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Rational solutions and interaction solutions for (2 + 1)-dimensional nonlocal Schrödinger equation*

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A chain of novel higher order rational solutions with some parameters and interaction solutions of a (2+1)-dimensional reverse space–time nonlocal Schrödinger (NLS) equation was derived by a generalized Darboux transformation (DT) which is derived by Taylor expansion and determinants. We obtained a series of higher-order rational solutions by one spectral parameter and we could get the periodic wave solution and three kinds of interaction solutions , singular breather and periodic wave interaction solution, singular breather and traveling wave interaction solution, bimodal breather and periodic wave interaction solution by two spectral parameters. We found a general formula for these solutions in the form of determinants. We also analyzed the complex wave structures of the dynamic behaviors and the effects of special parameters and presented exact solutions for the (2+1)-dimensional reverse space–time nonlocal NLS equation.

Keywords: Darboux transformation, nonlocal Schrödinger equation, rational solutions, interaction solutions

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1. Introduction

The nonlinear Schrödinger (NLS) equation is an important nonlinear integrable model. It appears in many nonlinear fields, such as Bose–Einstein condensation, DNA, finance, oceanography, etc. With the progress of science and technology, there are more and more methods to solve the NLS equation, such as bilinear method,[1–3] approximate symmetry reduction,[4,5] inverse scattering transform,[6,7] Painlevé analysis,[8,9] Darboux transform (DT),[10–14] etc.

The Darboux transformation is particularly important for the development of NLS equations. The DT is modified to obtain the higher-order rogue wave (RW) solutions for the focusing NLS equation,[15] but it cannot give an exact formula of multi-RW solutions. Recently, the Matveev's generalized DT method[16] was developed to construct the higher-order RW solutions of the NLS and KN equations.[12,13] Li and Xu studied the interactions of dark and antidark solitons in the nonlocal NLS equation with the self-induced $\mathcal{PT}$-symmetric potential.[17] Wen studied higher-order rational solitons of the (1+1)-dimensional nonlocal NLS equation with a $\mathcal{PT}$-symmetric dipole.[18] Yan has done a lot of work on the NLS equation with the self-induced $\mathcal{PT}$-symmetric potential.[19–21] Zhu and Zuo decomposed the (2+1)-dimensional coupled fractured soliton type system under its reductions and obtained many nonlocal integrable systems[22] and we studied the reverse space–time NLS of Ref. [22] by using the generalized Darboux transformations.

The (2+1)-dimensional NLS equation, which is also called breaking soliton-type (BS-type) equation,[23,24] is given as follows:

$$\begin{aligned} &\mathrm{i}q_t + q_{xy} - 2q\partial_x^{-1}(qr)_y = 0,\\ &\mathrm{i}r_t - r_{xy} + 2r\partial_x^{-1}(qr)_y = 0, \end{aligned} \tag{1}$$

where q and r are respect to x, y, and t, ∂_x^{-1} represents the inverse operator of $\partial_x = \partial/\partial_x$, which can be defined as

$$(\partial_x^{-1} f)(x) = \frac{1}{2}\int_{-\infty}^{x} f(v)\,\mathrm{d}v - \frac{1}{2}\int_{x}^{+\infty} f(v)\,\mathrm{d}v.$$

If (q,r) satisfies the coupled NLS system

$$q_y - \mathrm{i}\beta q_{xx} + 2\mathrm{i}q^2 r = 0, \quad r_y + \mathrm{i}\beta r_{xx} - 2\mathrm{i}\beta q r^2 = 0, \tag{2}$$

and the coupled mKdV system

$$q_t - 6\beta q q_x r + \beta q_{xxx} = 0, \quad r_t - 6\beta q r r_x + \beta r_{xxx} = 0, \tag{3}$$

where $\beta = \pm 1$, then (q,r) satisfies the system (1).

We choose one of the decomposition cases with the reduction $r = \sigma q^*(-x,y,-t)$, the systems (1)–(3) are new reverse space–time nonlocal systems and $r = \sigma q^*(-x,y,-t)$ is a special form of the two kinds of reductions in Ref. [22]. Here $\sigma = \pm 1$, the subscript stands for the partial derivative with respect to the variables, and the star denotes the complex conjugation.

The soliton solutions of equation

$$\mathrm{i}q_t(x,t) - \frac{1}{2}q_{xx}(x,t) - \sigma q^2(x,t)q^*(-x,t) = 0, \quad (\sigma = \pm 1) \tag{4}$$

are studied in Ref. [17], and the higher-order rational solutions are studied in Ref. [18]. But the dynamics behaviors of the higher-order rational solutions for the (2+1)-dimensional reverse space–time nonlocal NLS equation were not considered before.

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Under the reduction $r=\sigma q^*(-x,y,-t)$, the system (1) reduces to

$$\begin{aligned}&\mathrm{i}q_t(x,y,t)+q_{xy}(x,y,t)\\&-2\sigma q(x,y,t)\partial_x^{-1}(q(x,y,t)q^*(-x,y,-t))_y=0,\end{aligned}\tag{5}$$

the system (2) reduces to

$$\mathrm{i}q_y(x,y,t)+\beta q_{xx}(x,y,t)-2\sigma q^2(x,y,t)q^*(-x,y,-t)=0,\tag{6}$$

and the system (3) reduces to

$$\begin{aligned}&q_t(x,y,t)-6\beta q(x,y,t)q_x(x,y,t)q^*(-x,y,-t)\\&+\beta q_{xxx}(x,y,t)=0.\end{aligned}\tag{7}$$

We will get the higher-order solutions of Eq. (5) by study solutions to Eqs. (6) and (7). Equations (5)–(7) can be regarded as integrable extension models of systems (1)–(3), respectively.

In the many higher-order solutions of the nonlocal NLS equation, the solutions of the following two cases are also the solutions of the local NLS equation.

(i) If $q^*(-x,y,-t)=q(x,y,t)$, then the solutions of Eqs. (6) and (7) are even functions for x and t, then the solutions must be ones of the systems (2) and (3) with $r=q^*(x,y,t)$.

(ii) If $q^*(-x,y,-t)=-q(x,y,t)$, then equation (5) equals to system (1) with $r=q^*(x,y,t)$, $\sigma=-\sigma$. That is, if the solutions of system (1) with $r=q^*(x,y,t)$, $\sigma=-\sigma$ are odd functions for x or t, then the solutions must be ones of Eq. (5). However, the higher-order rational solutions with respect to x or t are neither even nor odd functions and dynamical behaviors of Eq. (5) were not explored before.

In this paper, we will try to come up with a new generalized perturbation N-fold Darboux transformation to obtain the higher-order rational solutions of Eq. (5). We use the Taylor series expansion for the parameter and a limit procedure presenting the nonlocal version of the local (m,N)-fold Darboux transformation to get the higher-order rational solutions and interaction solutions of Eq. (5). On the one hand, we structure a new Darboux matrix and obtain the new solutions in terms of determinants without complicated iterations and the relationship between the seed solution and the new solutions. On the other hand, we will discuss the dynamic behavior of the soliton solutions for this nonlocal equation.

The main content of this article is as follows. In Section 2, for the (2+1)-dimensional reverse space–time nonlocal NLS equation, an generalized perturbation $(1,N-1)$-fold Darboux transformation and (n,M)-fold Darboux transformation are constructed. In Section 3, the higher-order rational solutions with some free parameters of the (2+1)-dimensional reverse space–time nonlocal NLS equation are obtained and the general-purpose formulas of these solutions are given in the form of determinant. We also discuss the important effects of the different values of these free parameters on the solutions. Then, we obtain some interaction solutions of the (2+1)-dimensional reverse space–time nonlocal NLS equation and the dynamic behavior of these solutions is analyzed. In the last section, we give some conclusions and discussion.

2. The generalized Darboux transformation of the (2+1)-dimensional nonlocal NLS equation

2.1. Lax pair and generalized perturbation $(1,N-1)$-fold Darboux transformation

The Lax pair (i.e., linear spectral problems) of Eqs. (6) and (7) is

$$\Phi_x=M\Phi,\quad \Phi_y=N\Phi,\quad \Phi_t=W\Phi,\tag{8}$$

with

$$\begin{aligned}&M=\begin{pmatrix}-\mathrm{i}\lambda & q\\ \sigma\bar{q} & \mathrm{i}\lambda\end{pmatrix},\quad N=\beta\begin{pmatrix}-2\mathrm{i}\lambda^2-\mathrm{i}q\sigma\bar{q} & 2\lambda q+\mathrm{i}q_x\\ 2\lambda\sigma\bar{q}-\mathrm{i}\sigma\bar{q}_x & 2\mathrm{i}\lambda^2+\mathrm{i}q\sigma\bar{q},\end{pmatrix},\quad W=\beta U,\\&U=\begin{pmatrix}-4\mathrm{i}\lambda^3-2\mathrm{i}\lambda q\sigma\bar{q}-(q\sigma\bar{q}_x-\sigma\bar{q}q_x) & 4\lambda^2q+2\mathrm{i}\lambda q_x+2q^2\sigma\bar{q}-q_{xx}\\ 4\lambda^2\sigma\bar{q}-2\mathrm{i}\lambda\sigma\bar{q}_x+2q\sigma\bar{q}^2-\sigma\bar{q}_{xx} & 4\mathrm{i}\lambda^3+2\mathrm{i}\lambda q\sigma\bar{q}+(q\sigma\bar{q}_x-\sigma\bar{q}q_x)\end{pmatrix},\end{aligned}$$

where $q=q(x,y,t)$, $\bar{q}=q^*(-x,y,-t)$, the vector $\Phi=(\phi,\varphi)^{\mathrm{T}}$ is an eigenfunction associated with spectral parameter λ, and $\mathrm{i}^2=-1$. It is easy to verify that equations (6) and (7) can be generated by two zero curvature equations

$$M_y-N_x+[M,N]=0,\quad M_t-W_x+[M,W]=0.$$

We introduce the gauge transformation with the Darboux matrix $T(\lambda)$ in the form of

$$T(\lambda)=\begin{pmatrix}(-\mathrm{i}\lambda)^N+\sum\limits_{i=0}^{N-1}A_i(x,y,t)(-\mathrm{i}\lambda)^i & \sum\limits_{i=0}^{N-1}B_i(x,y,t)(-\mathrm{i}\lambda)^i\\ -\sigma\sum\limits_{i=0}^{N-1}B_i^*(-x,y,-t)(-\mathrm{i}\lambda)^i & (-\mathrm{i}\lambda)^N+\sum\limits_{i=0}^{N-1}A_i^*(-x,y,-t)(-\mathrm{i}\lambda)^i\end{pmatrix},\tag{9}$$

and it satisfies

$$\widetilde{\Phi} = T(\lambda)\Phi. \tag{10}$$

Here $\widetilde{\Phi} = (\widetilde{\phi}, \widetilde{\varphi})^{\mathrm{T}}$, and $\widetilde{\Phi}$ satisfies the Lax pair (8), where $q(x,y,t)$, $q^*(-x,y,-t)$ are corresponding to $\widetilde{q}(x,y,t)$, $\widetilde{q^*}(-x,y,-t)$, respectively,

$$\widetilde{\Phi}_x = \widetilde{M}\widetilde{\Phi}, \quad \widetilde{\Phi}_y = \widetilde{N}\widetilde{\Phi}, \quad \widetilde{\Phi}_t = \widetilde{W}\widetilde{\Phi}. \tag{11}$$

Here T, M, $\widetilde{M}$, N, $\widetilde{N}$, W, and $\widetilde{W}$ satisfy

$$T_x + [T, \{M, \widetilde{M}\}] = 0, \quad T_y + [T, \{N, \widetilde{N}\}] = 0,$$

$$T_t + [T, \{W, \widetilde{W}\}] = 0, \tag{12}$$

where the generalized bracket for the square matrix is defined by $[F, \{G, \widetilde{G}\}] = FG - \widetilde{G}F$, and we can obtain

$$\widetilde{M}_y - \widetilde{N}_x + [\widetilde{M}, \widetilde{N}] = T(M_y - N_x + [M, N])T^{-1}, \tag{13}$$

$$\widetilde{M}_t - \widetilde{W}_x + [\widetilde{M}, \widetilde{W}] = T(M_t - W_x + [M, W])T^{-1}. \tag{14}$$

The usual N-order Darboux matrix $T(\lambda_j)$ is supposed in the form of

$$T(\lambda_j) = \begin{pmatrix} (-\mathrm{i}\lambda_j)^N + \sum\limits_{i=0}^{N-1} A_i(x,y,t)(-\mathrm{i}\lambda_j)^i & \sum\limits_{i=0}^{N-1} B_i(x,y,t)(-\mathrm{i}\lambda_j)^i \\ -\sigma \sum\limits_{i=0}^{N-1} B_i^*(-x,y,-t)(-\mathrm{i}\lambda_j)^i & (-\mathrm{i}\lambda_j)^N + \sum\limits_{i=0}^{N-1} A_i^*(-x,y,-t)(-\mathrm{i}\lambda_j)^i \end{pmatrix}, \tag{15}$$

where N is a positive integer, A_i, B_i $(0 \leqslant i \leqslant N-1)$ are $2N$ undetermined complex functions and satisfy the following $2N$ equations:

$$\begin{aligned} &((-\mathrm{i}\lambda_j)^N + \sum_{i=0}^{N-1} A_i(-\mathrm{i}\lambda_j)^i)\phi(\lambda_j) \\ &+ \sum_{i=0}^{N-1} B_i(-\mathrm{i}\lambda_j)^i\varphi(\lambda_j) = 0, \\ &((-\mathrm{i}\lambda_j)^N + \sum_{i=0}^{N-1} A_i^*(-\mathrm{i}\lambda_j)^i)\varphi(\lambda_j) \\ &-\sigma \sum_{i=0}^{N-1} B_i^*(-\mathrm{i}\lambda_j)^i\phi(\lambda_j) = 0, \end{aligned} \tag{16}$$

where $\Phi(\lambda_j) = (\phi(\lambda_j),\ \varphi(\lambda_j))^{\mathrm{T}}$, $j = 1,2,\ldots,N$ are the solutions of Lax pair (8) for the spectral parameters λ_j and the initial solution ψ_0. There are $2N$ distinct parameters λ_j, $\varphi(\lambda_j)$ $(\lambda_j \neq \lambda_i, j \neq i, \lambda_j \neq 0, j = 1,2,\ldots,N)$ that satisfy Eq. (15), and they are the roots of

$$\det(T(\lambda)) = 0. \tag{17}$$

It is easy to know that when the spectral parameter $\lambda = \lambda_1$ of system (9), the condition $T(\lambda_1)\Phi(\lambda_1) = 0$ leads to the linear algebraic system

$$\begin{aligned} &((-\mathrm{i}\lambda_1)^N + \sum_{i=0}^{N-1} A_i(-\mathrm{i}\lambda_1)^i)\phi(\lambda_1) \\ &+ \sum_{i=0}^{N-1} B_i(-\mathrm{i}\lambda_1)^i\varphi(\lambda_1) = 0, \\ &((-\mathrm{i}\lambda_1)^N + \sum_{i=0}^{N-1} A_i^*(-\mathrm{i}\lambda_1)^i)\varphi(\lambda_1) \\ &-\sigma \sum_{i=0}^{N-1} B_i^*(-\mathrm{i}\lambda_1)^i\phi(\lambda_1) = 0, \end{aligned} \tag{18}$$

where $(\phi(\lambda_1), \varphi(\lambda_1))^{\mathrm{T}}$ is a solution of the linear spectral problem (8). Next we can consider the Taylor expression at $\varepsilon = 0$,

$$\Phi(\lambda_1 + \varepsilon) = \Phi^{(0)}(\lambda_1) + \Phi^{(1)}(\lambda_1)\varepsilon + \Phi^{(2)}(\lambda_1)\varepsilon^2 + \cdots, \tag{19}$$

with $\Phi^{(k)}(\lambda_1) = \frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}\Phi(\lambda)|_{\lambda=\lambda_1}$ and

$$T(\lambda_1 + \varepsilon) = \sum_{k=0}^{N} T^{(k)}(\lambda_1)\varepsilon^k, \tag{20}$$

where

$$T^{(0)}(\lambda_1) = T(\lambda_1), \quad T^{(N)}(\lambda_1) = \begin{pmatrix} (-\mathrm{i})^N & 0 \\ 0 & (-\mathrm{i})^N \end{pmatrix},$$

$$T^{(k)}(\lambda_1) = \begin{pmatrix} (-\mathrm{i})^N C_N^k \lambda_1^{N-k} + \sum\limits_{i=k}^{N-1} (-\mathrm{i})^i C_i^k A_i \lambda_1^{i-k} & \sum\limits_{i=k}^{N-1} (-\mathrm{i})^i C_i^k B_i \lambda_1^{i-k} \\ -\sigma \sum\limits_{i=k}^{N-1} (-\mathrm{i})^i C_i^k B_i^* \lambda_1^{i-k} & (-\mathrm{i})^N C_N^k \lambda_1^{N-k} + \sum\limits_{i=k}^{N-1} (-\mathrm{i})^i C_i^k A_i^* \lambda_1^{i-k} \end{pmatrix},$$

with $C_i^k = \frac{i!}{k!(i-k)!}$ $(k = 0,1,\ldots,N)$. From Eqs. (19) and (20) we have

$$T(\lambda_1 + \varepsilon)\Phi(\lambda_1 + \varepsilon) = \sum_{k=0}^{N}\sum_{i=0}^{k} T^{(i)}(\lambda_1)\Phi^{(k-i)}(\lambda_1)\varepsilon^k. \tag{21}$$

Let

$$\lim_{\varepsilon\to 0} \frac{T(\lambda_1 + \varepsilon)\Phi(\lambda_1 + \varepsilon)}{\varepsilon^k} = 0, \quad (k = 0,1,\ldots,N-1),$$

we can determine the $2N$ unknown functions A_i, B_i, $(0 \leqslant i \leqslant N-1)$ in Eq. (14). So based on Eq. (15), we can obtain $2N$ linear algebraic systems

$$\begin{aligned}
&T^{(0)}(\lambda_1)\Phi^{(0)}(\lambda_1) = 0,\\
&T^{(0)}(\lambda_1)\Phi^{(1)}(\lambda_1) + T^{(1)}(\lambda_1)\Phi^{(0)}(\lambda_1) = 0,\\
&\cdots,\\
&\sum_{i=0}^{N-1} T^{(i)}(\lambda_1)\Phi^{(N-1-i)}\Phi^{(N-1-i)}(\lambda_1) = 0.
\end{aligned} \tag{22}$$

In order to guarantee that the coefficient determinant of system (22) is non-zero, we can choose a suitable spectral parameter λ_1 . So the transformation matrix T can be uniquely determined by system (22). Then we can obtain the new DT with the same spectral parameter $\lambda = \lambda_1$ by new distinct functions A_i, B_i obtained in the N-order Darboux matrix T.

Theorem 1 Let $\Phi(\lambda_1) = (\phi(\lambda_1), \varphi(\lambda_1))^{\mathrm{T}}$ be a solution of Eq. (8), and ψ_0 is the seed solution of Eqs. (6) and (7), then the $(1, N-1)$-fold DT of Eq. (5) is

$$\widetilde{u}_N = \widetilde{\psi}_N = \psi_0 - 2B^{(N-1)}, \tag{23}$$

where $B^{(N-1)} = \frac{\Delta B^{(N-1)}}{\Delta_N}$, with

$$\Delta_N = \begin{vmatrix}
(-\mathrm{i})^{N-1}\lambda_1^{N-1}\phi^{(0)} & (-\mathrm{i})^{N-2}\lambda_1^{N-2}\phi^{(0)} & \cdots & \phi^{(0)} & (-\mathrm{i})^{N-1}\lambda_1^{N-1}\varphi^{(0)} & (-\mathrm{i})^{N-2}\lambda_1^{N-2}\varphi^{(0)} & \cdots & \varphi^{(0)}\\
\Delta_{2,1} & \Delta_{2,2} & \cdots & \phi^{(1)} & \Delta_{2,N+1} & \Delta_{2,N+2} & \cdots & \varphi^{(1)}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\Delta_{N,1} & \Delta_{N,2} & \cdots & \phi^{(N-1)} & \Delta_{N,N+1} & \Delta_{N,N+2} & \cdots & \varphi^{(N-1)}\\
(\mathrm{i})^{N-1}\lambda_1^{*N-1}\varphi^{*(0)} & (\mathrm{i})^{N-2}\lambda_1^{*N-2}\varphi^{*(0)} & \cdots & \varphi^{*(0)} & -\sigma(\mathrm{i})^{N-1}\lambda_1^{*N-1}\phi^{*(0)} & -\sigma(\mathrm{i})^{N-2}\lambda_1^{*N-2}\phi^{*(0)} & \cdots & -\sigma\phi^{*(0)}\\
\Delta_{N+2,1} & \Delta_{N+2,2} & \cdots & \varphi^{*(1)} & \Delta_{N+2,N+1} & \Delta_{N+2,N+2} & \cdots & -\sigma\phi^{*(1)}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\Delta_{2N,1} & \Delta_{2N,2} & \cdots & \varphi^{*(N-1)} & \Delta_{2N,N+1} & \Delta_{2N,N+2} & \cdots & -\sigma\phi^{*(N-1)}
\end{vmatrix},$$

$$\Delta_{i,k} = \begin{cases}
(-\mathrm{i})^{N-1}\sum\limits_{h=0}^{i-1} C_{N-k}^{h}\lambda_1^{N-k-h}\varphi^{(i-1-h)}, & (1 \leqslant i,\ k \leqslant N),\\
(-\mathrm{i})^{N-1}\sum\limits_{h=0}^{i-1} C_{2N-k}^{h}\lambda_1^{2N-k-h}\phi^{(i-1-h)}, & (1 \leqslant i \leqslant N,\ N+1 \leqslant k \leqslant 2N),\\
(\mathrm{i})^{N-1}\sum\limits_{h=0}^{i-N-1} C_{N-k}^{h}\lambda_1^{*N-k-h}\varphi^{*(i-N-1-h)}, & (N+1 \leqslant i \leqslant 2N,\ 1 \leqslant k \leqslant N),\\
-\sigma(\mathrm{i})^{N-1}\sum\limits_{h=0}^{i-N-1} C_{2N-k}^{h}\lambda_1^{*2N-k-h}\phi^{*(i-N-1-h)}, & (N+1 \leqslant i,\ k \leqslant 2N),
\end{cases}$$

and $\Delta B^{(N-1)}$ is the $(N+1)$-th column of the Δ_N determinant generated by replacing the vector $b = (b_i)_{2N\times 1}$,

$$b_i = \begin{cases}
-(-\mathrm{i})^{N-1}\sum\limits_{h=0}^{i-1} C_N^h\lambda_1^{N-h}\phi^{(i-1-h)}, & (1 \leqslant i \leqslant N),\\
(\mathrm{i})^{N+1}\sum\limits_{h=0}^{i-N-1} C_N^h\lambda_1^{*N-h}\phi^{*(i-N-1-h)}, & (N+1 \leqslant i \leqslant 2N).
\end{cases}$$

2.2. The generalized perturbation (n,M)-fold Darboux transformation

Now we investigate the Darboux matrix (15) and assume that the eigenfunctions $\Phi(\lambda_i)$ $(i = 1,2,\ldots,n)$ are the solutions of Eq. (8) with λ_i and seed solution ψ_0 of Eqs. (6) and (7). So we obtain

$$T(\lambda_i+\varepsilon)\Phi(\lambda_i+\varepsilon) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} T^{(j)}(\lambda_i)\Phi^{(k-j)}(\lambda_i)\varepsilon^k, \quad (i = 1,2,\ldots,n),$$

with $\Phi^{(k)}(\lambda_i) = \frac{1}{k!}\frac{\partial^k}{\partial\lambda_i^k}\Phi(\lambda_i), T^{(j)}(\lambda_i)$ $(j = 1,2,\ldots)$. And

$$\lim_{\varepsilon\to 0}\frac{T(\lambda_i+\varepsilon)\Phi(\lambda_i+\varepsilon)}{\varepsilon^k} = 0,$$

with $i = 1,2,\ldots,n$ and $k_i = 0,1,\ldots,m_i$. According to the idea of (2.1), we can get $2N$ systems $(N = n+\sum_{i=1}^{n} m_i = n+M)$

$$\begin{aligned}
&T^{(0)}(\lambda_i)\Phi^{(0)}(\lambda_i) = 0,\\
&T^{(0)}(\lambda_i)\Phi^{(1)}(\lambda_i) + T^{(1)}(\lambda_i)\Phi^{(0)}(\lambda_i) = 0,\\
&\cdots,\\
&\sum_{j=0}^{m_i} T^{(j)}(\lambda_i)\Phi^{(m_i-j)}(\lambda_i) = 0,
\end{aligned} \tag{24}$$

where $i = 1,2,\ldots,n$.

Theorem 2 Let $\Phi(\lambda_i) = (\phi(\lambda_i), \varphi(\lambda_i))^{\mathrm{T}}$ $(i = 1,2,\ldots,n)$ be a distinct column vector solution of the problem (8) for the corresponding spectral parameters λ_i $(i = 1,2,\ldots,n)$ and seed solution ψ_0 of Eqs. (6) and (7), respectively. The (n,M)-fold Darboux transformation of Eq. (5) is as follows:

$$\widetilde{u}_N = \widetilde{\psi}_N = \psi_0 - 2B^{(N-1)}, \tag{25}$$

where $B^{(N-1)} = \frac{\Delta B^{(N-1)}}{\Delta_N^{(n)}}$, with $\Delta_N^{(n)} = \det([\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(n)}]^{\mathrm{T}})$ and

$$\Delta^{(i)}=\begin{pmatrix}(-\mathrm{i})^{N-1}\lambda_i^{N-1}\phi_i^{(0)} & (-\mathrm{i})^{N-2}\lambda_i^{N-2}\phi_i^{(0)} & \cdots & \phi_i^{(0)} & (-\mathrm{i})^{N-1}\lambda_i^{N-1}\varphi_i^{(0)} & (-\mathrm{i})^{N-2}\lambda_i^{N-2}\varphi_i^{(0)} & \cdots & \varphi_i^{(0)}\\ \Delta^{(i)}_{2,1} & \Delta^{(i)}_{2,2} & \cdots & \phi_i^{(1)} & \Delta^{(i)}_{2,N+1} & \Delta^{(i)}_{2,N+2} & \cdots & \varphi_i^{(1)}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \Delta^{(i)}_{m_i+1,1} & \Delta^{(i)}_{m_i+1,2} & \cdots & \phi_i^{(m_i)} & \Delta^{(i)}_{m_i+1,N+1} & \Delta^{(i)}_{m_i+1,N+2} & \cdots & \varphi_i^{(m_i)}\\ (\mathrm{i})^{N-1}\lambda_i^{*N-1}\varphi_i^{*(0)} & (\mathrm{i})^{N-2}\lambda_i^{*N-2}\varphi_i^{*(0)} & \cdots & \varphi_i^{*(0)} & -\sigma(\mathrm{i})^{N-1}\lambda_i^{*N-1}\phi_i^{*(0)} & -\sigma(\mathrm{i})^{N-2}\lambda_i^{*N-2}\phi_i^{*(0)} & \cdots & -\sigma\phi_i^{*(0)}\\ \Delta^{(i)}_{m_i+3,1} & \Delta^{(i)}_{m_i+3,2} & \cdots & \varphi_i^{*(1)} & \Delta^{(i)}_{m_i+3,N+1} & \Delta^{(i)}_{m_i+3,N+2} & \cdots & -\sigma\phi_i^{*(1)}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \Delta^{(i)}_{2(m_i+1),1} & \Delta^{(i)}_{2(m_i+2),2} & \cdots & \varphi_i^{*(m_i)} & \Delta^{(i)}_{2(m_i+1),N+1} & \Delta^{(i)}_{2(m_i+1),N+2} & \cdots & -\sigma\phi_i^{*(m_i)}\end{pmatrix}.$$

Here $\Delta^{(i)}_{j,k}$ $(1\leqslant i,k\leqslant 2N)$ are given as follows:

$$\Delta^{(i)}_{j,k}=\begin{cases}(-\mathrm{i})^{N-1}\sum\limits_{h=0}^{j-1}C^h_{N-k}\lambda_i^{N-k-h}\phi_i^{(j-1-h)}, & (1\leqslant j\leqslant m_i+1,\ 1\leqslant k\leqslant N),\\ (-\mathrm{i})^{N-1}\sum\limits_{h=0}^{j-1}C^h_{2N-k}\lambda_i^{2N-k-h}\varphi_i^{(j-1-h)}, & (1\leqslant j\leqslant m_i+1,\ N+1\leqslant k\leqslant 2N),\\ (\mathrm{i})^{N-1}\sum\limits_{h=0}^{j-N-1}C^h_{N-k}\lambda_i^{*N-k-h}\varphi_i^{*(j-N-1-h)}, & (m_i+2\leqslant j\leqslant 2(m_i+1),\ 1\leqslant k\leqslant N),\\ -\sigma(\mathrm{i})^{N-1}\sum\limits_{h=0}^{j-N-1}C^h_{2N-k}\lambda_i^{*2N-k-h}\phi_i^{*(j-N-1-h)}, & (m_i+2\leqslant j\leqslant 2(m_i+1),\ N+1\leqslant k\leqslant 2N),\end{cases}$$

and $\Delta B^{(N-1)}$ is obtained from the determinant $\Delta^{(n)}_N$ by replacing its $(N+1)$-th column with the vector $(b^{(1)},\ldots,b^{(n)})^{\mathrm{T}}$, where $b^{(i)}=(b^{(i)}_j)_{2(m_i+1)\times 1}$ with

$$b^{(i)}_j=\begin{cases}(-\mathrm{i})^{N-1}\sum\limits_{h=0}^{j-1}C^h_N\lambda_i^{N-h}\phi_i^{(j-1-h)}, & (1\leqslant j\leqslant m_i+1),\\ -(\mathrm{i})^{N-1}\sum\limits_{h=0}^{j-N-1}C^h_N\lambda_i^{*N-h}\varphi_i^{*(j-N-1-h)}, & (m_i+2\leqslant j\leqslant 2(m_i+1)).\end{cases}$$

By solving the nontrival solution to Eq. (24), we can obtain the new functions A_i and B_i in the N-order Darboux matrix which are different from the usual Darboux transformation. So we can derive some new solutions by the new $(1,N-1)$-fold DT with the same spectral parameter $\lambda=\lambda_1$ or the (n,M)-fold DT with n distinct spectral parameters λ_i $(i=1,2,\ldots,n)$. We call Eqs. (10) and (25) as a generalized perturbation (n,M)-fold DT of Eq. (5).

Comparing system (22) with system (24), it is easy to see that some equations for every index i are the same, but if we require that there exists at least one index $m_i\neq 0$, then they are different.

3. The solutions of the (2+1)-dimensional nonlocal NLS equation

3.1. The higher-order rational solutions of the (2+1)-dimensional nonlocal NLS equationn

In this section, we will use the generalized perturbation $(1,N-1)$-fold DT to present some higher-order rational solutions and RW solutions of Eq. (5). We select a plane wave solution as the seed solution of Eqs. (6) and (7) as follows:

$$\psi_0(x,y,t)=\mathrm{e}^{-2\mathrm{i}\sigma\beta y} \tag{26}$$

where $\beta\neq 0$, $\sigma=\pm 1$. We set the plane-wave solution ψ_0 into the Lax pair (8), then we can obtain the solution of Eq. (8) with the spectral parameter $\lambda=\lambda_1$ as follows:

$$\Phi=\begin{pmatrix}(C_2\frac{(\lambda_1+\sqrt{\lambda_1^2-\sigma})}{\sigma}\mathrm{e}^{-Q}-C_1\frac{(\lambda_1-\sqrt{\lambda_1^2-\sigma})}{\sigma}\mathrm{e}^{Q})\mathrm{e}^{-\mathrm{i}\sigma\beta y}\\ (-\mathrm{i}C_1\mathrm{e}^{Q}+\mathrm{i}C_2\mathrm{e}^{-Q})\mathrm{e}^{\mathrm{i}\sigma\beta y}\end{pmatrix}, \tag{27}$$

with

$$Q=\mathrm{i}\sqrt{\lambda_1-\sigma}(4\beta\lambda_1^2t+2\beta\lambda y+2\beta\sigma t+x+\Theta(\varepsilon)),$$
$$\Theta(\varepsilon)=\sum_{i=1}^{N}(p_i+\mathrm{i}q_i)\varepsilon^{2i},$$

where C_1, C_2, p_i, q_i $(i=1,2,\ldots,N)$ are all real parameters and ε is a small parameter.

Next, by setting the eigenvalue $\lambda=1+\varepsilon^2$, $\sigma=1$ and expanding the vector function $\Phi(\varepsilon^2)$ in Eq. (27) as a Taylor series at $\varepsilon=0$, we derive the following results:

$$\Phi^{(0)}=\begin{pmatrix}\phi^{(0)}\\ \varphi^{(0)}\end{pmatrix}=\begin{pmatrix}(C_2-C_1)\mathrm{e}^{-\mathrm{i}\beta y}\\ (\mathrm{i}C_2-\mathrm{i}C_1)\mathrm{e}^{\mathrm{i}\beta y}\end{pmatrix},$$
$$\Phi^{(1)}=\begin{pmatrix}\phi^{(1)}\\ \varphi^{(1)}\end{pmatrix},$$

where

$$\phi^{(1)}=12\left(3\left(t+\frac{y}{3}\right)^2\beta^2+\left(t+\frac{y}{3}\right)(\mathrm{i}+x)\beta\right.$$

$$+\frac{1}{6}\mathrm{i}x+\frac{1}{12}x^2-\frac{1}{12}\Big)(-C_2+C_1)\mathrm{e}^{-\mathrm{i}\beta y},$$
$$\varphi^{(1)} = (-6\beta t+2\beta y+x)^2\mathrm{i}(-C_2+C_1)\mathrm{e}^{\mathrm{i}\beta y},$$

$(\phi^{(i)},\varphi^{(i)})^{\mathrm{T}}$ $(i=2,3,\ldots)$ are cumbersome, so we omit them here. It should be pointed out that $(\phi^{(i)},\varphi^{(i)})^{\mathrm{T}}$ has the same factor (C_2-C_1), which has no effect on the solutions of Eq. (25).

Next according to Theorem 1, we discuss the solutions (25) from the following three cases ($N=1$, 2, 3) to comprehend the obtained exact solutions of Eq. (5). When $N=1$, we can only give the trivial constant solution. In this case, we do not use the derivatives of $T(\lambda_1)\Phi(\lambda_1)$ to determine $B^{(0)}$ so we can not obtain a new solution.

Case 1 For $N=2$, from the generalized $(1,1)$-fold DT, we can obtain the first-order rational solution of Eq. (5)

$$\widetilde{u_1}(x,y,t)=\psi_0-2B_1=\frac{F}{H} \tag{28}$$

with

$$F = \psi_0((72t^2-8y^2)\beta^2+((12\mathrm{i}+24x)t -8\mathrm{i}y)\beta+2\mathrm{i}x+2x^2+1),$$
$$H = (72t^2-8y^2)\beta^2+(12\mathrm{i}+24x)t\beta+2\mathrm{i}x+2x^2-1,$$

which contains a free parameter β that also appears in Eq. (5). Based on Eq. (31) we can obtain the maximum and minimum amplitudes of the modulus of $\widetilde{u_1}$. When $x=-6\beta t-\frac{\mathrm{i}}{2}$ and $y=-\frac{\mathrm{i}}{4\beta}(1+2\sqrt{3})$, we have the maximum amplitude

$$\widetilde{u_1}=\frac{(1+\sqrt{3})^2}{2}\mathrm{e}^{-\frac{1+2\sqrt{3}}{2}}$$

and when $x=-6\beta t-\mathrm{i}/2$ and $y=-4\mathrm{i}/\beta$, we have the minimum amplitude $\widetilde{u_1}=\mathrm{e}^{-1/2}$, here we consider t as an arbitrary real number. When x or $t\to\infty$, $|\widetilde{u_1}|\to 1$.

When $y=0$,

$$u(x,y,t)=\frac{72\beta^2t^2+12\,(\mathrm{i}+2x)t\beta+2\,\mathrm{i}x+2x^2+1}{72\beta^2t^2+12\,(\mathrm{i}+2x)t\,\beta+2\,\mathrm{i}x+2x^2-1}$$

is the solution of the (1+1)-dimensional nonlocal complex mKdV equation. In other words, we can use the reduction $y=0$ of the rational solution of the (2+1)-dimensional NNLS equation to obtain the rational solution of the (1+1)-dimensional nonlocal complex mKdV equation. Without loss of generality, we set $\beta=1$, the profile of the regular rational solution (31) is shown in Fig. 1.

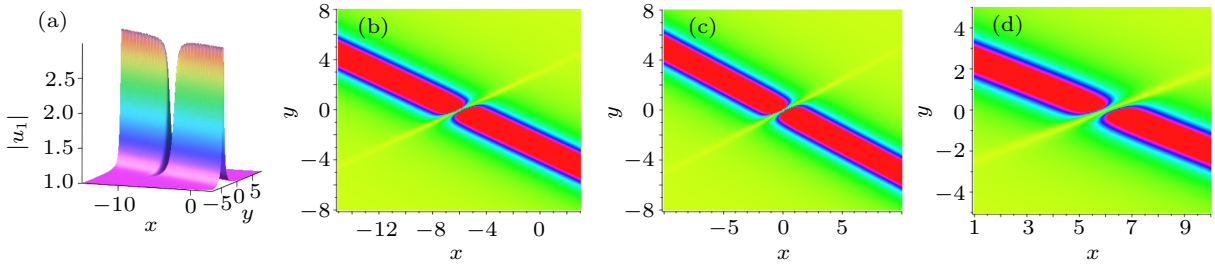


Fig. 1. $|\widetilde{u_1}|$ is the modulus of the first-order rational solution of Eq. (5). (a) $t=1$, $\beta=1$, (b) $t=1$, $\beta=1$, (c) $t=0$, $\beta=1$, (d) $t=-1$, $\beta=1$.

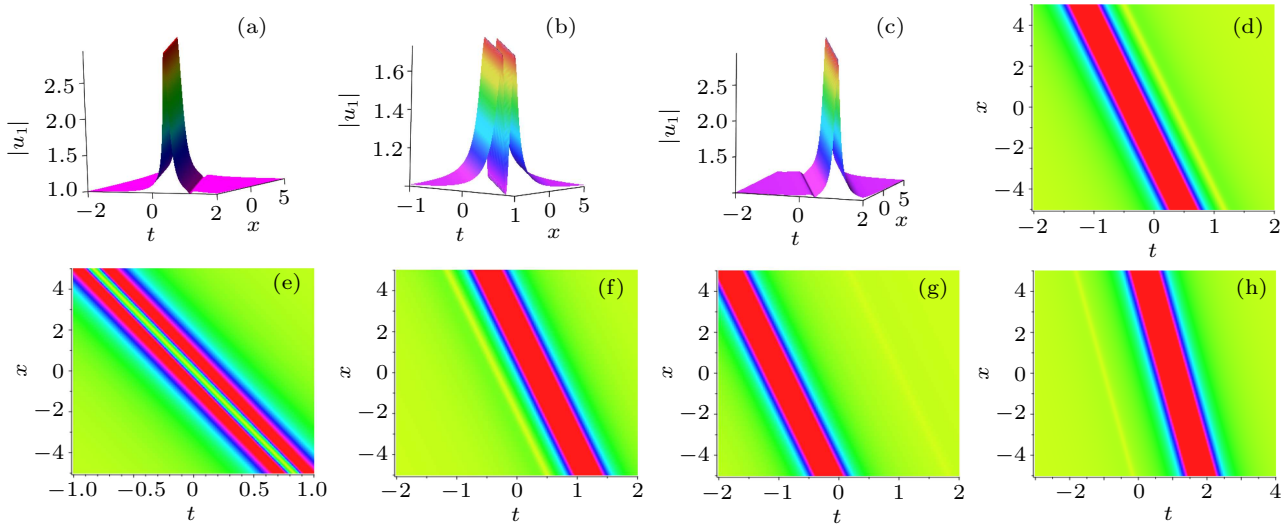


Fig. 2. $|\widetilde{u_1}|$ is the modulus of the first-order rational solution of Eq. (5). (a) $y=1$, $\beta=1$, (b) $y=0$, $\beta=1$, (c) $y=-1$, $\beta=1$, (d) $y=1$, $\beta=1$, (e) $y=0$, $\beta=1$, (f) $y=-1$, $\beta=1$, (g) $y=3$, $\beta=1$, (h) $y=-3$, $\beta=1$.

Through tedious calculations, it can be found that when $t=1, x=-6-\mathrm{i}/2$ and $y=-\mathrm{i}+\mathrm{i}2\sqrt{3}/4, t=0, x=-\mathrm{i}/2$ and $y=-\mathrm{i}+\mathrm{i}2\sqrt{3}/4, t=-1, x=6-\mathrm{i}/2$ and $y=-\mathrm{i}+\mathrm{i}2\sqrt{3}/4$, the amplitude of the soliton solution is $\frac{\mathrm{e}^{-\frac{1}{2}+\sqrt{3}}(3+\sqrt{3})}{3-\sqrt{3}}$. From Figs. 1(a) and 1(b), it can be regarded as elastic interactions occurred between the bright soliton and dark soliton and the interacting solitons can fully recover their shapes and velocities when interacting, and the energy does not decrease after the collision. If $q^*(-x,y,-t)\neq q^*(x,y,t)$ then the solution (31) does not satisfy the system (1) with $r=q^*(x,y,t)$. Keep $y=0$, the wave profiles of solution (31) in (x,y)-space under fixed time are obviously different. From Figs. 1(b)–1(d), we can obtain that when time t increases, the first-order rational soliton moves to the left on the x-axis.

From Fig. 2, when $y=0$, the shape of the first-order rational soliton in (x,t)-space is shown as the M-shaped solitary wave and the rational soliton consists of two same bright solitons and one dark soliton. After a series of experiments we reach the following conclusion. When $y>0$, the bright soliton on the right disappears and when $y<0$, the bright soliton on the left disappears. When $y>0$, through the increase of y, the bright soliton moves to the left along the x-axis, and the dark soliton moves to the right along the x-axis. When $y<0$, through reduction of y, the bright soliton moves to the right along the x-axis, and the dark soliton moves to the left along the x-axis.

Case 2 For $N=3$, on the basis of the generalized (1, 2)-fold DT, the second-order rational soliton solution of Eq. (5) can be derived as follows:

$$\widetilde{u_2}(x,y,t)=\psi_0-2B_2=\frac{F_1}{G_1} \tag{29}$$

which includes two parameters b_1 and c_1. Here

$$\begin{aligned}
F_1 =\ & -(-9-72x^2c_1+311040t^4x^2+82944t^3y+384yx^3+576tx^5-192tx^3+864tb_1-192y^2x^4-1152y^2x^2\\
& -288y^2c_1+8640t^2x^4+1728t^2x^2-2592t^2c_1-1536y^3x+69120t^3x^3-248832t^4y^2+27648t^2y^4\\
& -9216y^3t+41472t^2xy-9216txy^2-165888t^3xy^2+9216txy^4+6912tyx^2-3456tyc_1-13824y^2t^2\\
& -41472y^2t^2x^2-4608y^2tx^3-288yx^2b_1-576yxc_1-864txc_1+144yb_1+311040\mathrm{i}t^4x-248832\mathrm{i}t^4y\\
& +103680\mathrm{i}t^3x^2-82944\mathrm{i}t^3y^2+17280\mathrm{i}t^2x^3+55296\mathrm{i}t^2y^3+1440\mathrm{i}tx^4+4608\mathrm{i}ty^4-192\,\mathrm{i}yx^4-384\mathrm{i}y^2x^3\\
& +1536\mathrm{i}y^3x^2+768\mathrm{i}y^4x+10368\mathrm{i}t^3c_1+48\mathrm{i}x^3c_1+9504\mathrm{i}t^2x+27648\mathrm{i}t^2y-5184\mathrm{i}t^2b_1+1008\mathrm{i}tx^2-6336\mathrm{i}ty^2\\
& -144\mathrm{i}x^2b_1-1440\mathrm{i}y^2x-576\mathrm{i}y^2b_1-288\mathrm{i}tc_1-144\,\mathrm{i}xc_1-288\,\mathrm{i}yc_1+373248\mathrm{i}t^5-3072\mathrm{i}y^5+25920\mathrm{i}t^3\\
& +24\mathrm{i}x^3-768\mathrm{i}y^3-792\mathrm{i}t-36\,\mathrm{i}x+288\,\mathrm{i}y+72\mathrm{i}b_1+1920y^4-7200t^2+864y^2+51840t^4+72c_1+746496t^6\\
& -1024y^6+16x^6-384y^3b_1-72x^2+3456\mathrm{i}y^2tc_1+576\mathrm{i}y^2xc_1+4608\mathrm{i}tyx-1728\mathrm{i}txb_1-1728\mathrm{i}tyb_1\\
& -288\,\mathrm{i}yxb_1+5184\mathrm{i}t^2xc_1+864\mathrm{i}tx^2c_1-165888\mathrm{i}t^3xy-41472\,\mathrm{i}yt^2x^2-41472\mathrm{i}y^2t^2x-4608\mathrm{i}tyx^3-6912\mathrm{i}y^2tx^2\\
& +18432\mathrm{i}y^3tx-3456tyxb_1-10368t^2yb_1-2016tx-576ty+20736t^3x+746496t^5x+768y^4x^2+144xb_1\\
& +288xy-36{c_1}^2-24x^4-36{b_1}^2+48\mathrm{i}x^5)\mathrm{e}^{-2\,\mathrm{i}y},\\
G_1 =\ & -9-72x^2c_1+311040t^4x^2+576tx^5-1344tx^3-192y^2x^4-288y^2c_1+8640t^2x^4-8640t^2x^2-2592t^2c_1\\
& +69120t^3x^3-248832t^4y^2+27648t^2y^4+4608txy^2-165888t^3xy^2+9216txy^4+27648y^2t^2-41472y^2t^2x^2\\
& -4608y^2tx^3-288yx^2b_1-864txc_1-144yb_1+311040\mathrm{i}t^4x+103680\mathrm{i}t^3x^2-82944\mathrm{i}t^3y^2+17280\mathrm{i}t^2x^3\\
& +1440\mathrm{i}tx^4+4608\mathrm{i}ty^4-384\mathrm{i}y^2x^3+768\mathrm{i}y^4x+10368\mathrm{i}t^3c_1+48\mathrm{i}x^3c_1+373248\mathrm{i}t^5+5184\mathrm{i}t^3-72\mathrm{i}x^3+216\mathrm{i}t\\
& -1920y^4+4896t^2-288y^2-10368t^4+746496t^6-1024y^6+16x^6-384y^3b_1-864\mathrm{i}t^2x-720\mathrm{i}tx^2\\
& +576\mathrm{i}ty^2-288\mathrm{i}y^2x+576\mathrm{i}tc_1+36\,\mathrm{i}x+72x^2+3456\mathrm{i}y^2tc_1+576\mathrm{i}y^2xc_1-1728\mathrm{i}tyb_1-288\,\mathrm{i}yxb_1+5184\mathrm{i}t^2xc_1\\
& +864\mathrm{i}tx^2c_1-41472\mathrm{i}y^2t^2x-6912\mathrm{i}y^2tx^2-3456tyxb_1-10368t^2yb_1+864tx-20736t^3x+746496t^5x\\
& +768y^4x^2-36{c_1}^2-72x^4-36{b_1}^2+48\mathrm{i}x^5.
\end{aligned}$$

After a series of numerical experiments, we find that when $b_1>0$, $c_1>0$, the shape of the solution (29) is basically unchanged but the position changes with time t. While $c_1>0$, the solution (29) is not a singular solution, and $c_1<0$, the solution (29) is a singular solution.

From Fig. 3, solution (29) exhibits the elastic interaction of two bright solitons and two dark solitons, which is not localized. From Figs. 3(b)–3(d), we know that when $t>0$, the elastic collision occurs in the lower left and upper left spaces. When $t<0$, the elastic collision occurs in the upper right and lower right spaces and the phase changes. It is worth noting that no matter how t changes, the phase of a bright soliton and a dark soliton in different directions does not change after an elastic collision.

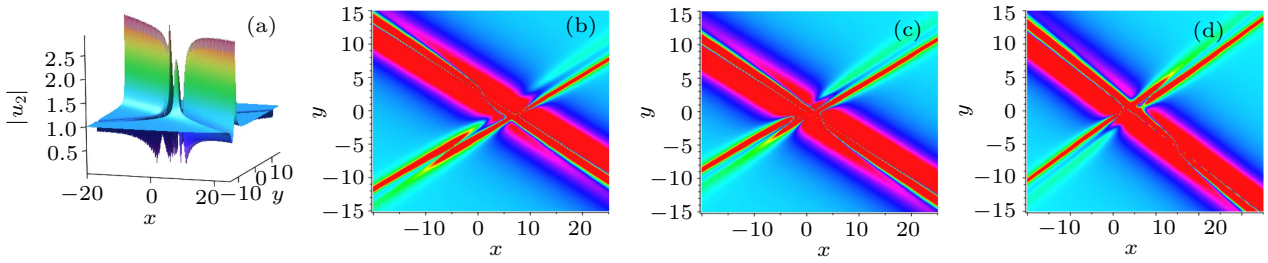


Fig. 3. $|\widetilde{u}_2|$ is the modulus of the second-order rational solution of Eq. (5). (a) $t=0$, $b_1=3$, $c_1=1$, (b) $t=-1$, $b_1=3$, $c_1=1$, (c) $t=0$, $b_1=3$, $c_1=1$, (d) $t=1$, $b_1=3$, $c_1=1$.

It can be seen from Figs. 4(a)–4(c) that when $b_1=0$ and $c_1=0$, the solution (29) is a singular solution. An inelastic collision occurs between a bright soliton and a dark soliton. From Figs. 4(a)–4(c), we know that when $t>0$, two dark solitons and one bright soliton collide elastically at the upper left of (x,y)-space, and the phase does not change. A dark soliton and a bright soliton occur elastic collision in (x,y)-space at the bottom right, but their phase changes. When $t<0$, two dark solitons and one bright soliton collide elastically at the bottom right of (x,y)-space, and the phase does not change. A dark soliton and a bright soliton occur elastic collision in (x,y)-space at the upper left, but their phase changes. From Figs. 4(d)–4(f), we can see that in (x,y)-space when $b_1>0$ and $c_1>0$, the second-order rational solitons are the interaction of two parallel bright solitons and two parallel dark solitons, and the waveform is center-symmetrical. When time $t>0$, the collision position moves to the left along the x-axis as time increases, and when time $t<0$, the collision position moves to the right along the x-axis as time decreases.

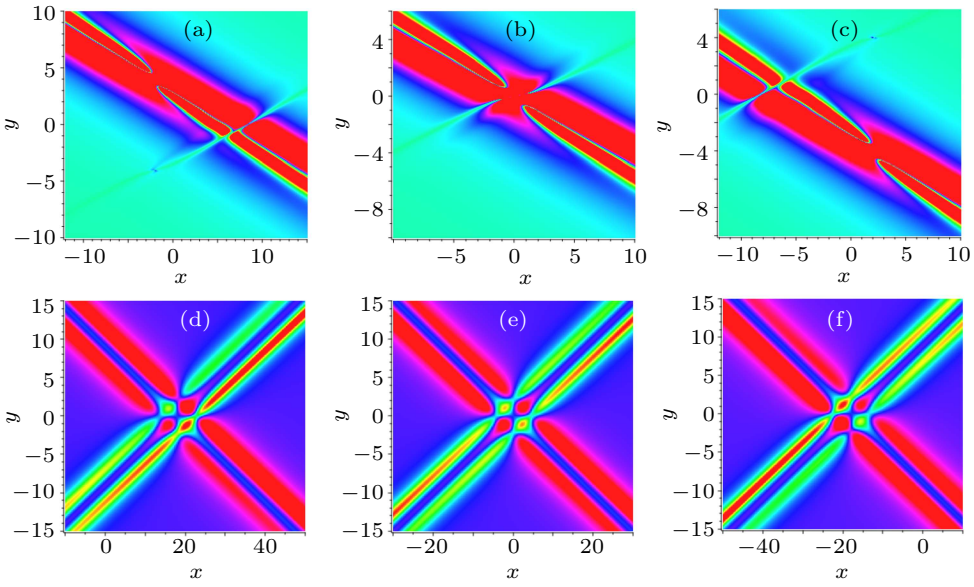


Fig. 4. $|\widetilde{u}_2|$ is the second-order rational solution of Eq. (5). (a) $t=-1$, $b_1=0$, $c_1=0$, (b) $t=0$, $b_1=0$, $c_1=0$, (c) $t=1$, $b_1=0$, $c_1=0$, (d) $t=-3$, $b_1=3$, $c_1=80$, (e) $t=0$, $b_1=3$, $c_1=80$, (f) $t=3$, $b_1=3$, $c_1=80$.

It is necessary to point out that for $N>3$, we can find that the higher-order rational solutions of Eq. (5) have many soliton structures and dynamic behaviors. Because of its complexity, we omit it here, but it is controlled by four parameters (b_1,b_2,c_1,c_2).

3.2. The interaction solutions of the (2+1)-dimensional nonlocal NLS equation

In this section we will analyze the the interaction between the interaction solutions of Eq. (5) which is given by the generalized $(2,N-1)$-fold DTs with two spectral parameters λ_1 and λ_2. The seed solution and eigenfunction are the same as Eqs. (26) and (27). Next, we will discuss the second order DT and the third order DT respectively.

Case 1 Second order DT ($N=2$).

We could select a special equation of system (24)

$$T^{(0)}(\lambda_1)\Phi_i^{(0)}(\lambda_1)=0,\quad T(\lambda_2)\Phi(\lambda_2)=0. \tag{30}$$

For different λ_2, the wave structure and dynamic behavior of the solution

$$\widetilde{u_1}(x,y,t)=\psi_0-2B_1 \tag{31}$$

show many interesting phenomena.

If we choose $\lambda_2=3/2$, the periodic wave solution solution of Eq. (5) can be expressed as

$$\widetilde{u_1} = q_1 = \mathrm{e}^{-2i\zeta y} + \frac{2(3\mathrm{e}^{\mathrm{i}\zeta y 3\sqrt{5}-2}\sqrt{5} - 3\mathrm{e}^{-\mathrm{i}\zeta y\left(3\sqrt{5}+2\right)}\sqrt{5} - 5\mathrm{e}^{\mathrm{i}\zeta y\left(3\sqrt{5}-2\right)} - 5\mathrm{e}^{-\mathrm{i}\zeta y\left(3\sqrt{5}+2\right)})}{3\left(\sqrt{5}+3\right)\mathrm{e}^{-\mathrm{i}\sqrt{5}(11\zeta t+x)} - 3\left(\sqrt{5}-3\right)\mathrm{e}^{\mathrm{i}\sqrt{5}(11\zeta t+x)} - 4\mathrm{e}^{3\mathrm{i}\sqrt{5}\zeta y} - 4\mathrm{e}^{-3\mathrm{i}\sqrt{5}\zeta y}}$$

and shown in Fig. 5. The solution is a very stable periodic solution and the amplitude and wave profile of the periodic wave remain unchanged over time. From the expression and Fig. 5 we can know the interaction of the wave profile motion on the y-axis and the x-axis.

When the general parameter $\lambda_2 = (1+\mathrm{i})/2$, the solution of Eq. (5) is the interaction of two strange breathing waves. The expression for the solution is so complicated that we omit it. When $\lambda_2 = (1+\mathrm{i})/2$, we can get two strange breathing solutions. As can be seen from Figs. 6(a)–6(c), as the absolute value of y increases, the two strange breathing waves gradually move away from each other on the x–t plane and the wave profile remains parallel. Figure 5(f) is the wave profile evolution diagram at different time when $y = 6$. From Figs. 6(a)–6(d), it can be seen that the amplitude of the solitary wave changes significantly during the propagation process, and there is energy loss or increase.

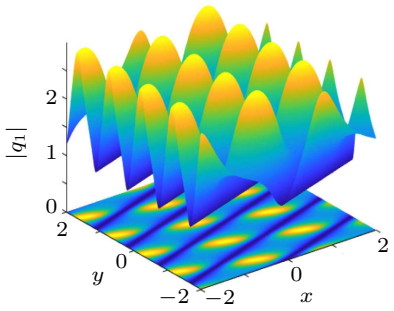


Fig. 5. $|q_1|$ is the solution of Eq. (5). $\lambda_2 = 3/2$, $t = 1$.

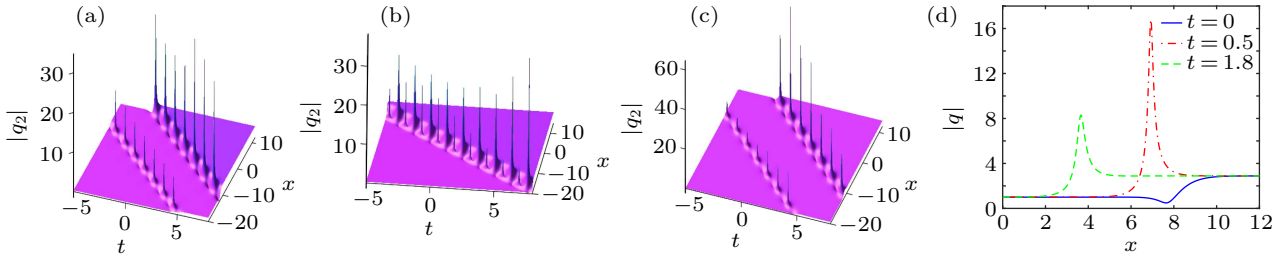


Fig. 6. $|q_1|$ is the solution of Eq. (5). (a) $\lambda_2 = \frac{1+i}{2}$, $y = -6$, (b) $\lambda_2 = \frac{1+i}{2}$, $y = 0$, (c) $\lambda_2 = \frac{1+i}{2}$, $y = 6$, (d) $\lambda_2 = \frac{1+i}{2}$, $y = 6$.

Case 2 Third order DT ($N = 3$).

We could select two special equations of system (24)

$$\begin{aligned} &T^{(0)}(\lambda_1)\Phi_i^{(0)}(\lambda_1) = 0, \\ &T^{(1)}(\lambda_1)\Phi_i^{(0)}(\lambda_1) + T^{(0)}(\lambda_1)\Phi_i^{(1)}(\lambda_1) = 0, \end{aligned} \tag{32}$$

and

$$T(\lambda_2)\Phi(\lambda_2) = 0. \tag{33}$$

For different λ_1, the wave structure and dynamic behavior of the solution

$$\widetilde{u_2}(x,y,t) = q_2 = \psi_0 - 2B_2 \tag{34}$$

show many interesting phenomena.

When we select $\lambda_2 = 6$, $\widetilde{u_2}$ is the interaction solution of the interaction between periodic waves and bimodal breather (a breather having two peaks). The expression for the solution is so complicated that we omit it. When $x > 0$, the wave profile moves to the left along the time t axis in the (y, t) plane as x increases and when $x < 0$, the wave profile moves to the right along the time t axis as x decreases (see Figs. 7(a)–7(c)). It can be seen from Fig. 7(f) that in the (y, t) plane, the amplitude of each bimodal breather is different. The amplitude in the $y > 0$ half plane is greater than the magnitude in the $y < 0$ half plane. We can also see that although their amplitudes are different, the wave profiles and amplitudes do not change much at different time, and it can be seen from Figs. 7(d) and 7(e) that the phase shifts when the collision occurs, but the propagation path does not change, so the collision is an elastic one.

When we select $\lambda_2 = 1 - \mathrm{i}/2$, $\widetilde{u_2}$ is the interaction solution of two singular breathers and two traveling waves. It can be seen from Figs. 8(g) and 8(i) that when $y \neq 0$, the two parallel solitary waves gradually move away. When $y < 0$, after the collision, the amplitude of the traveling wave is obviously larger or smaller, and the propagation direction does not change, so the collision is an inelastic one. While when $y > 0$, the amplitude of the traveling wave remains unchanged after the collision, and the propagation direction has not changed, so the collision is an elastic one.

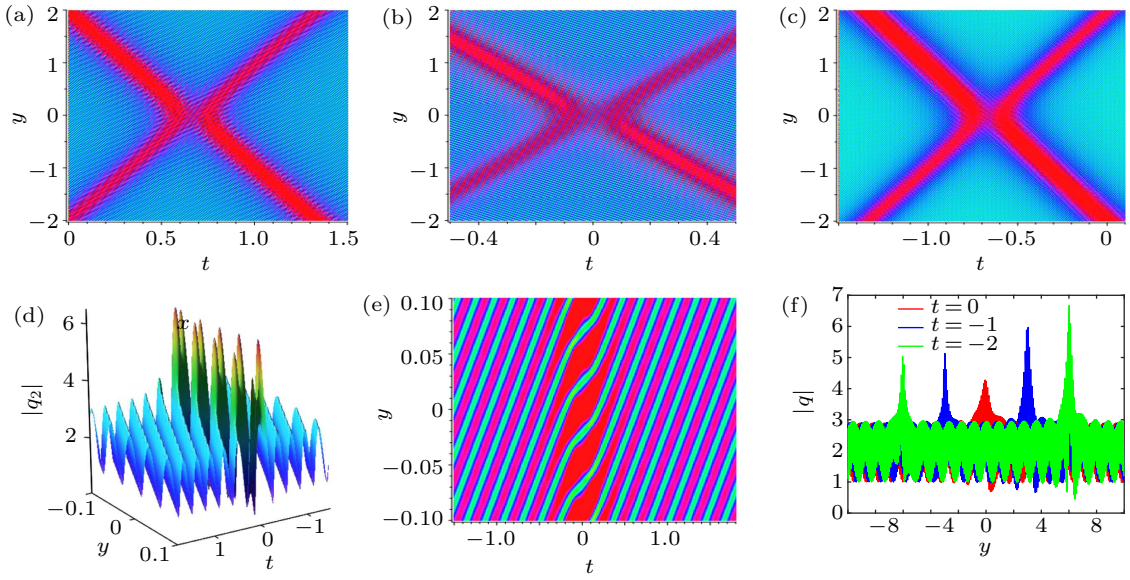


Fig. 7. $|q_2|$ is the solution of Eq. (5). (a) $\lambda_2=6, x=-4$, (b) $\lambda_2=6, x=0$, (c) $\lambda_2=6, x=4$, (d) $\lambda_2=6, x=0$ is a partially enlarged image of (b), (e) $\lambda_2=6, x=0$ is the projected image of (d), (f) $\lambda_2=6, x=0$.

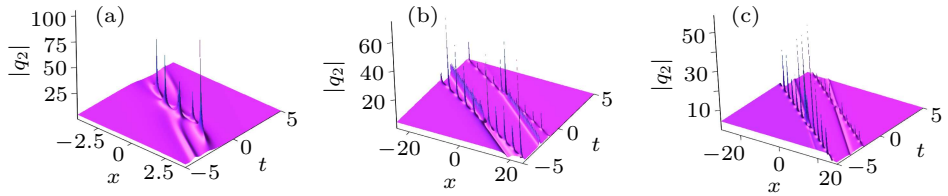


Fig. 8. $|q_2|$ is the interaction solution of two singular breathing waves and two traveling waves of Eq. (5). (a) $\lambda_2=1-\frac{\mathrm{i}}{2}, y=0$, (b) $\lambda_2=1-\frac{\mathrm{i}}{2}$, $y=-5$, (c) $\lambda_2=1-\frac{\mathrm{i}}{2}, y=5$.

4. Summary and discussion

In summary, we used the Taylor expansion and a limit procedure to present a new generalized perturbation (n, M)-fold Darboux transformation of the (2+1)-dimensional reverse space–time NLS equation. we chose the generalized $(1, N-1)$-fold DTs with only one spectral parameter to obtain higher-order rational solutions of Eq. (5) with some parameters. We discussed the wave profiles of these solutions with distinct parameters. Some higher-order rational solutions' propagation and interaction were discussed and demonstrated, which display rich and interesting wave structures. We obtained the periodic solution and interaction solution of Eq. (5) through $(2, N-1)$-fold DTs. We have studied the interactions of singular breather and periodic waves, singular breather and traveling waves, bimodal breather and periodic waves, and obtained abundant wave profile structures. The idea of this article can applied to other physically nonlinear wave models. There are more free parameters in these solutions, general temporal-spatial structures of those solutions could be studied in the future.

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