(\lambda_i)\varphi^{(M_i-j)}(\lambda_i) = 0, \end{cases} \quad (15)$$

where $T^{(i)}$ is derived based on binomial expansion by $T(\lambda_i+\varepsilon) = T^{(0)} + T^{(1)}\varepsilon + \cdots + T^{(M_i)}\varepsilon^{M_i}$, $\varphi^{(k)}$ is determined based on Taylor series expansion by $\varphi_n(\lambda_i+\varepsilon) = \varphi^{(0)}(\lambda_i) + \varphi^{(1)}(\lambda_i)\varepsilon + \varphi^{(2)}(\lambda_i)\varepsilon^2 + \cdots$, in which $\varphi^{(k)}(\lambda_i) = \frac{1}{k!}\frac{\partial^k}{\partial\lambda_i^k}\varphi_n(\lambda_i)$.

Based on the above analysis and referring to the discrete local version of generalized $(m,N-m)$-fold DT in Ref. [21], we can conclude that Eq. (3) admits the following discrete nonlocal version of generalized $(m,N-m)$-fold DT:

Theorem 1 Let $\varphi_i(\lambda_i) = (\phi_i(\lambda_i), \psi_i(\lambda_i))^{\mathrm{T}}$ be column vector solutions with distinct spectral parameters λ_i $(i = 1,2,\ldots,m)$ of Eqs. (4) and (5) starting from the seed solutions (I_n, V_n) to Eq. (3), as considered above, then the generalized $(m,N-m)$-fold DT for Eq. (3) can be given by

$$\widetilde{I}_n = \frac{I_n - \sigma b_{-n}^{(N-1)}}{a_n^{(0)}}, \quad \widetilde{V}_n = b_{n+1}^{(0)} + V_n a_{n+1}^{(0)}, \quad (16)$$

with

$$\begin{aligned} a_n^{(0)} &= \frac{\Delta a_n^{(0)}}{\Delta_N^{\varepsilon(m)}}, \quad b_n^{(0)} = \frac{\Delta b_n^{(0)}}{\Delta_N^{\varepsilon(m)}}, \quad b_n^{(N-1)} = \frac{\Delta b_n^{(N-1)}}{\Delta_N^{\varepsilon(m)}}, \\ \Delta_N^{\varepsilon(m)} &= \det([\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(m)}]^{\mathrm{T}}), \end{aligned} \quad (17)$$

where $\Delta^{(i)} = (\Delta_{j,s}^{(i)})_{2(M_i+1)\times 2N}$, in which $\Delta_{j,s}^{(i)}$ $(1 \le j \le 2M_i + 2,\ 1 \le s \le 2N,\ i = 1,2,\ldots,m)$ are given by following formulae:

$$\Delta_{j,s}^{(i)} = \begin{cases} \sum\limits_{k=0}^{j-1} C_{N-s}^k \lambda_i^{N-s-k}\phi_n^{(j-1-k)}, & \text{for } 1 \le j \le M_i+1,\ 1 \le s \le N, \\ \sum\limits_{k=0}^{j-1} C_{2N-s}^k \lambda_i^{2N-s-k}\psi_n^{(j-1-k)}, & \text{for } 1 \le j \le M_i+1,\ N+1 \le s \le 2N, \\ \sum\limits_{k=0}^{j-(N+1)} C_s^k \lambda_i^{s-k}\psi_{-n}^{(j-1-N-k)}, & \text{for } M_i+2 \le j \le 2(M_i+1),\ 1 \le s \le N, \\ \sigma\sum\limits_{k=0}^{j-(N+1)} C_{s-N}^k \lambda_i^{s-N-k}\phi_{-n}^{(j-N-1-k)}, & \text{for } M_i+2 \le j \le 2(M_i+1),\ N+1 \le s \le 2N, \end{cases}$$

and $\Delta a_n^{(0)}$, $\Delta b_n^{(0)}$ and $\Delta b_n^{(N-1)}$ are given by the determinant $\Delta_N^{\varepsilon(m)}$ replacing its N-th, $(2N)$-th and $(N+1)$-th columns by the column vector $b = (b_j)_{2N\times 1}$ with

$$b_j = \begin{cases} -\sum\limits_{k=0}^{j-1} C_N^k \lambda_i^{N-k}\phi_n^{(j-1-k)}, & \text{for } 1 \le j \le M_i+1, \\ -\psi_{-n}^{(j-N-1)}, & \text{for } M_i+2 \le j \le 2(M_i+1), \end{cases}$$

where $b_{-n}^{(1)}$ is obtained from $b_n^{(1)}$ by replacing n with $-n$.

Here we omit the proof of Theorem 1, the reader can refer to Refs. [9,12] for the complete proving process, we believe that the similar procedure also works for the discrete nonlocal systems. In the next section, we will use discrete nonlocal generalized $(m,N-m)$-fold DT to construct some exact solutions of Eq. (3). Before that, we give a simple remark.

Remark 1 In the discrete nonlocal generalized $(m,N-m)$-fold DT, m represents the number of the distinct spectral parameter, N denote the order number of DT, $N-m$ means the sum of the highest derivative order number in Darboux matrix T_n or the vector eigenfunction φ_n. It is worth noting that when $m=N$, we can derive the discrete generalized $(N,0)$-fold DT, if Taylor series expansion is not used for every λ_i, it is the N-fold DT which can be used to construct multi-soliton solutions. When $m=1$, the generalized $(m,N-m)$-fold DT in Theorem 1 can reduce to the discrete generalized $(1,N-1)$-fold DT which can be used to give high-order rational solutions to Eq. (3). For $1<m<N$, we shall obtain some new DTs which may give some interaction solutions between soliton solutions and rational solutions, we shall not discuss them here. In the next section, we will give multi-soliton solutions in zero seed background via the generalized $(1,N-1)$-fold DT and rational solutions from nonzero-seed background via N-fold DT, respectively.

3.1. N-soliton solutions via the discrete nonlocal N-fold DT

In this subsection, we shall use the N-fold DT to give multi-soliton solutions of Eq. (3) with $\sigma=-1$ in zero background and discuss related soliton propagation phenomena, and we here omit to discuss the exact solutions of Eq. (3) with $\sigma=1$ due to their singularities. Substituting the seed solutions $I_n=V_n=0$ into Eqs. (4) and (5) gives one solution with $\lambda=\lambda_k$ as $\varphi_{k,n}=(\phi_{1,n}(\lambda_k),\,\phi_{2,n}(\lambda_k))^{\mathrm{T}}=(\lambda_k^n\,\mathrm{e}^{\lambda_k t},\,\lambda_k^{-n}\,\mathrm{e}^{\frac{t}{\lambda_k}})^{\mathrm{T}}$. Based on Eq. (16), multi-soliton solutions of Eq. (3) can be obtained. Next we discuss the following four cases $N=1,2,3,4$.

3.1.1. One-soliton solutions

When $N=1$, from Theorem 1, we can derive the 1-fold exact solutions given by

$$\widetilde{I}_n=b_{-n}^{(0)}/a_n^{(0)},\qquad \widetilde{V}_n=b_{n+1}^{(0)}, \tag{18}$$

where $a_n^{(0)}=\frac{\Delta a_n^{(0)}}{\Delta_1}$, $b_{-n}^{(0)}=\frac{\Delta b_{-n}^{(0)}}{\Delta_{1,-n}}$ and $b_{n+1}^{(0)}=\frac{\Delta b_{n+1}^{(0)}}{\Delta_{1,n+1}}$ along with

$$\Delta a_n^{(0)}=\begin{vmatrix}-\lambda_1\phi_{1,n} & \psi_{1,n}\\ -\psi_{1,-n} & -\lambda_1\phi_{1,-n}\end{vmatrix},$$
$$\Delta b_{-n}^{(0)}=\begin{vmatrix}\phi_{1,-n} & -\lambda_1\phi_{1,-n}\\ \lambda_1\psi_{1,n} & -\psi_{1,n}\end{vmatrix},$$
$$\Delta b_{n+1}^{(0)}=\begin{vmatrix}\phi_{1,n+1} & -\lambda_1\phi_{1,n+1}\\ \lambda_1\psi_{1,-n+1} & -\psi_{1,-n+1}\end{vmatrix},$$
$$\Delta_1=\begin{vmatrix}\phi_{1,n} & \psi_{1,n}\\ \lambda_1\psi_{1,-n} & -\lambda_1\phi_{1,-n}\end{vmatrix},$$
$$\Delta_{1,-n}=\begin{vmatrix}\phi_{1,-n} & \psi_{1,-n}\\ \lambda_1\psi_{1,n} & -\lambda_1\phi_{1,n}\end{vmatrix},$$
$$\Delta_{1,n+1}=\begin{vmatrix}\phi_{1,n+1} & \psi_{1,n+1}\\ \lambda_1\psi_{1,-n+1} & -\lambda_1\phi_{1,-n+1}\end{vmatrix},$$

where $b_{-n}^{(0)}$ is obtained from $b_n^{(0)}$ by replacing n with $-n$. For the sake of analysis, the solutions (18) can be rewritten in the form

$$\widetilde{I}_n=\frac{\lambda_1^2-1}{2\lambda_1^{2n+1}}\operatorname{sech}(\xi_1),$$
$$\widetilde{V}_n=\frac{1}{2}(1-\lambda_1^2)\lambda_1^{2n-1}\operatorname{sech}(\xi_1+\ln\lambda_1), \tag{19}$$

with $\xi_1=(\lambda_1-\frac{1}{\lambda_1})t+\ln\lambda_1$, in which $\lambda_1>0$ is a real constant. It is obvious that the solutions (19) have no singularity. From solutions (19), we can observe that the components $\widetilde{I}_n$ and $\widetilde{V}_n$ are space soliton solutions whose amplitudes are only related to variable n. If $0<\lambda_1<1(\lambda_1>1)$, the absolute value of $\widetilde{I}_n$ decreases (increases) when n varies from $-\infty$ to $+\infty$, and the center position of its amplitude is along a straight line $t=\frac{\lambda_1\ln(\lambda_1)}{1-\lambda_1^2}$. If $0<\lambda_1<1$ $(\lambda_1>1)$, the absolute value of $\widetilde{V}_n$ decays (grows) when n changes from $-\infty$ to $+\infty$, and its maximum amplitude is along a straight line $t=\frac{\lambda_1\ln(\lambda_1)}{1-\lambda_1^2}$. To illustrate our results, we plot two components $\widetilde{I}_n$ and $\widetilde{V}_n$ for different values of parameters as depicted in Fig. 1. Figures 1(a) and 1(b) show the unstable space soliton structures with $\lambda_1=1/2$, while Figures 1(c) and 1(d) exhibits the unstable space soliton structures with $\lambda_1=2$.

Next, we investigate some quantities such as the potential terms $\widetilde{I}_n\widetilde{I}_{-n}$, $\widetilde{V}_n\widetilde{V}_{-n}$ and $\widetilde{I}_n\widetilde{V}_n$, which can be expressed as

$$\widetilde{I}_n\widetilde{I}_{-n}=\frac{(\lambda_1^2-1)^2}{4\lambda_1^2}\operatorname{sech}^2\xi_1,$$
$$\widetilde{V}_n\widetilde{V}_{-n}=\frac{(1-\lambda_1^2)^2}{4\lambda_1^2}\operatorname{sech}^2(\xi_1+\ln\lambda_1),$$
$$\widetilde{I}_n\widetilde{V}_n=\frac{-(\lambda_1^2-1)^2}{2\lambda_1^2}\frac{1}{\cosh(2\xi_1+\ln\lambda_1)+\cosh\ln\lambda_1}. \tag{20}$$

From Eq. (20), we can see that three potential terms are independent of space variable n and only dependent of time t. In Fig. 2, we present the plots for Eq. (20) with $\lambda_1=1/2$. Figures 2(a) and 2(b) give the stable space bell-shaped one-soliton structures for the potential terms $\widetilde{I}_n\widetilde{I}_{-n}$ whose amplitude is along the straight line $t=\frac{\lambda_1\ln(\lambda_1)}{1-\lambda_1^2}$ and $\widetilde{V}_n\widetilde{V}_{-n}$ whose amplitude is along the straight line $t=\frac{2\lambda_1\ln(\lambda_1)}{1-\lambda_1^2}$. Figure 2(c) shows the stable space anti-bell-shaped one-soliton structure for $\widetilde{I}_n\widetilde{V}_n$ whose amplitude is along the straight line $t=\frac{3\lambda_1\ln(\lambda_1)}{2(1-\lambda_1^2)}$. The possible reason for the formation of these stable structures is due to balanced gain and loss of two components, and that also shows from another aspect that the energy of this system is conserved.

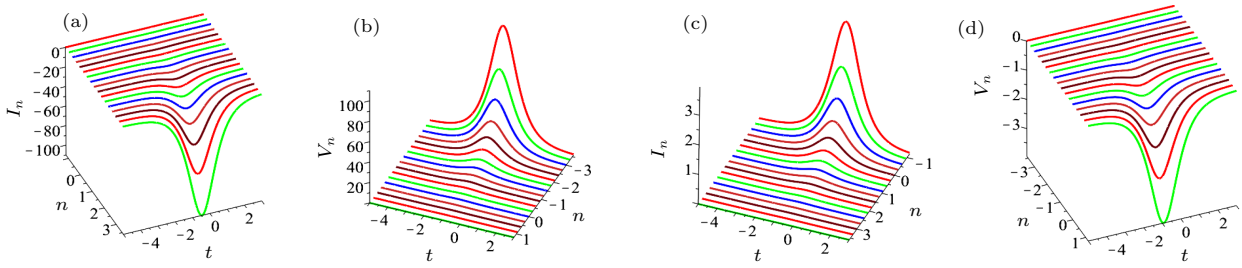


Fig. 1. [(a), (b)] Unstable one-soliton solutions for Eq. (3) with $\lambda_1 = 1/2$. [(c), (d)] Unstable one-soliton solutions for Eq. (3) with $\lambda_1 = 2$.

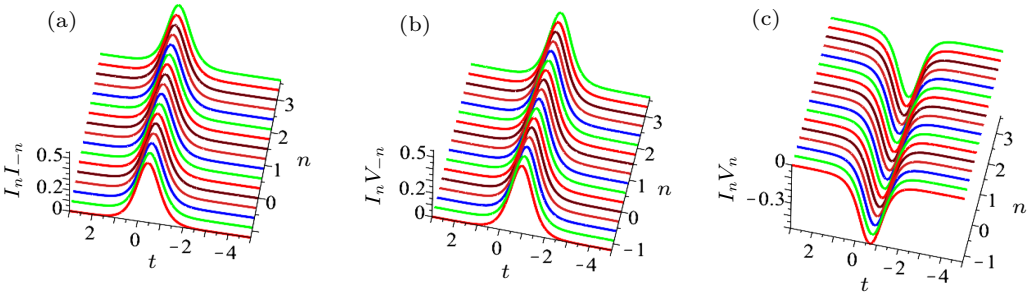


Fig. 2. Stable one-soliton solutions for potential terms (a) $\widetilde{I}_n\widetilde{I}_{-n}$, (b) $\widetilde{V}_n\widetilde{V}_{-n}$ and (c) $\widetilde{I}_n\widetilde{V}_n$ with $\lambda_1 = 1/2$.

3.1.2. Two-soliton solutions

When $N = 2$, from Theorem 1, we can derive the 2-fold exact solutions given by

$$\widetilde{I}_n = b_{-n}^{(1)}/a_n^{(0)}, \quad \widetilde{V}_n = b_{n+1}^{(0)}, \tag{21}$$

where $a_n^{(0)} = \frac{\Delta a_n^{(0)}}{\Delta_2}$, $b_{-n}^{(1)} = \frac{\Delta b_{-n}^{(1)}}{\Delta_{2,-n}}$ and $b_{n+1}^{(0)} = \frac{\Delta b_{n+1}^{(0)}}{\Delta_{2,n+1}}$ along with

$$\Delta a_n^{(0)} = \begin{vmatrix} -\lambda_1^2\phi_{1,n} & \lambda_1\phi_{1,n} & \psi_{1,n} & \lambda_1\psi_{1,n} \\ -\lambda_2^2\phi_{2,n} & \lambda_2\phi_{2,n} & \psi_{2,n} & \lambda_2\psi_{2,n} \\ -\psi_{1,-n} & \lambda_1\psi_{1,-n} & -\lambda_1^2\phi_{1,-n} & -\lambda_1\phi_{1,-n} \\ -\psi_{2,-n} & \lambda_2\psi_{2,-n} & -\lambda_2^2\phi_{2,-n} & -\lambda_2\phi_{2,-n} \end{vmatrix}, \quad \Delta b_{-n}^{(1)} = \begin{vmatrix} \phi_{1,-n} & \lambda_1\phi_{1,-n} & \psi_{1,-n} & -\lambda_1^2\phi_{1,-n} \\ \phi_{2,-n} & \lambda_2\phi_{2,-n} & \psi_{2,-n} & -\lambda_2^2\phi_{2,-n} \\ \lambda_1^2\psi_{1,n} & \lambda_1\psi_{1,n} & -\lambda_1^2\phi_{1,n} & -\psi 1,n \\ \lambda_2^2\psi_{2,n} & \lambda_2\psi_{2,n} & -\lambda_2^2\phi_{2,n} & -\psi 2,n \end{vmatrix},$$

$$\Delta b_{n+1}^{(0)} = \begin{vmatrix} \phi_{1,n+1} & \lambda_1\phi_{1,n+1} & -\lambda_1^2\phi_{1,n+1} & \lambda_1\psi_{1,n+1} \\ \phi_{2,n+1} & \lambda_2\phi_{2,n+1} & -\lambda_2^2\phi_{2,n+1} & \lambda_2\psi_{2,n+1} \\ \lambda_1^2\psi_{1,-n+1} & \lambda_1\psi_{1,-n+1} & -\psi_{1,-n+1} & -\lambda_1\phi_{1,-n+1} \\ \lambda_2^2\psi_{2,-n+1} & \lambda_2\psi_{2,-n+1} & -\psi_{2,-n+1} & -\lambda_2\phi_{2,-n+1} \end{vmatrix}, \quad \Delta_{2,-n} = \begin{vmatrix} \phi_{1,-n} & \lambda_1\phi_{1,-n} & \psi_{1,-n} & \lambda_1\psi_{1,-n} \\ \phi_{2,-n} & \lambda_2\phi_{2,-n} & \psi_{2,-n} & \lambda_2\psi_{2,-n} \\ \lambda_1^2\psi_{1,n} & \lambda_1\psi_{1,n} & -\lambda_1^2\phi_{1,n} & -\lambda_1\phi 1,n \\ \lambda_2^2\psi_{2,n} & \lambda_2\psi_{2,n} & -\lambda_2^2\phi_{2,n} & -\psi 2,n \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} \phi_{1,n} & \lambda_1\phi_{1,n} & \psi_{1,n} & \lambda_1\psi_{1,n} \\ \phi_{2,n} & \lambda_2\phi_{2,n} & \psi_{2,n} & \lambda_2\psi_{2,n} \\ \lambda_1^2\psi_{1,-n} & \lambda_1\psi_{1,-n} & -\lambda_1^2\phi_{1,-n} & -\lambda_1\phi_{1,-n} \\ \lambda_2^2\psi_{2,-n} & \lambda_2\psi_{2,-n} & -\lambda_2^2\phi_{2,-n} & -\lambda_2\phi_{2,-n} \end{vmatrix}, \quad \Delta_{2,n+1} = \begin{vmatrix} \phi_{1,n+1} & \lambda_1\phi_{1,n+1} & \psi_{1,n+1} & \lambda_1\psi_{1,n+1} \\ \phi_{2,n+1} & \lambda_2\phi_{2,n+1} & \psi_{2,n+1} & \lambda_2\psi_{2,n+1} \\ \lambda_1^2\psi_{1,-n+1} & \lambda_1\psi_{1,-n+1} & -\lambda_1^2\phi_{1,-n+1} & -\lambda_1\phi_{1,-n+1} \\ \lambda_2^2\psi_{2,-n+1} & \lambda_2\psi_{2,-n+1} & -\lambda_2^2\psi_{2,-n+1} & -\lambda_2\phi_{2,-n+1} \end{vmatrix}.$$

For convenience of analysis, they can be rewritten as

$$\begin{aligned} \widetilde{I}_n &= \frac{(\lambda_1\lambda_2-1)(\lambda_1-\lambda_2)\left[(1-\lambda_2^2)\lambda_1^{2n+1}\cosh\xi_1 + (\lambda_1^2-1)\lambda_2^{2n+1}\cosh\xi_2\right]}{\lambda_1^{2n+1}\lambda_2^{2n+1}\left[(\lambda_1^2-1)(\lambda_2^2-1)\cosh(2n\ln\frac{\lambda_1}{\lambda_2}) - (\lambda_1-\lambda_2)^2\cosh(\xi_1+\xi_2) - (\lambda_1\lambda_2-1)^2\cosh(\xi_1-\xi_2)\right]}, \\ \widetilde{V}_n &= \frac{(\lambda_1\lambda_2-1)(\lambda_1-\lambda_2)\left[(1-\lambda_2^2)\lambda_2^{2n-1}\cosh(\xi_1+\ln\lambda_1) + (\lambda_1^2-1)\lambda_1^{2n-1}\cosh(\xi_2+\ln\lambda_2)\right]}{(\lambda_1^2-1)(1-\lambda_2^2)\cosh(2n\ln\frac{\lambda_1}{\lambda_2}) + (\lambda_1-\lambda_2)^2\cosh(\xi_1+\xi_2+\ln\lambda_1\lambda_2) + (\lambda_1\lambda_2-1)^2\cosh(\xi_1-\xi_2+\ln\frac{\lambda_1}{\lambda_2})}, \end{aligned} \tag{22}$$

where $\xi_k = (\lambda_k - \frac{1}{\lambda_k})t + \ln\lambda_k \ \ (k = 1, 2)$. Similar to one-soliton solutions, we also consider the potential terms $\widetilde{I}_n\widetilde{I}_{-n}$, $\widetilde{V}_n\widetilde{V}_{-n}$ and $\widetilde{I}_n\widetilde{V}_n$, which can be expressed as

$$\widetilde{I}_n\widetilde{I}_{-n} = F_1/F_2, \qquad \widetilde{V}_n\widetilde{V}_{-n} = G_1/G_2, \qquad \widetilde{I}_n\widetilde{V}_n = H_1/H_2, \tag{23}$$

where

$$F_1 = (\lambda_1\lambda_2-1)^2(\lambda_2-\lambda_1)^2\{\lambda_1\lambda_2(\lambda_2^2-1)(1-\lambda_1^2)\cosh\left(2n\ln\frac{\lambda_1}{\lambda_2}\right)[\cosh(\xi_1+\xi_2)+\cosh(\xi_1-\xi_2)]$$

$$
\begin{aligned}
&+\lambda_1^2(\lambda_2^2-1)^2\cosh^2\xi_1+\lambda_2^2(\lambda_1^2-1)^2\cosh^2\xi_2\},\\
F_2 &= \lambda_1^2\lambda_2^2\{(\lambda_1-\lambda_2)^4\cosh^2(\xi_1+\xi_2)+(\lambda_1\lambda_2-1)^4\cosh^2(\xi_1-\xi_2)+(1-\lambda_1^2)^2(1-\lambda_2^2)^2\cosh^2\left(2n\ln\frac{\lambda_1}{\lambda_2}\right)\\
&-2(1-\lambda_1^2)(1-\lambda_2^2)\cosh\left(2n\ln\frac{\lambda_1}{\lambda_2}\right)\left[(\lambda_1-\lambda_2)^2\cosh(\xi_1+\xi_2)+(\lambda_1\lambda_2-1)^2\cosh(\xi_1-\xi_2)\right]\\
&+2(\lambda_1-\lambda_2)^2(\lambda_1\lambda_2-1)^2\cosh(\xi_1+\xi_2)\cosh(\xi_1-\xi_2)\},\\
G_1 &= (\lambda_1\lambda_2-1)^2(\lambda_2-\lambda_1)^2\{\lambda_1\lambda_2(\lambda_2^2-1)(1-\lambda_1^2)\cosh\left(2n\ln\frac{\lambda_1}{\lambda_2}\right)\left[\cosh(\xi_1+\xi_2+\ln(\lambda_1\lambda_2))+\cosh\left(\xi_1-\xi_2+\ln\frac{\lambda_1}{\lambda_2}\right)\right]\\
&+\lambda_1^2(\lambda_2^2-1)^2\cosh^2(\xi_1+\ln\lambda_1)+\lambda_2^2(\lambda_1^2-1)^2\cosh^2(\xi_2+\ln\lambda_2)\},\\
G_2 &= \lambda_1^2\lambda_2^2\Big\{2(\lambda_1-\lambda_2)^2(\lambda_1\lambda_2-1)^2\cosh(\xi_1+\xi_2+\ln(\lambda_1\lambda_2))\cosh\left(\xi_1-\xi_2+\ln\frac{\lambda_1}{\lambda_2}\right)+(1-\lambda_1^2)^2(1-\lambda_2^2)^2\cosh^2\left(2n\ln\frac{\lambda_1}{\lambda_2}\right)\\
&-2(1-\lambda_1^2)(1-\lambda_2^2)\cosh\left(2n\ln\frac{\lambda_1}{\lambda_2}\right)\left[(\lambda_1-\lambda_2)^2\cosh(\xi_1+\xi_2+\ln(\lambda_1\lambda_2))+(\lambda_1\lambda_2-1)^2\cosh\left(\xi_1-\xi_2+\ln\frac{\lambda_1}{\lambda_2}\right)\right]\\
&+(\lambda_1-\lambda_2)^4\cosh^2(\xi_1+\xi_2+\ln(\lambda_1\lambda_2)+(\lambda_1\lambda_2-1)^4\cosh^2\left(\xi_1-\xi_2+\ln\frac{\lambda_1}{\lambda_2}\right)\Big\},\\
H_1 &= -(\lambda_1\lambda_2-1)^2(\lambda_2-\lambda_1)^2\{2\lambda_1^2(\lambda_2^2-1)^2\cosh\xi_1\cosh(\xi_1+\ln\lambda_1)+2\lambda_2^2(\lambda_1^2-1)^2\cosh\xi_2\cosh(\xi_2+\ln\lambda_2)\\
&-(\lambda_2^2-1)(\lambda_1^2-1)\lambda_1^{2n+1}\lambda_2^{-2n+1}\left[\cosh(\xi_1+\xi_2+\ln\lambda_2)+\cosh(\xi_1-\xi_2-\ln\lambda_2)\right]\\
&-(\lambda_2^2-1)(\lambda_1^2-1)\lambda_1^{-2n+1}\lambda_2^{2n+1}\left[\cosh(\xi_1+\ln\lambda_1+\xi_2)+\cosh(\xi_1+\ln\lambda_1-\xi_2)\right]\},\\
H_2 &= \lambda_1^2\lambda_2^2\Big\{2(\lambda_1-\lambda_2)^4\cosh(\xi_1+\xi_2)\cosh(\xi_1+\xi_2+\ln(\lambda_1\lambda_2))+2(\lambda_1\lambda_2-1)^4\cosh(\xi_1-\xi_2)\cosh\left(\xi_1-\xi_2+\ln\frac{\lambda_1}{\lambda_2}\right)\\
&-2(1-\lambda_1^2)(1-\lambda_2^2)\cosh\left(2n\ln\frac{\lambda_1}{\lambda_2}\right)\left[(\lambda_1-\lambda_2)^2\cosh(\xi_1+\xi_2+\ln(\lambda_1\lambda_2))+(\lambda_1\lambda_2-1)^2\cosh\left(\xi_1-\xi_2+\ln\frac{\lambda_1}{\lambda_2}\right)\right]\\
&+2(\lambda_1-\lambda_2)^2(\lambda_1\lambda_2-1)^2\left[\cosh(\xi_1+\xi_2)\cosh\left(\xi_1-\xi_2+\ln\frac{\lambda_1}{\lambda_2}\right)+\cosh(\xi_1-\xi_2)\cosh\left(\xi_1+\xi_2+\ln(\lambda_1\lambda_2)\right)\right]\\
&-2(1-\lambda_1^2)(1-\lambda_2^2)\cosh\left(2n\ln\frac{\lambda_1}{\lambda_2}\right)\left[(\lambda_1-\lambda_2)^2\cosh(\xi_1+\xi_2)+(\lambda_1\lambda_2-1)^2\cosh(\xi_1-\xi_2)\right]\\
&+2(1-\lambda_1^2)^2(1-\lambda_2^2)^2\cosh^2\left(2n\ln\frac{\lambda_1}{\lambda_2}\right)\Big\}.
\end{aligned}
$$

Next, to investigate the structures of solutions (22) and potential terms (23), we perform the asymptotic analysis of $\widetilde{I}_n$, $\widetilde{V}_n$, $\widetilde{I}_n\widetilde{I}_{-n}$, $\widetilde{V}_n\widetilde{V}_{-n}$ and $\widetilde{I}_n\widetilde{V}_n$ at $t\to\mp\infty$. In the following, we discuss the four cases with regard to the different values of λ_1 and λ_2, which leads to the following different asymptotic state expressions.

Case (i) if $0<\lambda_1<1$, $\lambda_2>1$, $\lambda_1\lambda_2<1$, as $t\to\mp\infty$

$$
\begin{aligned}
&I_1^{\mp}=\frac{1-\lambda_1^2}{2\lambda_1^{2n+1}}\operatorname{sech}\left(\xi_1\mp\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_1\sim 0,\ \ \xi_2\to\mp\infty),\\
&I_2^{\mp}=\frac{\lambda_2^2-1}{2\lambda_2^{2n+1}}\operatorname{sech}\left(\xi_2\pm\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim 0,\ \ \xi_1\to\pm\infty),\\
&V_1^{\mp}=\frac{1}{2}(\lambda_1^2-1)\lambda_1^{2n-1}\operatorname{sech}\left(\xi_1+\ln\lambda_1\mp\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_1\sim 0,\ \ \xi_2\to\mp\infty),\\
&V_2^{-}=\frac{1}{2}(1-\lambda_2^2)\lambda_2^{2n-1}\operatorname{sech}\left(\xi_2+\ln\lambda_2\pm\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim 0,\ \ \xi_1\to\pm\infty),\\
&II_{-n1}^{\mp}=\frac{(\lambda_1^2-1)^2}{4\lambda_1^2}\operatorname{sech}^2\left(\xi_1\mp\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_1\sim 0,\ \ \xi_2\to\mp\infty),\\
&II_{-n2}^{\mp}=\frac{(\lambda_2^2-1)^2}{4\lambda_2^2}\operatorname{sech}^2\left(\xi_2\pm\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim 0,\ \ \xi_1\to\pm\infty),\\
&VV_{-n1}^{\mp}=\frac{(\lambda_1^2-1)^2}{4\lambda_1^2}\operatorname{sech}^2\left(\xi_1+\ln\lambda_1\mp\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_1\sim 0,\ \ \xi_2\to\mp\infty),\\
&VV_{-n2}^{\mp}=\frac{(\lambda_2^2-1)^2}{4\lambda_2^2}\operatorname{sech}^2\left(\xi_2+\ln\lambda_2\pm\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim 0,\ \ \xi_1\to\pm\infty),
\end{aligned}
$$

$$IV_1^{\mp}=\frac{-(\lambda_1^2-1)^2}{2\lambda_1^2}\frac{1}{\cosh\left[2\xi_1+\ln\lambda_1\mp\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1\lambda_2-1)^2}\right]+\cosh(\ln\lambda_1)\pm\sinh(\ln\lambda_1)},\quad(\xi_1\sim0,\ \xi_2\to\mp\infty),$$
$$IV_2^{\mp}=\frac{-(\lambda_2^2-1)^2}{2\lambda_2^2}\frac{1}{\cosh\left[2\xi_2+\ln\lambda_2\pm\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1\lambda_2-1)^2}\right]+\cosh(\ln\lambda_2)\mp\sinh(\ln\lambda_2)},\quad(\xi_2\sim0,\ \xi_1\to\pm\infty).\tag{24}$$

Case (ii) if $0<\lambda_1<1$, $\lambda_2>1$, $\lambda_1\lambda_2>1$, as $t\to\mp\infty$

$$I_1^{\mp}=\frac{\lambda_1^2-1}{2\lambda_1^{2n+1}}\mathrm{sech}\left(\xi_1\mp\ln\frac{\lambda_2-\lambda_1}{\lambda_2\lambda_1-1}\right),\quad(\xi_1\sim0,\ \xi_2\to\mp\infty),$$
$$I_2^{\mp}=\frac{1-\lambda_2^2}{2\lambda_2^{2n+1}}\mathrm{sech}\left(\xi_2\pm\ln\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim0,\ \xi_1\to\pm\infty),$$
$$V_1^{\mp}=\frac{1}{2}(1-\lambda_1^2)\lambda_1^{2n-1}\mathrm{sech}\left(\xi_1+\ln\lambda_1\mp\ln\frac{\lambda_2-\lambda_1}{\lambda_2\lambda_1-1}\right),\quad(\xi_1\sim0,\ \xi_2\to\mp\infty),$$
$$V_2^{\mp}=\frac{1}{2}(\lambda_2^2-1)\lambda_2^{2n-1}\mathrm{sech}\left(\xi_2+\ln\lambda_2\pm\ln\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim0,\ \xi_1\to\pm\infty),$$
$$II_{-n1}^{\mp}=\frac{(\lambda_1^2-1)^2}{4\lambda_1^2}\mathrm{sech}^2\left(\xi_1\mp\ln\frac{\lambda_2-\lambda_1}{\lambda_2\lambda_1-1}\right),\quad(\xi_1\sim0,\ \xi_2\to\mp\infty),$$
$$II_{-n2}^{\mp}=\frac{(\lambda_2^2-1)^2}{4\lambda_2^2}\mathrm{sech}^2\left(\xi_2\pm\ln\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim0,\ \xi_1\to\pm\infty),$$
$$VV_{-n1}^{\mp}=\frac{(\lambda_1^2-1)^2}{4\lambda_1^2}\mathrm{sech}^2\left(\xi_1+\ln\lambda_1\mp\ln\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2-1}\right),\quad(\xi_1\sim0,\ \xi_2\to\mp\infty),$$
$$VV_{-n2}^{\mp}=\frac{(\lambda_2^2-1)^2}{4\lambda_2^2}\mathrm{sech}^2\left(\xi_2+\ln\lambda_2\pm\ln\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim0,\ \xi_1\to\pm\infty),$$
$$IV_1^{\mp}=\frac{-(\lambda_1^2-1)^2}{2\lambda_1^2}\frac{1}{\cosh\left[2\xi_1+\ln\lambda_1\mp\ln\frac{(\lambda_1-\lambda_2-1)^2}{(\lambda_1\lambda_2-1)^2}\right]+\cosh(\ln\lambda_1)\pm\sinh(\ln\lambda_1)},\quad(\xi_1\sim0,\ \xi_2\to\mp\infty),$$
$$IV_2^{\mp}=\frac{-(\lambda_2^2-1)^2}{2\lambda_2^2}\frac{1}{\cosh\left[2\xi_2+\ln\lambda_2\pm\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1\lambda_2-1)^2}\right]+\cosh(\ln\lambda_2)\mp\sinh(\ln\lambda_2)},\quad(\xi_2\sim0,\ \xi_1\to\pm\infty).\tag{25}$$

Case (iii) if $\lambda_1,\lambda_2>1$, $\lambda_1>\lambda_2$, as $t\to\mp\infty$

$$I_1^{\mp}=\frac{1-\lambda_1^2}{2\lambda_1^{2n+1}}\mathrm{sech}\left(\xi_1\mp\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_1\sim0,\ \xi_2\to\mp\infty),$$
$$I_2^{\mp}=\frac{\lambda_2^2-1}{2\lambda_2^{2n+1}}\mathrm{sech}\left(\xi_2\mp\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim0,\ \xi_1\to\mp\infty),$$
$$V_1^{\mp}=\frac{1}{2}(\lambda_1^2-1)\lambda_1^{2n-1}\mathrm{sech}\left(\xi_1+\ln\lambda_1\mp\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_1\sim0,\ \xi_2\to\mp\infty),$$
$$V_2^{\mp}=\frac{1}{2}(1-\lambda_2^2)\lambda_2^{2n-1}\mathrm{sech}\left(\xi_2+\ln\lambda_2\mp\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim0,\ \xi_1\to\mp\infty),$$
$$II_{-n1}^{\mp}=\frac{(\lambda_1^2-1)^2}{4\lambda_1^2}\mathrm{sech}^2\left(\xi_1\mp\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_1\sim0,\ \xi_2\to\mp\infty),$$
$$II_{-n2}^{\mp}=\frac{(\lambda_2^2-1)^2}{4\lambda_2^2}\mathrm{sech}^2\left(\xi_2\mp\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim0,\ \xi_1\to\mp\infty),$$
$$VV_{-n1}^{\mp}=\frac{(\lambda_1^2-1)^2}{4\lambda_1^2}\mathrm{sech}^2\left(\xi_1+\ln\lambda_1\mp\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_1\sim0,\ \xi_2\to\mp\infty),$$
$$VV_{-n2}^{\mp}=\frac{(\lambda_2^2-1)^2}{4\lambda_2^2}\mathrm{sech}^2\left(\xi_2+\ln\lambda_2\mp\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim0,\ \xi_1\to\mp\infty),$$
$$IV_1^{\mp}=\frac{-(\lambda_1^2-1)^2}{2\lambda_1^2}\frac{1}{\cosh\left[2\xi_1+\ln\lambda_1\mp\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1\lambda_2-1)^2}\right]+\cosh(\ln\lambda_1)\pm\sinh(\ln\lambda_1)},\quad(\xi_1\sim0,\ \xi_2\to\mp\infty),$$

$$IV_2^{\mp}=\frac{-(\lambda_2^2-1)^2}{2\lambda_2^2}\frac{1}{\cosh\left[2\xi_2+\ln\lambda_2\mp\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1\lambda_2-1)^2}\right]+\cosh(\ln\lambda_2)\pm\sinh(\ln\lambda_2)},\quad(\xi_2\sim 0,\ \ \xi_1\to\mp\infty).\tag{26}$$

Case (iv) if $0<\lambda_1,\lambda_2<1$, $\lambda_1<\lambda_2$, as $t\to\mp\infty$

$$\begin{aligned}
&I_1^{\mp}=\frac{1-\lambda_1^2}{2\lambda_1^{2n+1}}\mathrm{sech}\left(\xi_1\pm\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_1\sim 0,\ \ \xi_2\to\pm\infty),\\
&I_2^{\mp}=\frac{\lambda_2^2-1}{2\lambda_2^{2n+1}}\mathrm{sech}\left(\xi_2\pm\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim 0,\ \ \xi_1\to\pm\infty),\\
&V_1^{\mp}=\frac{1}{2}(\lambda_1^2-1)\lambda_1^{2n-1}\mathrm{sech}\left(\xi_1+\ln\lambda_1\pm\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_1\sim 0,\ \ \xi_2\to\pm\infty),\\
&V_2^{\mp}=\frac{1}{2}(1-\lambda_2^2)\lambda_2^{2n-1}\mathrm{sech}\left(\xi_2+\ln\lambda_2\pm\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim 0,\ \ \xi_1\to\pm\infty),\\
&II_{-n1}^{\mp}=\frac{(\lambda_1^2-1)^2}{4\lambda_1^2}\mathrm{sech}^2\left(\xi_1\pm\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_1\sim 0,\ \ \xi_2\to\pm\infty),\\
&II_{-n2}^{\mp}=\frac{(\lambda_2^2-1)^2}{4\lambda_2^2}\mathrm{sech}^2\left(\xi_2\pm\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim 0,\ \ \xi_1\to\pm\infty),\\
&VV_{-n1}^{\mp}=\frac{(\lambda_1^2-1)^2}{4\lambda_1^2}\mathrm{sech}^2\left(\xi_1+\ln\lambda_1\pm\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_1\sim 0,\ \ \xi_2\to\pm\infty),\\
&VV_{-n2}^{\mp}=\frac{(\lambda_2^2-1)^2}{4\lambda_2^2}\mathrm{sech}^2\left(\xi_2+\ln\lambda_2\pm\ln\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2-1}\right),\quad(\xi_2\sim 0,\ \ \xi_1\to\pm\infty),\\
&IV_1^{\mp}=\frac{-(\lambda_1^2-1)^2}{2\lambda_1^2}\frac{1}{\cosh\left[2\xi_1+\ln\lambda_1\pm\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1\lambda_2-1)^2}\right]+\cosh(\ln\lambda_1)\mp\sinh(\ln\lambda_1)},\quad(\xi_1\sim 0,\ \ \xi_2\to\pm\infty),\\
&IV_2^{\mp}=\frac{-(\lambda_2^2-1)^2}{2\lambda_2^2}\frac{1}{\cosh\left[2\xi_2+\ln\lambda_2\pm\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1\lambda_2-1)^2}\right]+\cosh(\ln\lambda_2)\mp\sinh(\ln\lambda_2)},\quad(\xi_2\sim 0,\ \ \xi_1\to\pm\infty).
\end{aligned}\tag{27}$$

Here $I_1^-, I_2^-, V_1^-, V_2^-, II_{-n1}^-, II_{-n2}^-, VV_{-n1}^-, VV_{-n2}^-, IV_1^-, IV_2^-$ denote the asymptotic expressions of two-soliton solutions $\widetilde{I}_n$, $\widetilde{V}_n$, $\widetilde{I}_n\widetilde{I}_{-n}$, $\widetilde{V}_n\widetilde{V}_{-n}$ and $\widetilde{I}_n\widetilde{V}_n$ before the interactions, respectively, while I_1^+, I_2^+, V_1^+, V_2^+, II_{-n1}^+, II_{-n2}^+, VV_{-n1}^+, VV_{-n2}^+, IV_1^+, IV_2^+ represent the asymptotic expressions of two-soliton solutions after the interactions, respectively. From the above analysis, we can see that the shapes of two-soliton solutions $\widetilde{I}_n$ and $\widetilde{V}_n$ are related to the parameters λ_1,λ_2 which are summarized in Table 1.

Table 1. The relationship between parameters and shapes of two solitons.

Solitons	Parameters	Shapes
I_n	$\lambda_1>1, 0<\lambda_2<1$ and $\lambda_1>1/\lambda_2$	UABS-UABS
	$0<\lambda_k<1$ or $\lambda_k>1$ for $k=1,2$	UBS-UABS
	$\lambda_1>1, 0<\lambda_2<1$ and $\lambda_1<1/\lambda_2$	UBS-UBS
V_n	$\lambda_1>1, 0<\lambda_2<1$ and $\lambda_1>1/\lambda_2$	UBS-UBS
	$0<\lambda_k<1$ or $\lambda_k>1$ for $k=1,2$	UBS-UABS
	$\lambda_1>1, 0<\lambda_2<1$ and $\lambda_1<1/\lambda_2$	UABS-UABS

UBS: unstable Bell soliton, UABS: unstable anti-Bell soliton.

To illustrate the above-mentioned results, we plot two components $\widetilde{I}_n$ and $\widetilde{V}_n$ and three potential terms $\widetilde{I}_n\widetilde{I}_{-n}$, $\widetilde{V}_n\widetilde{V}_{-n}$, $\widetilde{I}_n\widetilde{V}_n$ as shown in Figs. 3 and 4, respectively. Figures 3(a) and 3(b) show the unstable two-soliton structures with parameters $\lambda_1=1/4$, $\lambda_2=5$, the absolute value of $\widetilde{I}_n$ increases when n varies from $-\infty$ to $+\infty$, while the absolute value of $\widetilde{V}_n$ decreases when n varies from $-\infty$ to $+\infty$. Figures 3(c) and 3(d) exhibit the unstable two-soliton structures with parameters $\lambda_1=4$, $\lambda_2=1/5$, the absolute value of $\widetilde{I}_n$ decays when n varies from $-\infty$ to $+\infty$, while the absolute value of $\widetilde{V}_n$ grows when n varies from $-\infty$ to $+\infty$. From Fig. 3, we can see that two unstable solitons are approaching to each other with time increasing. At a certain moment the interaction between two solitons takes place and energy exchange occurs. After the interaction, they begin to move away from each other. The amplitude of one soliton increases and the other decreases. Figure 4 gives the plots for Eq. (23) with parameters $\lambda_1=1/4$, $\lambda_2=5$. Figures 2(a) and 2(b) present the stable bell-shaped two-soliton structures for the potential terms $\widetilde{I}_n\widetilde{I}_{-n}$ and $\widetilde{V}_n\widetilde{V}_{-n}$. Figure 2(c) shows the stable anti-bell-shaped two-soliton structure for $\widetilde{I}_n\widetilde{V}_n$. From Fig. 4, we can see that the solitonic shapes and amplitudes have not changed before and after the interactions, their interactions are elastic. The possible reason for such stable structures is due to the balance of certain gain and loss of two components.

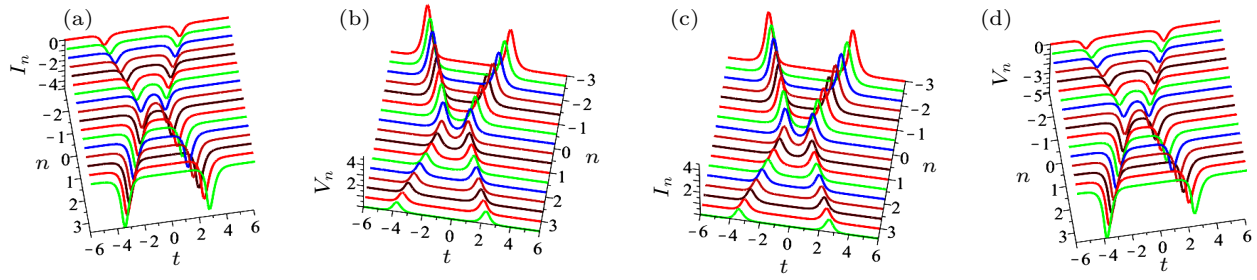


Fig. 3. [(a), (b)] Unstable two-soliton solutions for Eq. (3) with $\lambda_1 = 1/4$ and $\lambda_2 = 5$. [(c), (d)] Unstable two-soliton solutions for Eq. (3) with $\lambda_1 = 4$ and $\lambda_2 = 1/5$.

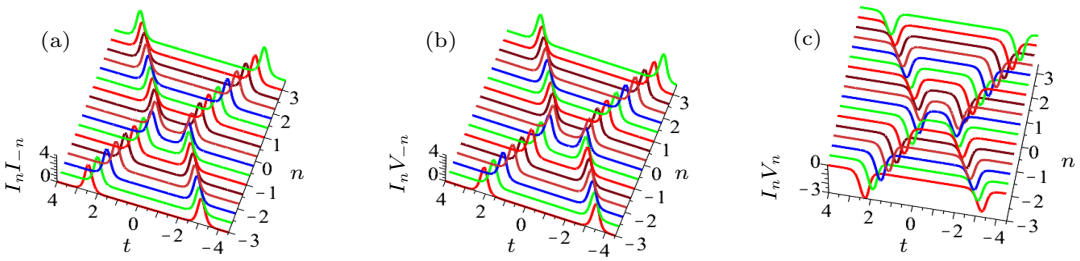


Fig. 4. Stable two-soliton solutions for potential terms (a) $\widetilde{I}_n\widetilde{I}_{-n}$, (b) $\widetilde{V}_n\widetilde{V}_{-n}$ and (c) $\widetilde{I}_n\widetilde{V}_n$ with $\lambda_1 = 1/4$ and $\lambda_2 = 5$.

3.1.3. Three-soliton solutions

When $N = 3$, from Theorem 1, we can derive the 3-fold exact solutions given by

$$\widetilde{I}_n = \frac{b_{-n}^{(2)}}{a_n^{(0)}}, \qquad \widetilde{V}_n = b_{n+1}^{(0)}, \tag{28}$$

where $a_n^{(0)}$, $b_{-n}^{(2)}$, $b_{n+1}^{(0)}$ are omitted to the same space here. Similar to two-soliton solutions' asymptotic analysis, we can also discuss their interactions phenomena, which are omitted to discuss here. If choosing the parameters $\lambda_1 = 1/3$, $\lambda_2 = 2$ and $\lambda_3 = 3/2$, when $n \to +\infty$, we can realize the decaying amplitude of three-soliton solution $|\widetilde{I}_n|$ and the growing amplitude of three-soliton solution $|\widetilde{V}_n|$ as shown Figs. 5(a) and 5(b). If taking the parameters $\lambda_1 = 1/3$, $\lambda_2 = 2, \lambda_3 = 3/2$, when $n \to +\infty$, we can obtain the increasing amplitude of three-soliton solution $|\widetilde{I}_n|$ and the decreasing amplitude of three-soliton solution $|\widetilde{V}_n|$ as shown Figs. 5(c) and 5(d). Figure 6 provides the plots for the potential terms $\widetilde{I}_n\widetilde{I}_{-n}$ and $\widetilde{V}_n\widetilde{V}_{-n}$ in Eq. (28). Figures 6(a) and 6(b) give the stable bell-shaped three-soliton structures for $\widetilde{I}_n\widetilde{I}_{-n}$, $\widetilde{V}_n\widetilde{V}_{-n}$. Figure 6(c) exhibits the stable anti-bell-shaped three-soliton structure for $\widetilde{I}_n\widetilde{V}_n$. Three solitons preserve their shapes and amplitudes before and after the interactions, so their interactions are elastic.

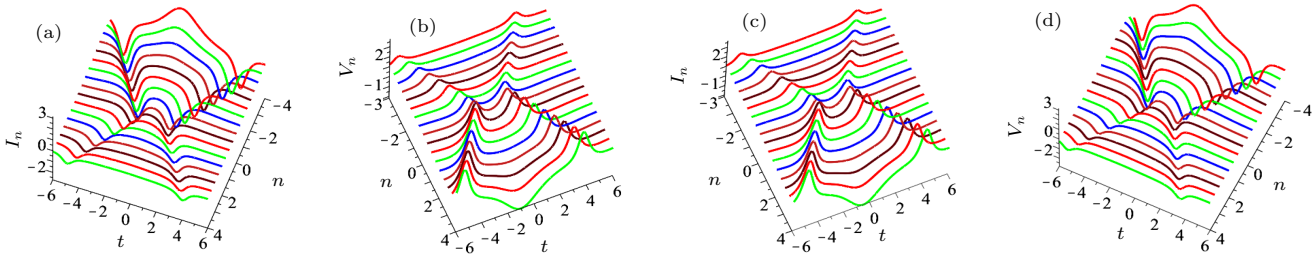


Fig. 5. [(a), (b)] Unstable three-soliton solutions for Eq. (3) with $\lambda_1 = 1/3$, $\lambda_2 = 2$ and $\lambda_3 = 3/2$. [(c), (d)] Unstable three-soliton solutions for Eq. (3) with $\lambda_1 = 3$, $\lambda_2 = 1/2$ and $\lambda_3 = 2/3$.

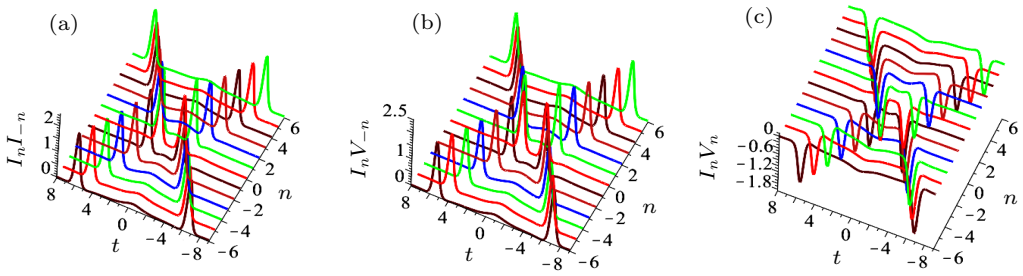


Fig. 6. Stable three-soliton solutions for potential terms (a) $\widetilde{I}_n\widetilde{I}_{-n}$, (b) $\widetilde{V}_n\widetilde{V}_{-n}$ and (c) $\widetilde{I}_n\widetilde{V}_n$ with $\lambda_1 = 1/3$, $\lambda_2 = 2$ and $\lambda_3 = 3/2$.

3.1.4. Four-soliton solutions

When $N=4$, from Theorem 1, the 4-fold exact solutions can be given by

$$\widetilde{I}_n=\frac{b^{(3)}_{-n}}{a^{(0)}_n},\quad \widetilde{V}_n=b^{(0)}_{n+1}, \tag{29}$$

where $a^{(0)}_n$, $b^{(3)}_{-n}$, $b^{(0)}_{n+1}$ are omitted to the same space here. When the parameters are chosen appropriately, the unstable four-soliton solutions $\widetilde{I}_n$ and $\widetilde{V}_n$ including two decaying solitons and two growing solitons are shown in Fig. 7. If choosing the parameters $\lambda_1=1/5$, $\lambda_2=4$, $\lambda_3=1/6$, $\lambda_4=7$, we can obtain the four-soliton structures as shown in Figs. 7(a) and 7(b). If taking the parameters $\lambda_1=5$, $\lambda_2=1/4$, $\lambda_3=6$, $\lambda_4=1/7$, we can realize the different four-soliton structures as shown in Figs. 7(c) and 7(d). Figure 8 provides the plots for the potential terms $\widetilde{I}_n\widetilde{I}_{-n}$ and $\widetilde{V}_n\widetilde{V}_{-n}$ in Eq. (29). Figures 8(a) and 8(b) present the stable bell-shaped bright four-soliton structures for $\widetilde{I}_n\widetilde{I}_{-n}$, $\widetilde{V}_n\widetilde{V}_{-n}$. Figure 8(c) exhibits the stable anti-bell-shaped four-soliton structure for $\widetilde{I}_n\widetilde{V}_n$. Four solitons preserve their shapes and amplitudes before and after the interactions due to balanced gain and loss so that their interactions are elastic.

With the aid of symbolic computation Maple, one-, two-, three- and four-soliton solutions obtained above have been verified by substituting them into Eq. (3).

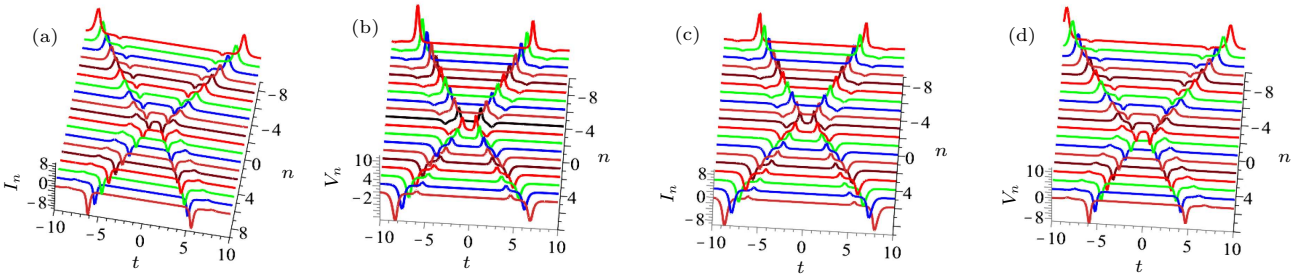


Fig. 7. (a) and (b) Unstable four-soliton solutions for Eq. (3) with $\lambda_1=1/5$, $\lambda_2=4$, $\lambda_3=1/6$ and $\lambda_4=7$; (c) and (d) Unstable four-soliton solutions for Eq. (3) with $\lambda_1=5$, $\lambda_2=1/4$, $\lambda_3=6$ and $\lambda_4=1/7$.

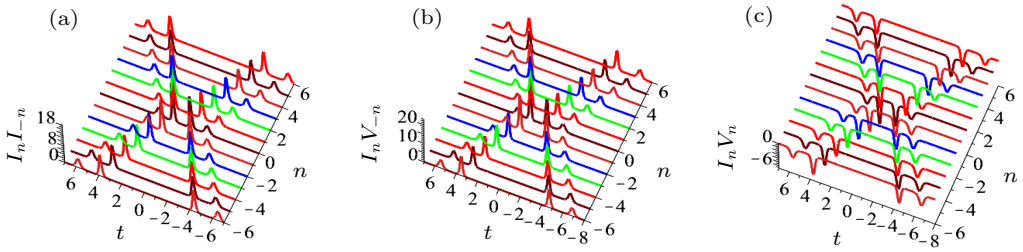


Fig. 8. (a), (b), (c) Stable four-soliton solutions for potential terms $\widetilde{I}_n\widetilde{I}_{-n}$, $\widetilde{V}_n\widetilde{V}_{-n}$ and $\widetilde{I}_n\widetilde{V}_n$ with $\lambda_1=1/5$, $\lambda_2=4$, $\lambda_3=1/6$ and $\lambda_4=7$.

3.2. N-rational solutions via the discrete nonlocal generalized $(1,N-1)$-fold DT

In this subsection, we shall derive rational solutions to Eq. (3) with $\sigma=-1$ via the discrete nonlocal generalized $(1,N-1)$-fold DT. For $\sigma=1$, we have the similar process which will not be discussed here. Substitution of nonzero seed solutions $I_n=a, V_n=0$ into Eqs. (4) and (5) yields the following basic solution:

$$\varphi_n(\lambda)=\begin{pmatrix}\phi_n\\ \psi_n\end{pmatrix}=\begin{pmatrix}C_1\tau_1^n\mathrm{e}^{\tau_1 t}+C_2\tau_2^n\mathrm{e}^{\tau_2 t}\\ C_1\frac{\lambda-\tau_1}{a}\tau_1^n\mathrm{e}^{\tau_1 t}+C_2\frac{\lambda-\tau_2}{a}\tau_2^n\mathrm{e}^{\tau_2 t}\end{pmatrix}, \tag{30}$$

in which $\tau_1=\tau_+$, $\tau_2=\tau_-$ with $\tau_\pm=\frac{1}{2\lambda}[\lambda^2+1\pm\sqrt{-(2a\lambda+\lambda^2-1)(2a\lambda-\lambda^2+1)}]$, where C_1 and C_2 are arbitrary constants.

Next, we fix the spectral parameter in Eq. (30) as $\lambda=\lambda_1+\varepsilon^2$ with $\lambda_1=a+\sqrt{a^2+1}$, and expand vector function φ_n into Taylor series around $\varepsilon=0$, explicitly calculating the coefficients of the expansion in φ_n. In particular, choosing $C_1=-C_2=1/\varepsilon$, $a=4/3$ (i.e., $\lambda_1=3$), we have

$$\begin{aligned}\varphi_n(\varepsilon)&=\sum_{j=0}^{\infty}\varphi_n^{(j)}\varepsilon^j\\&=\varphi_n^{(0)}+\varphi_n^{(1)}\varepsilon+\varphi_n^{(2)}\varepsilon^2+\varphi_n^{(3)}\varepsilon^3+\cdots.\end{aligned} \tag{31}$$

Here

$$\varphi_n^{(0)}=\begin{pmatrix}\phi_n^{(0)}\\ \psi_n^{(0)}\end{pmatrix}=\left(\frac{5}{3}\right)^n\frac{\sqrt{30}}{45}\mathrm{e}^{\frac{5}{3}t}\begin{pmatrix}4(5t+3n)\\ 4(5t+3n)-15\end{pmatrix},$$

$$\varphi_n^{(1)}=\begin{pmatrix}\phi_n^{(1)}\\ \psi_n^{(1)}\end{pmatrix}=\left(\frac{5}{3}\right)^n\frac{16\sqrt{30}}{675}\mathrm{e}^{\frac{5}{3}t}\begin{pmatrix}\frac{125}{27}t^3+(\frac{25}{3}n+\frac{25}{3})t^2+(5n^2+5n+\frac{105}{64})t+n^3-\frac{1}{64}n\\ n^3+(5t-\frac{15}{4})n^2+(\frac{25}{3})t^2-(\frac{15}{2}t+\frac{299}{64})n+\frac{125}{27}t^3-\frac{25}{12}t^2+\frac{205}{64}t-\frac{315}{256}\end{pmatrix},$$

where we omit the rest $\varphi^{(k)}(k \geq 2)$.

According to the discrete nonlocal generalized $(1, N-1)$-fold DT in Theorem 1, we shall discuss the two cases: $N = 1, 2$.

3.2.1. First-order rational solutions

For $N = 1$, according to Theorem 1, based on the discrete generalized $(1,0)$-fold DT, the first-order rational solutions to Eq. (3) are given as

$$\widetilde{I}_n = \frac{a + b_{-n}^{(0)}}{a_n^{(0)}}, \quad \widetilde{V}_n = b_{n+1}^{(0)}, \tag{32}$$

where $a_n^{(0)} = \frac{\Delta a_n^{(0)}}{\Delta_1^{\varepsilon}}$, $b_{-n}^{(0)} = \frac{\Delta b_{-n}^{(0)}}{\Delta_{1,-n}^{\varepsilon}}$ and $b_{n+1}^{(0)} = \frac{\Delta b_{n+1}^{(0)}}{\Delta_{1,n+1}^{\varepsilon}}$ with

$$\Delta a_n^{(0)} = \begin{vmatrix} -\lambda_1\phi_n^{(0)} & \psi_n^{(0)} \\ -\psi_{-n}^{(0)} & -\lambda_1\phi_{-n}^{(0)} \end{vmatrix}, \quad \Delta b_{-n}^{(0)} = \begin{vmatrix} \phi_{-n}^{(0)} & -\lambda_1\phi_{-n}^{(0)} \\ \lambda_1\psi_n^{(0)} & -\psi_n^{(0)} \end{vmatrix},$$

$$\Delta b_{n+1}^{(0)} = \begin{vmatrix} \phi_{1,n+1} & -\lambda_1\phi_{1,n+1} \\ \lambda_1\psi_{1,-n+1} & -\psi_{1,-n+1} \end{vmatrix},$$

$$\Delta_1^{\varepsilon} = \begin{vmatrix} \phi_n^{(0)} & \psi_n^{(0)} \\ \lambda_1\psi_{-n}^{(0)} & -\lambda_1\phi_{-n}^{(0)} \end{vmatrix}, \quad \Delta_{1,-n}^{\varepsilon} = \begin{vmatrix} \phi_{-n}^{(0)} & \psi_{-n}^{(0)} \\ \lambda_1\psi_n^{(0)} & -\lambda_1\phi_n^{(0)} \end{vmatrix},$$

$$\Delta_{1,n+1}^{\varepsilon} = \begin{vmatrix} \phi_{1,n+1}^{(0)} & \psi_{1,n+1}^{(0)} \\ \lambda_1\psi_{1,-n+1}^{(0)} & -\lambda_1\phi_{1,-n+1}^{(0)} \end{vmatrix},$$

while $b_{-n}^{(0)}$ is obtained from $b_n^{(0)}$ by replacing n with $-n$. Through the calculation, the solutions (32) can be rewritten as

$$\widetilde{I}_n = \frac{180 - 288n}{288n^2 - 800(t - \frac{3}{40})^2 - \frac{81}{2}},$$
$$\widetilde{V}_n = -\frac{4}{3} - \frac{5(288n + 180)}{864n^2 - 2400(t + \frac{9}{40})^2 - \frac{675}{2}}, \tag{33}$$

from which we can see that $\widetilde{I}_n$ possesses singularity at the hyperbola $\frac{64n^2}{9} - \frac{1600(t-\frac{3}{40})^2}{81} = 1$, while $\widetilde{V}_n$ possesses singularity at the hyperbola $\frac{64n^2}{25} - \frac{64(t+\frac{9}{40})^2}{9} = 1$. We can see that $\widetilde{I}_n \to 0$, $\widetilde{V}_n \to -4/3$ for $n \to \pm\infty$ or $t \to \pm\infty$.

3.2.2. Second-order rational solutions

When $N = 2$, according to Theorem 1, based on the discrete generalized $(1,1)$-fold DT, the second-order rational solutions to Eq. (3) are given by

$$\widetilde{I}_n = \frac{a + b_{-n}^{(1)}}{a_n^{(0)}}, \quad \widetilde{V}_n = b_{n+1}^{(0)}, \tag{34}$$

where $a_n^{(0)} = \frac{\Delta a_n^{(0)}}{\Delta_2^{\varepsilon}}$, $b_{-n}^{(1)} = \frac{\Delta b_{-n}^{(1)}}{\Delta_{2,-n}^{\varepsilon}}$ and $b_{n+1}^{(0)} = \frac{\Delta b_{n+1}^{(0)}}{\Delta_{2,n+1}^{\varepsilon}}$ with

$$\Delta_2^{\varepsilon} = \begin{vmatrix} \lambda_1\phi_n^{(0)} & \phi_n^{(0)} & \lambda_1\psi_n^{(0)} & \psi_n^{(0)} \\ \lambda_1\phi_n^{(1)} + \phi_n^{(0)} & \phi_n^{(1)} & \lambda_1\psi_n^{(1)} + \psi_n^{(0)} & \psi_n^{(1)} \\ \lambda_1\psi_{-n}^{(0)} & \lambda_1^2\psi_{-n}^{(0)} & -\lambda_1\phi_{-n}^{(0)} & -\lambda_1^2\phi_{-n}^{(0)} \\ \lambda_1\psi_{-n}^{(1)} + \psi_{-n}^{(0)} & \lambda_1^2\psi_{-n}^{(1)} + 2\lambda_1\psi_{-n}^{(0)} & -\lambda_1\phi_{-n}^{(1)} - \phi_{-n}^{(0)} & -\lambda_1^2\phi_{-n}^{(1)} - 2\lambda_1\phi_{-n}^{(0)} \end{vmatrix},$$

$$\Delta_{2,-n}^{\varepsilon} = \begin{vmatrix} \lambda_1\phi_{-n}^{(0)} & \phi_{-n}^{(0)} & \lambda_1\psi_{-n}^{(0)} & \psi_{-n}^{(0)} \\ \lambda_1\phi_{-n}^{(1)} + \phi_{-n}^{(0)} & \phi_{-n}^{(1)} & \lambda_1\psi_{-n}^{(1)} + \psi_{-n}^{(0)} & \psi_{-n}^{(1)} \\ \lambda_1\psi_n^{(0)} & \lambda_1^2\psi_n^{(0)} & -\lambda_1\phi_n^{(0)} & -\lambda_1^2\phi_n^{(0)} \\ \lambda_1\psi_n^{(1)} + \psi_n^{(0)} & \lambda_1^2\psi_n^{(1)} + 2\lambda_1\psi_n^{(0)} & -\lambda_1\phi_n^{(1)} - \phi_n^{(0)} & -\lambda_1^2\phi_n^{(1)} - 2\lambda_1\phi_n^{(0)} \end{vmatrix},$$

$$\Delta_{2,n+1}^{\varepsilon} = \begin{vmatrix} \lambda_1\phi_{n+1}^{(0)} & \phi_{n+1}^{(0)} & \lambda_1\psi_{n+1}^{(0)} & \psi_{n+1}^{(0)} \\ \lambda_1\phi_{n+1}^{(1)} + \phi_{n+1}^{(0)} & \phi_{n+1}^{(1)} & \lambda_1\psi_{n+1}^{(1)} + \psi_{n+1}^{(0)} & \psi_{n+1}^{(1)} \\ \lambda_1\psi_{-n+1}^{(0)} & \lambda_1^2\psi_{-n+1}^{(0)} & -\lambda_1\phi_{-n+1}^{(0)} & -\lambda_1^2\phi_{-n+1}^{(0)} \\ \lambda_1\psi_{-n+1}^{(1)} + \psi_{-n+1}^{(0)} & \lambda_1^2\psi_{-n+1}^{(1)} + 2\lambda_1\psi_{-n+1}^{(0)} & -\lambda_1\phi_{-n+1}^{(1)} - \phi_{-n+1}^{(0)} & -\lambda_1^2\phi_{-n+1}^{(1)} - 2\lambda_1\phi_{-n+1}^{(0)} \end{vmatrix},$$

where $\Delta a_n^{(0)}$ is given by the determinant Δ_2^{ε} replacing its second column by the column vector $(-\lambda_1^2\phi_n^{(0)}, -\lambda_1^2\phi_n^{(1)} - 2\lambda_1\phi_n^{(0)}, -\psi_{-n}^{(0)}, -\psi_{-n}^{(1)})^T$, $\Delta b_{-n}^{(1)}$ is given by the determinant $\Delta_{2,-n}^{\varepsilon}$ replacing its third column by the column vector $(-\lambda_1^2\phi_{-n}^{(0)}, -\lambda_1^2\phi_{-n}^{(1)} - 2\lambda_1\phi_{-n}^{(0)}, -\psi_n^{(0)}, -\psi_n^{(1)})^{\mathrm{T}}$, and $\Delta b_n^{(n+1)}$ is given by the determinant $\Delta_{2,n+1}^{\varepsilon}$ replacing its fourth column by the column vector $(-\lambda_1^2\phi_{n+1}^{(0)}, -\lambda_1^2\phi_{n+1}^{(1)} - 2\lambda_1\phi_{n+1}^{(0)}, -\psi_{-n+1}^{(0)}, -\psi_{-n+1}^{(1)})^{\mathrm{T}}$. By the calculation, the solutions (34) can be rewritten in the following form:

$$\widetilde{I}_n = \frac{4}{3} + \frac{F_1}{G_1}, \quad \widetilde{V}_n = \frac{F_2}{G_2}, \tag{35}$$

where

$$\begin{aligned} F_1 = {} & (1152000000 - 1843200000n)t^4 + (552960000n - 345600000)t^3 + (1327104000n^3 - 2488320000n^2 \\ & - 393984000n + 1010880000)t^2 + (-199065600n^3 + 373248000n^2 - 136857600n - 29160000)t \\ & - 238878720n^5 + 746496000n^4 - 246343680n^3 - 214617600n^2 + 138801600n - 43740000, \end{aligned}$$

$$
\begin{aligned}
G_1 &= -1024000000t^6+460800000t^5+(1105920000n^2+34560000)t^4+(-331776000n^2+77760000)t^3\\
&\quad +(-398131200n^4+1368576000n^2-641520000)t^2+(59719680n^4-87713280n^2+21870000)t\\
&\quad +47775744n^6-306063360n^4+142814016n^2-4100625,\\
F_2 &= (-1105920000n-691200000)t^4+(-995328000n-622080000)t^3+(796262400n^3+1492992000n^2\\
&\quad -534988800n-793152000)t^2+(358318080n^3+671846400n^2-134369280n-290433600)t\\
&\quad -(143327232n^4+304570368n^3-49268736n^2-146686464n-10497600)(n+1),\\
G_2 &= -1024000000t^6-1382400000t^5+(1105920000n^2-380160000)t^4+(995328000n^2+119232000)t^3\\
&\quad +(-398131200n^4+1069977600n^2-235612800)t^2+(-179159040n^4+375114240n^2\\
&\quad -125096400)t+47775744n^6-144820224n^4+107542080n^2-23783625.
\end{aligned}
$$

From Eq. (35) we can also see that the second-order rational solutions $\widetilde{I}_n$ and $\widetilde{V}_n$ possess singularity. Similarly, we can also obtain third- or fourth-order rational solutions whose simplification form expressions are so complicated to be omitted here. With symbolic computation Maple, rational solutions (32) or (33), (34) or (35) have been verified by substituting them into Eq. (3).

Finally, we summarize a few mathematical features of previous discrete rational solutions for Eq. (3) as listed in Table 2. The first column in Table 2 shows the orders of the solutions, while the second, third, fifth and sixth columns exhibit the powers of the polynomials involved in each pair of solutions. The fourth and last columns give the background levels of the solutions. From Table 2, we can easily obtain the following information: provided that the order N of the rational solutions is odd, the highest powers in the numerator polynomial of the solutions $\widetilde{I}_n$ and $\widetilde{V}_n$ are $2n(2n-1)-1$ and $2n(2n-1)$, respectively, while the their highest powers in the denominator polynomial both are $2n(2n-1)$, and the background levels of the solutions $\widetilde{I}_n$ and $\widetilde{V}_n$ are zero and $-a$, respectively. On the other hand, supposing that the order N is even, the highest powers in the numerator polynomial of the solutions $\widetilde{I}_n$ and $\widetilde{V}_n$ are $2n(2n+1)$ and $2n(2n+1)-1$, respectively, while the highest powers in the denominator polynomial both are $2n(2n+1)$, and the background levels of the solutions $\widetilde{I}_n$ and $\widetilde{V}_n$ are a and zero, respectively.

Table 2. Main mathematical features of rational solutions $\widetilde{I}_n$ and $\widetilde{V}_n$ of order N.

N	HPN of $\widetilde{I}_n$	HPD of $\widetilde{I}_n$	Background of $\widetilde{I}_n$	HPN of $\widetilde{V}_n$	HPD of $\widetilde{V}_n$	Background of $\widetilde{V}_n$
1	1	2	0	2	2	$-a$
2	6	6	a	5	6	0
3	11	12	0	12	12	-a
4	20	20	a	19	20	0
...	...	...	...	...	...	...
$2n-1$	$2n(2n-1)-1$	$2n(2n-1)$	0	$2n(2n-1)$	$2n(2n-1)$	$-a$
$2n$	$2n(2n+1)$	$2n(2n+1)$	a	$2n(2n+1)-1$	$2n(2n+1)$	0

HPN and HPD denote the highest power in the numerator and denominator, respectively.

4. Conclusion

In summary, we have analytically investigated a reverse-space nonlinear self-dual network equation (3) with $\sigma=-1$, which may describe the propagation of electrical signals in a ladder-type nonlinear self-dual network. We summarize the main achievements of this paper as follows: (1) An infinite number of nonlocal conservation laws for Eq. (3) have been constructed. (2) The discrete nonlocal version of generalized $(m,N-m)$-fold DT is extended to nonlocal Eq. (3) in detail for the first time, which is a generalization of the usual discrete N-fold DT. (3) The asymptotic state analysis for two-soliton solutions are investigated, and the related properties of wave propagation are shown and discussed in detail. (4) It is found that individual components in this nonlocal equation (3) possess unstable soliton structures while the potential terms have the stable soliton structures, which also shows that the energy of this equation is conserved from another aspect. (5) Mathematical features of different-order rational solutions of Eq. (3) are summarized, from which we can see that the nonzero backgrounds for rational solutions appear alternately between two components in this system.

In theory, it is entirely possible to expand eigenfunctions at more spectral parameters to get more new solutions to Eq. (3) via the discrete generalized $(m,N-m)$-fold DT. However, in fact, the calculations are very complicated, we need further investigation in this field. We think that the key idea of this technique can be applied to other discrete local and nonlocal nonlinear integrable models as well. Finally, we need to point out that Eqs. (3) and (6) provide the possibility for designing nonlocal electrical circuits, and the results of this

paper also provide a theoretical basis for enhancing and weakening the propagation of electrical signals in electrical circuits. We hope that the results obtained in this paper may be useful for explaining the electrical signals' propagation phenomena in practical applications.

Acknowledgment

We would like to express our sincere thanks to other members of our discussion group for their valuable comments.

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