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Optimal parameter estimation of open quantum systems*

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In quantum information technologies, quantum weak measurement is beneficial for protecting coherence of systems. In order to further improve the protection effect of quantum weak measurement on coherence, we propose an optimization scheme of quantum Fisher information (QFI) protection in an open quantum system by combing no-knowledge quantum feedback control with quantum weak measurement. On the basis of solving the dynamic equations of a stochastic two-level quantum system under feedback control, we compare the effects of different feedback Hamiltonians on QFI and find that via no-knowledge quantum feedback, the observation operator σ_x (or σ_x and σ_z) can protect QFI for a long time. Namely, no-knowledge quantum feedback can improve the estimation precision of feedback coefficient as well as that of detection coefficient.

Keywords: no-knowledge quantum feedback, real-time parameters estimation, quantum Fisher information (QFI), open quantum system

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1. Introduction

Quantum precise measurement has become a new research hotpot in recent years.[1,2] It plays an important role in acquisition and transmission of quantum information, and has formed a new interdisciplinary-quantum metrology.[3] Precision of classical parameter estimation is limited by the quantum limit (or shot-noise limit). In order to enhance the precision of parameter estimation, quantum properties of probe state have been introduced. The precision of quantum parameter estimation directly relates to quantum Fisher information (QFI) of probe state,[4] so it is of great importance to study the intrinsic factors of QFI in quantum metrology.[5,6] The quantum Cramér–Rao theorem tells us that the upper bound of the precision of parameter estimation is given by the inverse of QFI.[7] Moreover, the actual precision of parameter estimation needs to consider the measurement process. In other words, the measurement process does not affect the limit of the parameter estimation precision, instead it affects the actual parameter estimation precision. Therefore, in order to improve the precision of parameter estimation, appropriate probe states or control means are necessary so that the QFI cannot be zero for a long time and takes a value as large as possible. The exact relationship between the quantum coherence and QFI has been found. It indicates that quantum coherence is the determinant of the precision of quantum parameter estimation,[8] so there is the generalized quantum coherence in high-dimensional quantum system.[9]

Any realistic quantum system inevitably interacts with its environment leading to decoherence and considerable decrease of parameter estimation precision.[10,11] In order to enhance QFI, many different quantum control strategies have been proposed and applied in engineering. For example, the combination of the non-Markovian memory effect with the classical drive can be used to protect QFI encoded on the phase parameters of a single qubit system to improve the precision of parameter estimation.[12] Interestingly, the precision of parameter estimation can be improved by adding dynamical decoupling pulses into the open quantum system.[13]

Except for the strategies mentioned above, several useful tools from classical control theory have been introduced to the control analysis and design of quantum systems. For example, an optimal control approach for quantum systems is to design a set of admissible controllers satisfying the system dynamic equations as well as minimizing a certain cost function.[14,15] A learning control method has been presented for guiding the control of chemical reactions.[16] Lyapunov control provides a systematic design method for some difficult quantum control tasks.[17] Quantum feedback control is also an important control strategy used to enhance QFI.[18,19] In the measurement-based feedback control, the controller is designed based on the state information of the feedback system. In Ref. [20], different measurement-based feedback types have been investigated to enhance QFI.

In general, quantum measurement with feedback control will change state of a system and bring unfavorable effects on the system, which needs to be dealt carefully in the design of closed-loop feedback controller.[21] Quantum weak measurement is an alternative effective mean to protect the coherence of the system, which has been widely used in quan-

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tum precise measurement.[22] In fact, the back action of measurement can also be used to develop new control strategies for system control.[23,24] For example, sliding mode control is used to suppress decoherence and protect quantum states.[25] Recently, Szigeti *et al.* proposed the no-knowledge feedback control strategy, which has been applied to cancel Markovian decoherence, and preserved the superposition state.[26,27] This paper focuses on the combination of no-knowledge quantum feedback with quantum weak measurement,[28] and explores the scheme of protecting QFI. Our research results show that under the same weak measurement, no-knowledge quantum feedback control can protect QFI more effectively than state information feedback.

The paper is organized as follows. In Section 2, we present a brief review of QFI. The dynamical evolution of the system, under measurement-based feedback control, is described in Section 3. In Sections 4 and 5, based on numerical simulations, we respectively explore the scheme of real-time estimation of the feedback coefficient and detection efficiency. Finally, a brief summary is given in Section 6.

2. Quantum Fisher information

In the classical measurement, the upper bound of the precision of unknown parameter estimation can be given by the classical Cramér–Rao inequality as[29]

$$(\delta\varphi)_{\mathrm{S}}^2 \geq \frac{1}{\mu F_{\mathrm{S}}(\varphi)}, \tag{1}$$

where $F_{\mathrm{S}}(\varphi)$ is the classical Fisher information, μ represents the total number of experiments.

With the development of quantum metrology, classical Fisher information has been applied to quantum field. The relationship between the upper bound of the precision of parameter estimation and QFI is called the quantum Cramér–Rao inequality[30]

$$(\delta\varphi)^2 \geq \frac{1}{\mu F_{\mathrm{Q}}(\varphi)}, \tag{2}$$

where $F_{\mathrm{Q}}(\varphi)$ represents the QFI. Equation (2) indicates that the influence of experiment times is the same as QFI on the precision of parameter estimation. However, increasing the number of measurements will inevitably consume more resources. It is necessary to use as few resources as possible to achieve as much measurement precision as possible in engineering practice. After the measurement times are determined, the theoretical upper limit of the precision of quantum parameter estimation is only determined by the QFI. The more the QFI is, the higher the precision is. It should be emphasized that the limit of parameter estimation is only in theory. In other words, although the actual measurement process does not affect the upper bound of the precision, it affects the measurement precision.[31] Therefore, based on the no-knowledge feedback control, this paper investigates the influence of different feedback processes on the parameter estimation of a dissipative qubit.

In the probe state $\rho(\varphi)$, the QFI is usually defined as[32]

$$F_{\mathrm{Q}}(\varphi) = \mathrm{Tr}\left[\rho(\varphi)L^2(\varphi)\right], \tag{3}$$

where $L(\varphi)$ is the symmetric logarithmic derivative operator defined implicitly by the equation, which reads

$$\frac{\partial}{\partial\varphi}\rho(\varphi) = \frac{L(\varphi)\rho(\varphi)+\rho(\varphi)L(\varphi)}{2}. \tag{4}$$

Typically, there are three methods to calculate the QFI. The most frequently used one is diagonalizing the N-dimension density matrix $\rho_{22}(t)$ as

$$\rho(\varphi) = \sum_{\lambda_m\in S}\lambda_m(\varphi)\left|\psi_m(\varphi)\right\rangle\left\langle\psi_m(\varphi)\right|,$$

where $\rho_{21}(t)$ and $|\psi_m(\varphi)\rangle$ are the eigenvalues and eigenvectors of the density matrix $\rho(\varphi)$, respectively. Here,

$$S := \{\lambda_m \in \{\lambda_m\}\,|\lambda_m \neq 0\}\ \ (S \leq N).$$

Then one can rewrite the QFI as[33–35]

$$\begin{aligned} F_{\mathrm{Q}}(\varphi) \;=\; & \sum_{\lambda_m\in S}\frac{\left(\partial_\varphi\lambda_m(\varphi)\right)^2}{\lambda_m(\varphi)} \\ & + \sum_{\lambda_m\in S} 4\lambda_m\left\langle\partial_\varphi\psi_m(\varphi)|\partial_\varphi\psi_m(\varphi)\right\rangle \\ & - \sum_{\lambda_m,\,\lambda_n\in S}\frac{8\lambda_m\lambda_n}{\lambda_m+\lambda_n}\left|\left\langle\partial_\varphi\psi_m(\varphi)|\psi_n(\varphi)\right\rangle\right|^2, \end{aligned} \tag{5}$$

where ∂_φ denotes the partial derivative $\partial_\varphi = \partial/\partial\varphi$.

Equation (5) suggests that QFI consists of three parts. The first term is the classical Fisher information determined by the probability distribution. The second term is a weighted average over QFI for all nonzero eigenstates. The last term, stemming from the mixture of pure states, reduces the QFI and hence the estimation precision is lower than the pure-state case.

For high-dimensional quantum systems, Eq. (5) is difficult to give the analytical expression of $F_{\mathrm{Q}}(\varphi)$. However, for the two-dimensional density matrix, a simple and explicit expression of QFI has been obtained explicitly as[36,37]

$$\begin{aligned} F_{\mathrm{Q}}(\varphi) \;=\; & \mathrm{Tr}\left[\left(\partial_\varphi\rho(\varphi)\right)^2\right] \\ & + \frac{1}{\mathrm{Det}\,\rho(\varphi)}\mathrm{Tr}\left[\left(\rho(\varphi)\partial_\varphi\rho(\varphi)\right)^2\right]. \end{aligned} \tag{6}$$

3. Investigation model

Consider a two-level system (qubit) with Hamiltonian H_0 that interacts with a Markovian reservoir via the coupling operator R. The system density operator $\rho(t)$ evolves according to the unconditional master equation ($\hbar = 1$)[38]

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = -\mathrm{i}\left[H_0,\rho(t)\right] + D[R]\rho(t), \tag{7}$$

where $D[X]\rho(t) = X\rho(t)X^+ - (X^+X\rho(t) + \rho(t)X^+X)/2$.

When one uses continuous measurement to obtain the parameter information of a quantum system, the evolution of quantum state can be described by the stochastic master equation. In this case, the system must be described by a conditional density matrix $\rho_t(t)$.[39] For homodyne measurement of the environment at angle θ, the conditional density matrix $\rho_t(t)$ is described as follows:[40]

$$\begin{aligned} \mathrm{d}\rho_t(t) &= -\mathrm{i}\,[H_0, \rho_t(t)]\,\mathrm{d}t + D[R]\rho_t(t)\,\mathrm{d}t + MD[O]\rho_t(t)\,\mathrm{d}t \\ &\quad + \sqrt{\eta M}\,\mathrm{d}WH[O\mathrm{e}^{\mathrm{i}\theta}]\rho_t(t)\,\mathrm{d}t, \qquad (8) \\ \mathrm{d}y &= \sqrt{\eta}\,\mathrm{Tr}(O\mathrm{e}^{\mathrm{i}\theta} + O^+\mathrm{e}^{-\mathrm{i}\theta})\,\mathrm{d}t + \mathrm{d}W(t), \qquad (9) \end{aligned}$$

where

$$H[X]\rho_t(t) = X\rho_t(t) + \rho_t(t)X^+ - \mathrm{Tr}\left(X\rho_t(t) + \rho_t(t)X^+\right)\rho_t(t);$$

θ is an auxiliary parameter introduced for the convenience of research, and the observable operator O is Hermitian operator. M is measurement strength, which is greater than zero; η is detection efficiency, $0 \le \eta \le 1$. The size of the stochastic terms is closely related to detection efficiency η, that is, the detection efficiency η directly affects the system characteristics.

In order to send the output signal back to the quantum system, we must apply feedback control to the system. The continuous measurement record can be described by the homodyne detection photocurrent[41]

$$I(t) = \frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{\eta}\,\mathrm{Tr}(O\mathrm{e}^{\mathrm{i}\theta} + O^+\mathrm{e}^{-\mathrm{i}\theta}) + \frac{\mathrm{d}W(t)}{\mathrm{d}t}, \qquad (10)$$

where $\xi(t) = \mathrm{d}W(t)/\mathrm{d}t$.

The feedback Hamiltonian can be written as

$$H_f(t) = I(t)F, \qquad (11)$$

where F is a feedback operator, which is an arbitrary Hermitian operator.

We find that the measurement output $I(t)$ in Eq. (10) has the form of no-knowledge measurement. Because the observable operator O is Hermitian, Eqs. (10) and (11) show that, when $\theta = \pi/2$, even if $\eta = 1$, there will be no effective signal output, the output is only the noise caused by detection. At this time, the noise fed back to the input is no-knowledge feedback process.[42–44] Considering the role of the feedback Hamiltonian, the corresponding conditional master equation in Eq. (8) is rewritten as

$$\begin{aligned} \mathrm{d}\rho_t(t) &= -\mathrm{i}\,[H_0, \rho_t(t)]\,\mathrm{d}t + D[R]\rho_t(t)\,\mathrm{d}t \\ &\quad - \mathrm{i}\sqrt{\eta}\left[F, O\mathrm{e}^{\mathrm{i}\theta}\rho_t(t) + \rho_t(t)O\mathrm{e}^{-\mathrm{i}\theta}\right]\mathrm{d}t \\ &\quad + MD[O]\rho_t(t)\,\mathrm{d}t + D[F]\rho_t(t)\,\mathrm{d}t \\ &\quad + \mathrm{d}WH[\sqrt{\eta M}O\mathrm{e}^{\mathrm{i}\theta} - \mathrm{i}F]\rho_t(t)\,\mathrm{d}t. \qquad (12) \end{aligned}$$

The corresponding unconditional master equation in Eq. (12) is

$$\begin{aligned} \frac{\mathrm{d}\rho(t)}{\mathrm{d}t} &= -\mathrm{i}\,[H_0, \rho(t)] + D[R]\rho(t) + MD[O]\rho(t) + D[F]\rho(t) \\ &\quad - \mathrm{i}\sqrt{\eta}\left[F, O\mathrm{e}^{\mathrm{i}\theta}\rho(t) + \rho(t)O\mathrm{e}^{-\mathrm{i}\theta}\right]. \qquad (13) \end{aligned}$$

If multiple observation operators need to be measured at the same time, the above equation can be extended to[45]

$$\begin{aligned} \frac{\mathrm{d}\rho(t)}{\mathrm{d}t} &= -\mathrm{i}\,[H_0, \rho(t)] + D[R]\rho(t) \\ &\quad + \sum_k \left(MD[O_k]\rho(t) + D[F_k]\rho(t)\right. \\ &\quad \left. - \mathrm{i}\sqrt{\eta}[F_k, O_k\mathrm{e}^{\mathrm{i}\theta}\rho(t) + \rho(t)O_k\mathrm{e}^{-\mathrm{i}\theta}]\right), \qquad (14) \end{aligned}$$

where O_k is an Hermitian observable operator that admits no-knowledge measurement. In the above formula, it is assumed that the detection efficiency is equal, namely, $\eta_1 = \eta_2 = \cdots = \eta$. Each of them generates homodyne detection photocurrent, and yields the total measurement signals

$$I(t) = \sum_k \left[\sqrt{\eta}\,\mathrm{Tr}(O_k\mathrm{e}^{\mathrm{i}\theta} + O_k^+\mathrm{e}^{-\mathrm{i}\theta}) + \xi_k(t)\right]. \qquad (15)$$

To facilitate control, we suppose that the feedback operator is proportional to the observable operator, i.e.,

$$F = fO, \qquad (16)$$

where f is the feedback coefficient. In order to maximize QFI, it is necessary to determine the appropriate feedback coefficient. Therefore, the feedback coefficient f, as well as the detection efficiency η, is to be estimated in this paper.[46] In the following study, based on no-knowledge quantum feedback control, we focus on the influence of feedback coefficient and detection efficiency on QFI. Without loss of generality, three Hermitian operators σ_x, σ_z and $\sigma_x + \sigma_z$ are selected as feedback operators to study the effects of different feedback types on QFI.

We assume that the free Hamiltonian is $H_0 = \omega_0\sigma_z/2$. In Markovian reservoir, $D[R]\rho(t)$ is rewritten as

$$\begin{aligned} D[R]\rho(t) &= \gamma(2\sigma_-\rho(t)\sigma_+ - \sigma_+\sigma_-\rho(t) - \rho(t)\sigma_+\sigma_-)/2 \\ &\quad + \Gamma(\sigma_z\rho(t)\sigma_z - \rho(t)), \end{aligned}$$

where $2\sigma_\pm = \sigma_x \pm \mathrm{i}\,\sigma_y$, $\sigma = \begin{pmatrix}\sigma_x & \sigma_y & \sigma_z\end{pmatrix}$ are standard Pauli matrices. The initial state is chosen as $\rho(0) = |\psi\rangle\langle\psi|$ ($|\psi\rangle = \cos\varphi\,|0\rangle + \sin\varphi\,|1\rangle$, $\varphi \in [0, \pi/2]$).

4. Enhancing the precision of feedback coefficient estimation

4.1. Feedback control with $f\sigma_z$

Firstly, we consider the observable operator $O = \sigma_z$, which is a Hermitian operator. In this case, Eq. (13) including the feedback operator $F = fO = f\sigma_z$ is given by

$$\frac{\mathrm{d}\rho_{11}(t)}{\mathrm{d}t} = -\gamma\rho_{11}(t), \qquad (17)$$

$$\frac{\mathrm{d}\rho_{22}(t)}{\mathrm{d}t} = \gamma\rho_{11}(t), \tag{18}$$

$$\frac{\mathrm{d}\rho_{12}(t)}{\mathrm{d}t} = \Big(-\mathrm{i}\omega_0 - 2(M+f^2 - 2\sqrt{\eta}f\sin\theta) - \frac{\gamma+4\Gamma}{2}\Big)\rho_{12}(t), \tag{19}$$

$$\frac{\mathrm{d}\rho_{21}(t)}{\mathrm{d}t} = \Big(\mathrm{i}\omega_0 - 2(M+f^2 - 2\sqrt{\eta}f\sin\theta) - \frac{\gamma+4\Gamma}{2}\Big)\rho_{21}(t). \tag{20}$$

The solution can be obtained as follows:

$$\rho_{11}(t) = \mathrm{e}^{-\gamma t}\rho_{11}(0), \tag{21}$$

$$\rho_{22}(t) = 1 - \mathrm{e}^{-\gamma t}\rho_{11}(0), \tag{22}$$

$$\rho_{12}(t) = \mathrm{e}^{-(i\omega_0+X_1)t}\rho_{12}(0), \tag{23}$$

$$\rho_{21}(t) = \mathrm{e}^{(i\omega_0-X_1)t}\rho_{21}(0), \tag{24}$$

where

$$X_1 = \frac{\gamma+4\Gamma+4(M+f^2-2\sqrt{\eta}f\sin\theta)}{2}. \tag{25}$$

Therefore, based on the reduced density matrix $\rho(t)$, assuming the feedback coefficient is unknown, we discuss how to improve the precision of quantum parameters estimation based on feedback control. Substituting the estimated state $\rho(t)$ into Eq. (6), we obtain the QFI of the parameters to be estimated. Thus, the QFI of feedback coefficient f is derived by Eq. (6),

$$F_{\mathrm{Q}}^{f}(t) = \frac{64|\rho_{12}(0)|^2t^2(f-\sqrt{\eta}\sin\theta)^2\mathrm{e}^{-2X_1t}\left[\rho_{11}(0)\mathrm{e}^{-\gamma t}-\rho_{11}^2(0)\mathrm{e}^{-2\gamma t}\right]}{\rho_{11}(0)\mathrm{e}^{-\gamma t}-\rho_{11}^2(0)\mathrm{e}^{-2\gamma t}-|\rho_{12}(0)|^2\mathrm{e}^{-2X_1t}}. \tag{26}$$

Accordingly, the precision of feedback coefficient f is given by

$$(\delta f)^2 \geq \frac{\rho_{11}(0)\mathrm{e}^{-\gamma t}-\rho_{11}^2(0)\mathrm{e}^{-2\gamma t}-|\rho_{12}(0)|^2\mathrm{e}^{-2X_1t}}{64|\rho_{12}(0)|^2t^2(f-\sqrt{\eta}\sin\theta)^2\mathrm{e}^{-2X_1t}\left[\rho_{11}(0)\mathrm{e}^{-\gamma t}-\rho_{11}^2(0)\mathrm{e}^{-2\gamma t}\right]}. \tag{27}$$

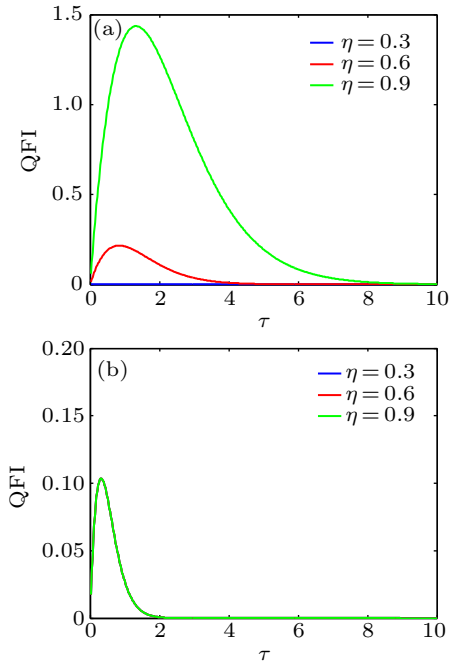


Fig. 1. Influence of the detection efficiency η on the QFI $F_{\mathrm{Q}}^{f}(t)$ under $f\sigma_z$ with $f = 0.5$, $M = 1$: (a) $\theta = \pi/2$, (b) $\theta = 0$.

From the above formula, in order to avoid zero QFI, the initial quantum state cannot be prepared in the eigenstate, which is helpful for improving the precision of real-time estimation of feedback coefficient. Under the feedback control $f\sigma_z$, time evolutions of $F_{\mathrm{Q}}^{f}(t)$ for different parameters f and ω_0 are drawn in Figs. 1–3. It should be stressed that in the following numerical simulations, we present the dynamical evolution of QFI with $\tau = \omega_0 t$. Based on current experimental technologies, two fine energy levels are commonly used in constituting qubits with real atoms and ions, where the transition frequency of the two-level system ω_0 (including the system of superconducting qubits) is at the GHz level. However, if a qubit is constructed using a mechanical oscillator, the frequency range varies from MHz to GHz. Thus, without loss of generality, in Figs. 1–7, time was plotted in units of $1/\omega_0$. At the same time, we set $\gamma/\omega_0 = 2.0\times10^{-6}$, $\Gamma/\omega_0 = 1.0\times10^{-6}$. For simplicity, the value of φ is assumed to be $\pi/4$.

Figures 1–3 clearly show that, under the action of $f\sigma_z$, the QFI $F_{\mathrm{Q}}^{f}(t)$ can be influenced by the detection efficiency η and feedback coefficient f. The QFI tends to be zero monotonically with time evolution for different values of η or f.

From Fig. 1, we can find that, under no-knowledge quantum feedback ($\theta = \pi/2$), the maximum value of QFI is enhanced with the increase of detection efficiency η. In the long-time limit, the decay rate of QFI with feedback is slowed down. Differently, for with-knowledge quantum feedback ($\theta = 0$), the detection efficiency η has little effect on QFI. Figure 2 shows that QFI is insensitive to the variation of feedback coefficient f. Comparing Fig. 1(a) with Fig. 1(b), and Fig. 2(a) with Fig. 2(b), we find that, utilizing no-knowledge measurement, QFI can be protected effectively.

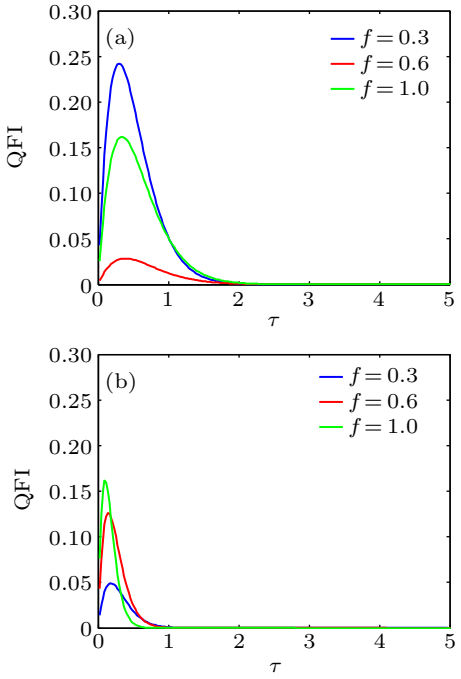


Fig. 2. Influence of the feedback coefficient f on the QFI $F_{\rm Q}^{f}(t)$ under $f\sigma_z$ with $\eta=0.5$, $M=1$: (a) $\theta=\pi/2$, (b) $\theta=0$.

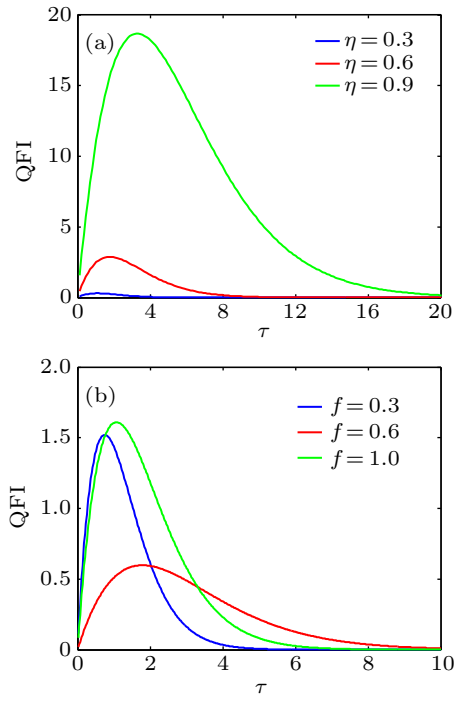


Fig. 3. Appropriately reducing the measurement strength is helpful to improve the QFI $F_{\rm Q}^{f}(t)$ under $f\sigma_z$ with $M=0.6$, $\theta=\pi/2$: (a) $f=0.3$, (b) $\eta=0.5$.

Another parameter that affects QFI and the protection effect is the measurement strength M. From Fig. 3, we know that reducing the measurement strength can significantly improve QFI $F_{\rm Q}^{f}(t)$ by no-knowledge measurement. Inversely, under the with-knowledge measurement, reducing the measurement strength has little effect on improving the QFI $F_{\rm Q}^{f}(t)$ (not shown). Our result indicates that, under no-knowledge feedback control, quantum weak measurement indeed enhances the QFI $F_{\rm Q}^{f}(t)$. Namely, combining no-knowledge feedback control with weak measurement will effectively improve the precision of unknown parameter estimation, which is the optimal way. In Refs. [47–49], it was proven that measurements, which returned a small amount of knowledge, can provide a good strategy for error correction and establish a tradeoff relation between information extraction and correction efficacy.[50] The no-knowledge feedback scheme with weak measurement proposed in this paper is consistent with those results.

4.2. The feedback control with $f\sigma_x$

As we know, zero QFI means that the information of unknown parameters cannot be obtained by measuring the output random variables (output state). Unfortunately, from Figs. 1–3, in a relatively short time, the QFI $F_{\rm Q}^{f}(t)$ firstly increases to a maximum value, then decreases and tends to be zero. In other words, the feedback $f\sigma_z$ cannot compensate the information of estimated parameter. In order to improve the protection effect of QFI $F_{\rm Q}^{f}(t)$, design of feedback Hamiltonian needs to be further optimized.

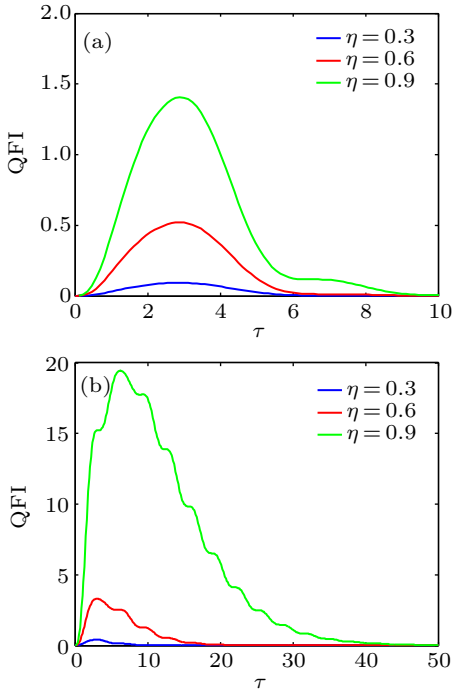


Fig. 4. Influence of the detection efficiency η on QFI $F_{\rm Q}^{f}(t)$ under $f\sigma_x$ with $f=0.3$, $\theta=\pi/2$: (a) $M=1$, (b) $M=0.6$.

Here we extend our study to no-knowledge quantum feedback. We choose $O=\sigma_x$ as the observable operator and the corresponding feedback is $F=f\sigma_x$. Then, the time-evolving density matrix of the qubit at any time can also be analytically given, see appendix A for detailed derivation. From Eqs. (6), (17), and (18), we can also obtain the analytical form of QFI $F_{\rm Q}^{f}(t)$. However, the analytical expression of QFI $F_{\rm Q}^{f}(t)$ is very cumbersome in this case. For this reason, we will analyze rel-

evant problems through numerical simulations. For simplicity, the following two sections only discuss the no-knowledge measurement.

In order to gain an insight into QFI of interest, the diagrams of QFI $F_{\mathrm{Q}}^{f}(t)$ as functions of time are shown in Figs. 4–5. Comparing Fig. 3(a) with Fig. 4(b), Fig. 2(a) with Fig. 5(a), Fig. 3(b) with Fig. 5(b), we can find that, significantly different from the case of the feedback operator $f\sigma_z$, the QFI $F_{\mathrm{Q}}^{f}(t)$ can remain greater than zero for a long time under the feedback operator $f\sigma_x$.

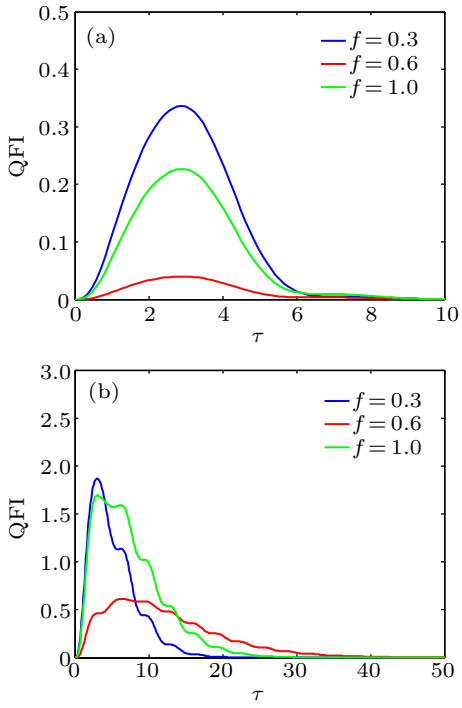


Fig. 5. Influence of the feedback coefficient f on QFI $F_{\mathrm{Q}}^{f}(t)$ under $f\sigma_x$ with $\eta=0.5$, $\theta=\pi/2$. (a) $M=1$, (b) $M=0.6$.

Above results show that QFI can be protected in a long time, which is conducive to improve the precision of quantum parameter estimation. In addition, as shown in Figs. 4 and 5, QFI can also be strengthened by carefully tuning the measurement strength M to an optimal value when the system reaches the dynamic balance. The results actually enlighten us that in the process of quantum measurement, when the observable operator is selective, we should choose the operator $f\sigma_x$ rather than operator $f\sigma_z$.

4.3. The feedback control with both $f\sigma_z$ and $f\sigma_x$

In order to further explore strategies to protect QFI, we observe both σ_x and σ_z at the same time. In this case, we can also obtain the reduced density matrix of the system, see appendix B for detailed derivation. Based on the results of numerical simulation (not shown), we find that the dynamical behaviors of QFI $F_{\mathrm{Q}}^{f}(t)$ with feedback $f\sigma_x$ and $f\sigma_z$ are very similar to that only with $f\sigma_x$.

5. Enhancing the precision of detection efficiency estimation

It is known that detection efficiency is an important physical quantity to judge the quality of detector. From the Eq. (13), we know that detection efficiency has a profound impact on the dynamic evolution of the system. Due to the mechanism of quantum feedback, in order to control the system effectively, it is more and more important to improve the precision of detection efficiency estimation in the feedback control. In this section, under no-knowledge quantum feedback control, we discuss how to improve the real-time estimation of detection efficiency η, and propose an optimized scheme to track the detection efficiency in real-time.

Notably, considering the observable operator$O=\sigma_z$, one can obtain the evolution of $F_{\mathrm{Q}}^{\eta}(t)$,

$$F_{\mathrm{Q}}^{\eta}(t)=\frac{(4|\rho_{12}(0)|ft\sin\theta)^2\,\mathrm{e}^{-2X_1 t}\left[\rho_{11}(0)\,\mathrm{e}^{-\gamma t}-\rho_{11}^2(0)\,\mathrm{e}^{-2\gamma t}\right]}{\eta\left(\rho_{11}(0)\,\mathrm{e}^{-\gamma t}-\rho_{11}^2(0)\,\mathrm{e}^{-2\gamma t}-|\rho_{12}(0)|^2\,\mathrm{e}^{-2X_1 t}\right)}. \tag{28}$$

Correspondingly, the precision of feedback coefficient η is given as

$$(\delta\eta)^2\geq\frac{\rho_{11}(0)\,\mathrm{e}^{-\gamma t}-\rho_{11}^2(0)\,\mathrm{e}^{-2\gamma t}-|\rho_{12}(0)|^2\,\mathrm{e}^{-2X_1 t}}{(4|\rho_{12}(0)|ft\sin\theta)^2\,\mathrm{e}^{-2X_1 t}\left[\rho_{11}(0)\,\mathrm{e}^{-\gamma t}-\rho_{11}^2(0)\,\mathrm{e}^{-2\gamma t}\right]}. \tag{29}$$

From Eqs. (28) and (29), QFI $F_{\mathrm{Q}}^{\eta}(t)$ monotonically tends to be zero, which indicates that the feedback operator $f\sigma_z$ cannot effectively protect the QFI $F_{\mathrm{Q}}^{\eta}(t)$ of detection efficiency. Next we focus on the effect of feedback operator $f\sigma_x$ on the QFI of detection efficiency.

Similar to the above process, for individual measurement σ_x, the QFI of parameters η and f are drawn in Figs. 6 and 7. Similar to Section 4, $F_{\mathrm{Q}}^{\eta}(t)$ is sensitive to the feedback coefficient and detection efficiency. Figures 6 and 7 further indicate that combining no-knowledge quantum feedback with weak measurement can effectively improve the precision of detection efficiency estimation. The effect of the feedback operator $f(\sigma_x+\sigma_z)$ on the QFI of detection efficiency is completely similar to that of $f\sigma_x$ and will not be discussed here.

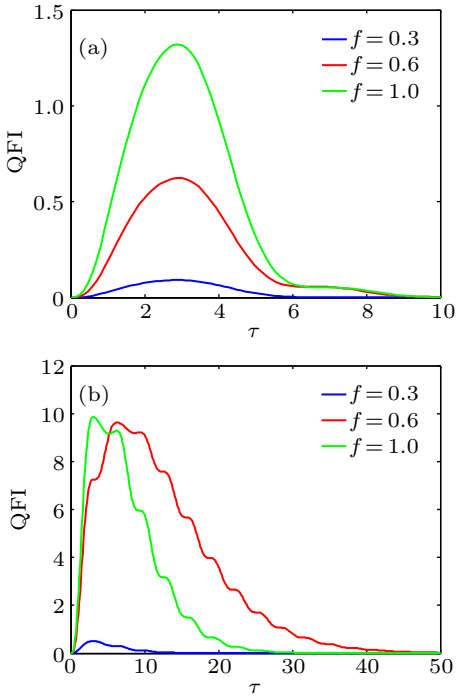


Fig. 6. Influence of the feedback coefficient f on QFI $F_{\mathrm{Q}}^{\eta}(t)$ under $f\sigma_x$ with $\eta=0.5$, $\theta=\pi/2$: (a) $M=1$, (b) $M=0.6$.

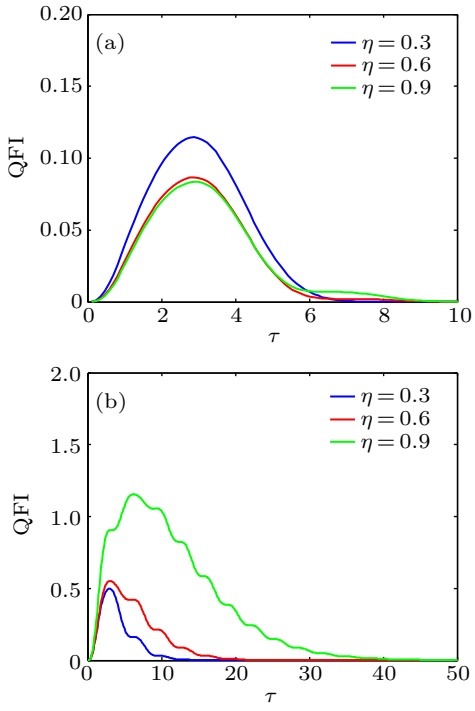


Fig. 7. Influence of the detection efficiency η on QFI $F_{\mathrm{Q}}^{\eta}(t)$ under $f\sigma_x$ with $f=0.3$, $\theta=\pi/2$: (a) $M=1$, (b) $M=0.6$.

6. Conclusions

First of all, it should be emphasized that quantum weak measurement and quantum feedback techniques have been considered to be useful decoherence suppression strategies. In order to protect QFI of the system, a control strategy which combines no-knowledge feedback with quantum weak measurement is proposed to enhance the suppression of decoherence. We choose that the feedback operator is proportional to the observable operator. Though exact numerical calculations, we research the effects of different feedback types on QFI. From Figs. 4–7, QFI of feedback coefficient or detection efficiency almost does not change after a period of time. In these cases, the information dissipation of parameter estimation is exactly equal to feedback compensation, and the system is in dynamical equilibrium. Our results show that both individual feedback $f\sigma_x$ and simultaneous feedback $f(\sigma_x+\sigma_z)$ are beneficial for improving the precision of feedback coefficient estimation, as well as that of detection efficiency estimation.

Combining no-knowledge quantum feedback control with quantum weak measurement, this paper makes a preliminary study on the optimization scheme of QFI protection in open quantum systems. The following two problems related to this optimization scheme need to be studied and solved.

(1) In this paper, the optimal estimations of the feedback coefficient and the detection coefficient are discussed in the case of single parameter estimation. Although the calculation of multi-parameter estimation is complicated,[51] it is still worth exploring estimations of both the feedback coefficient and the detection coefficient simultaneously and discuss whether there is an interaction between them.

(2) For simplicity, the feedback coefficient assumed in this paper is independent of time. In order to realize accurate quantum control, time-dependent control is needed in engineering practice. For example, the Gradient Ascent Pulse Engineering (GRAPE) algorithm is a gradient-based algorithm,[52–54] which was developed to search the optimal control for design of nuclear magnetic resonance pulse sequences, and now is extended to the scenario of quantum parameter estimation. Therefore, in order to improve the accuracy of quantum parameter estimation, time-dependent feedback control and the influence of feedback delay should be further considered.

Appendix A

If the selected observable operator is $O=\sigma_x$, the corresponding feedback operator is $F=f\sigma_x$. Substituting $O=\sigma_x$ and $F=f\sigma_x$ into Eq. (13), we can obtain the following differential equations:

$$\frac{\mathrm{d}\rho_{11}(t)}{\mathrm{d}t}=-(M+f^2-2\sqrt{\eta}f\sin\theta+\gamma)\rho_{11}(t)+(M+f^2-2\sqrt{\eta}f\sin\theta)\rho_{22}(t), \tag{A1}$$

$$\frac{\mathrm{d}\rho_{22}(t)}{\mathrm{d}t}=\left(M+f^2-2\sqrt{\eta}f\sin\theta+\gamma\right)\rho_{11}(t)-(M+f^2-2\sqrt{\eta}f\sin\theta)\rho_{22}(t), \tag{A2}$$

$$\frac{\mathrm{d}\rho_{12}(t)}{\mathrm{d}t}=-\left(\mathrm{i}\omega_0+\frac{\gamma+4\Gamma}{2}+M+f^2-2\sqrt{\eta}f\sin\theta\right)\rho_{12}(t)+(M+f^2-2\sqrt{\eta}f\sin\theta)\rho_{21}(t), \tag{A3}$$

$$\frac{\mathrm{d}\rho_{21}(t)}{\mathrm{d}t} = (M+f^2-2\sqrt{\eta}f\sin\theta)\rho_{12}(t) + \left(\mathrm{i}\omega_0 - \frac{\gamma+4\Gamma}{2} - M - f^2 + 2\sqrt{\eta}f\sin\theta g\right)\rho_{21}(t). \tag{A4}$$

We can find that Eqs. (A1)–(A4) are separated into two independent equation groups. For Eqs. (A1) and (A2), they satisfy the following matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\rho_{11}(t)\\ \rho_{22}(t)\end{pmatrix} = \begin{pmatrix}-(g+\gamma) & g\\ (g+\gamma) & -g\end{pmatrix}\begin{pmatrix}\rho_{11}(t)\\ \rho_{22}(t)\end{pmatrix} = G\begin{pmatrix}\rho_{11}(t)\\ \rho_{22}(t)\end{pmatrix}. \tag{A5}$$

with

$$g = M + f^2 - 2\sqrt{\eta}f\sin\theta. \tag{A6}$$

With the help of the eigenvectors and eigenvalues of the matrix G, the analytical solutions of $\rho_{11}(t)$ and $\rho_{22}(t)$ are obtained as follows:

$$\rho_{11}(t) = \frac{1}{2X_2}\left[(X_2-\gamma)+(X_2+\gamma)\mathrm{e}^{-X_2t}\right]\rho_{11}(0) + \frac{1}{2X_2}(X_2-\gamma)\left(1-\mathrm{e}^{-X_2t}\right)\rho_{22}(0), \tag{A7}$$

$$\rho_{22}(t) = \frac{1}{2X_2}(X_2+\gamma)\left(1-\mathrm{e}^{-X_2t}\right)\rho_{11}(0) + \frac{1}{2X_2}\left[(X_2+\gamma)+(X_2-\gamma)\mathrm{e}^{-X_2t}\right]\rho_{22}(0), \tag{A8}$$

where

$$X_2 = \gamma + 2(M+f^2-2\sqrt{\eta}f\sin\theta). \tag{A9}$$

Similarly, the analytical solutions of $\rho_{12}(t)$ and $\rho_{21}(t)$ are obtained as

$$\rho_{12}(t) = \mathrm{e}^{-(4\Gamma+X_2)t/2}\left[\frac{\Omega\cosh(\Omega t)-\mathrm{i}\omega_0\sinh(\Omega t)}{\Omega}\rho_{12}(0) + \frac{X_2-\gamma}{2\Omega}\sinh(\Omega t)\rho_{21}(0)\right], \tag{A10}$$

$$\rho_{21}(t) = \mathrm{e}^{-(4\Gamma+X_2)t/2}\left[\frac{X_2-\gamma}{2\Omega}\sinh(\Omega t)\rho_{12}(0) + \frac{\Omega\cosh(\Omega t)+i\omega_0\sinh(\Omega t)}{\Omega}\rho_{21}(0)\right], \tag{A11}$$

where

$$\Omega = \sqrt{(M+f^2-2\sqrt{\eta}f\sin\theta)^2-\omega_0^2}. \tag{A12}$$

Appendix B

Now the observable operators σ_z and σ_x are measured at the same time, and the output signals are sent back to the quantum system through the corresponding feedback operator $f\sigma_z$ and $f\sigma_x$ (here we choose the same feedback coefficient $f_z = f_x = f$). All the observable operators and feedback operators are replaced by Eq. (14), we have

$$\frac{\mathrm{d}\rho_{11}(t)}{\mathrm{d}t} = -\left(M+f^2-2\sqrt{\eta}f\sin\theta+\gamma\right)\rho_{11}(t) + (M+f^2-2\sqrt{\eta}f\sin\theta)\rho_{22}(t), \tag{B1}$$

$$\frac{\mathrm{d}\rho_{22}(t)}{\mathrm{d}t} = \left(M+f^2-2\sqrt{\eta}f\sin\theta+\gamma\right)\rho_{11}(t) - (M+f^2-2\sqrt{\eta}f\sin\theta)\rho_{22}(t), \tag{B2}$$

$$\frac{\mathrm{d}\rho_{12}(t)}{\mathrm{d}t} = -\left(\mathrm{i}\omega_0 + \frac{\gamma+4\Gamma}{2} + 3(M+f^2 - 2\sqrt{\eta}f\sin\theta)\right)\rho_{12}(t) + (M+f^2-2\sqrt{\eta}f\sin\theta)\rho_{21}(t), \tag{B3}$$

$$\frac{\mathrm{d}\rho_{21}(t)}{\mathrm{d}t} = (M+f^2-2\sqrt{\eta}f\sin\theta)\rho_{12}(t) + \left(\mathrm{i}\omega_0 - \frac{\gamma+4\Gamma}{2} - 3(M+f^2 - 2\sqrt{\eta}f\sin\theta)\right)\rho_{21}(t). \tag{B4}$$

Similarly to Appendix A, we obtain

$$\rho_{11}(t) = \frac{1}{2X_2}\left[(X_2-\gamma)+(X_2+\gamma)\mathrm{e}^{-X_2t}\right]\rho_{11}(0) + \frac{1}{2X_2}(X_2-\gamma)\left(1-\mathrm{e}^{-X_2t}\right)\rho_{22}(0), \tag{B5}$$

$$\rho_{22}(t) = \frac{1}{2X_2}(X_2+\gamma)\left(1-\mathrm{e}^{-X_2t}\right)\rho_{11}(0) + \frac{1}{2X_2}\left[(X_2+\gamma)+(X_2-\gamma)\mathrm{e}^{-X_2t}\right]\rho_{22}(0), \tag{B6}$$

$$\rho_{12}(t) = \mathrm{e}^{-(4\Gamma+3X_2-2\gamma)t/2}\left[\frac{\Omega\cosh(\Omega t)-\mathrm{i}\omega_0\sinh(\Omega t)}{\Omega}\rho_{12}(0) + \frac{X_2-\gamma}{2\Omega}\sinh(\Omega t)\rho_{21}(0)\right], \tag{B7}$$

$$\rho_{21}(t) = \mathrm{e}^{-(4\Gamma+3X_2-2\gamma)t/2}\left[\frac{X_2-\gamma}{\Omega}\sinh(\Omega t)\rho_{12}(0) + \frac{\Omega\cosh(\Omega t)+\mathrm{i}\omega_0\sinh(\Omega t)}{\Omega}\rho_{21}(0)\right]. \tag{B8}$$

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