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The (3+1)-dimensional generalized mKdV-ZK equation for ion-acoustic waves in quantum plasmas as well as its non-resonant multiwave solution*

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The quantum hydrodynamic model for ion-acoustic waves in plasmas is studied. First, we design a new disturbance expansion to describe the ion fluid velocity and electric field potential. It should be emphasized that the piecewise function perturbation form is new with great difference from the previous perturbation. Then, based on the piecewise function perturbation, a (3+1)-dimensional generalized modified Korteweg–de Vries Zakharov–Kuznetsov (mKdV-ZK) equation is derived for the first time, which is an extended form of the classical mKdV equation and the ZK equation. The (3+1)-dimensional generalized time-space fractional mKdV-ZK equation is constructed using the semi-inverse method and the fractional variational principle. Obviously, it is more accurate to depict some complex plasma processes and phenomena. Further, the conservation laws of the generalized time-space fractional mKdV-ZK equation are discussed. Finally, using the multi-exponential function method, the non-resonant multiwave solutions are constructed, and the characteristics of ion-acoustic waves are well described.

Keywords: ion-acoustic waves, piecewise function perturbation, (3+1)-dimensional generalized time-space fractional mKdV-ZK equation, non-resonant multiwave solution

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1. Introduction

Plasma research originated in the early twentieth century, and many scientists were interested in gas discharge. They believe that 90% of the tangible material in the universe exists in the form of plasmas. The traditional study of plasma physics is mainly carried out under the condition of high temperature and low density. In this case, quantum effect can be ignored. Since Pines and Robert[1] studied the properties of plasmons in low temperature and high density quantum plasmas in 1961, the practical application of plasma physics has attracted extensive attention and great interest of scientists. With the development of research on micro semiconductor devices, the quantum effect and boundary effect of plasmas[2] have attracted a great deal of attention from scientists. Quantum plasma has been widely used in different environments, such as ultra-small electronic equipment and microstructure systems, high-density astrophysical systems, high-density laser plasmas, nonlinear quantum optics and dust plasmas (Refs. [3,4]). Meanwhile, some ion-acoustic waves in quantum plasmas and quantum-related characteristics have also been deeply discussed.

With the in-depth study of singular perturbation theory, scholars have solved a lot of practical problems in the fields of boundary layers, solitons, marine science and so on. In the 21st century, there are some new hot issues (Refs. [5,6]) and research directions emerged in the theory and methods of singular perturbation. From ordinary differential equations to partial differential equations, the research on singular perturbation has entered a deeper level and a wider field.[7] Therefore, in this paper, we design a new perturbation expansion which is expressed in the form of piecewise function. The (3+1)-dimensional generalized mKdV-ZK equation is obtained for the first time. Thus, the application of the our new model may be particularly interesting in some fields of plasmas.

The KdV equation[8] is a fundamental equation in the field of mathematical physics. Research of the KdV equation promotes the development of nonlinear science, and a soliton theory[9] was formed to study nonlinear phenomena. Under the assumptions of approximation of long wave and finite amplitude, many wave equations describing weak nonlinear action[10] can be reduced to the KdV equation. Based on the nonlinear evolution equation, the mKdV equation[11] was studied using perturbation method or series expansion method. Meanwhile, the ZK equation, a generalization of the famous KdV equation in two-dimensional space, was derived in 1974

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by Zakharov and Kuznetsov[12] when they discussed the propagation of plane waves in magnetized plasma containing cold ions and hot isothermal electrons. Then, Shivamoggi also obtained it in the study of the quantum hydrodynamic. In this paper, we discuss a new generalized mKdV-ZK equation, which is closely related to plasma physics research and is beneficial to solve some practical problems.

In 1965, fractional derivative was proposed for the first time in the communication between Hospital and Leibniz.[13] Until the 1970s, Kirkby[14] presented the fact that there were a lot of fractional derivatives[15] in nature and many technical fields. The fractional derivative equation is more accurate and more suitable than traditional integer-order equations (Refs. [16,17]) to describe the complex ion-acoustic wave motion process[18] on the simulative results and characteristics. Therefore, it is necessary to study the (3+1)-dimensional generalized time-space fractional mKdV-ZK equation. In addition, it is of great significance to derive the conservation laws (Refs. [19,20]) and the symmetric analysis[21] of the equation in the plasma research.

With the development of science and technology, nonlinearity plays an increasingly essential role in the field of natural science and social science. The basic equations in many fields are nonlinear equations, such as in hydrodynamics,[22] high-energy electromagnetics, plasma physics, optical soliton communication (Refs. [23,24]) and other fields. Therefore, it is of great significance to study the exact solutions to nonlinear equations. For a long time, there is no unified solution for solving nonlinear evolution equations due to their complex and changeable nonlinearity. Many mathematicians and physicists have carried out a lot of work on those equations, and they found various sophisticated methods, such as the Darboux transformation,[25] Hirota bilinear method (Refs. [26,27]), Bell polynomials[28] and linear superposition principle, periodic wave solutions (Refs. [29–31]), travelling-wave solutions (Refs. [32,33]), solitary solution,[34] Painleve analysis, solitary solution (Refs. [35,36]), and Lie group method.[37]

In the 1950s, the concept of soliton is put forward. Furthermore, researchers studied the characteristics of solitons in the process of exploring nonlinear phenomena. With the development of soliton research, the Hirota bilinear method is one of the most classical and direct methods. Many physically important solutions[38] can be presented explicitly through a simple, specific form using the Hirota bilinear method,[39] such as rational solutions, solitons, positons, negatons, breathers, complexitons and interaction solutions.[40] The Hirota bilinear method provides an efficient technique to determine soliton solutions, and lump solutions[41] also exhibit diversity of exact solutions. In this paper, the non-resonant multiwave solution[42] of the equation is calculated using the method of multi-exponential function. The multi-exponential function method is a generalization of the Hirota bilinear method, which is used to construct multiwave solutions of nonlinear equations.

In this paper, a new (3+1)-dimensional generalized mKdV-ZK equation is yielded in quantum plasmas. The rest of this paper is organized as follows. In Section 2, we design a new piecewise function perturbation form. Based on the new perturbation form and multi-scale analysis, the generalized mKdV-ZK equation is obtained for the first time. In Section 3, starting from the (3+1)-dimensional generalized integer-order mKdV-ZK equation, the (3+1)-dimensional generalized time-space fractional mKdV-ZK equation is given with the help of the semi-inverse method, the Euler-Lagrangian equation and the fractional variational principle. In Section 4, the conservation laws of the (3+1)-dimensional generalized time-space fractional mKdV-ZK equation are considered with the help of Lie group analysis. The non-resonant multiwave solution to the (3+1)-dimensional generalized mKdV-ZK equation is founded using the multi-exponential function method in Section 5. In Section 6, the profiles of ion-acoustic waves with the variable coefficients are depicted. Finally, some conclusions are presented.

2. Derivation of the (3+1)-dimensional generalized mKdV-ZK equation

According to the continuity and momentum equation with the energy transport equation ignored, we can obtain the ion conservation of charge and momentum and Poisson's equation using the new variables.[43] Therefore, the relevant equations are given as follows:

$$\begin{cases} \dfrac{\partial n}{\partial t}+\dfrac{\partial (nu)}{\partial x}+\dfrac{\partial (nv)}{\partial y}+\dfrac{\partial (nw)}{\partial z}=0, \\ \dfrac{\partial n}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial w}{\partial z}=-\dfrac{\partial \Phi}{\partial x}-\dfrac{\partial \Phi}{\partial y}-\dfrac{\partial \Phi}{\partial z}, \\ \dfrac{\partial^2 \Phi}{\partial x^2}+\dfrac{\partial^2 \Phi}{\partial y^2}+\dfrac{\partial^2 \Phi}{\partial z^2}=n_{\mathrm{e}}-n, \end{cases} \tag{1}$$

where n and n_{e} are the ionic and electric number densities, Φ is the electric field potential, and u, v, w are the ion fluid velocities in the x, y and z directions.

Introducing the independent stretched variables as

$$\begin{cases} X=\varepsilon^{\frac{1}{2}}(x-t), \\ Y=\varepsilon^{\frac{1}{2}}(y-t), \\ Z=\varepsilon^{\frac{1}{2}}(z-t), \\ T=\varepsilon^{\frac{3}{2}}t, \end{cases} \tag{2}$$

where ε is a small parameter. Furthermore, we can obtain

$$\begin{cases} \dfrac{\partial}{\partial x} = \varepsilon^{\frac{1}{2}} \dfrac{\partial}{\partial X}, \\ \dfrac{\partial}{\partial y} = \varepsilon^{\frac{1}{2}} \dfrac{\partial}{\partial Y}, \\ \dfrac{\partial}{\partial z} = \varepsilon^{\frac{1}{2}} \dfrac{\partial}{\partial Z}, \\ \dfrac{\partial}{\partial t} = \varepsilon^{\frac{3}{2}} \dfrac{\partial}{\partial T} - \varepsilon^{\frac{1}{2}} \dfrac{\partial}{\partial X} - \varepsilon^{\frac{1}{2}} \dfrac{\partial}{\partial Y} - \varepsilon^{\frac{1}{2}} \dfrac{\partial}{\partial Z}. \end{cases} \tag{3}$$

In order to derive the objective equation, we design a new perturbation expansion which is expressed in the form of piecewise function. Furthermore, the ion fluid velocity and electric field potential are expanded with the help of a new perturbation expansion form. The form of piecewise function disturbance is as follows:

$$\psi = \begin{cases} \varepsilon^{\frac{3n+2}{4}} \psi_{2n+1}, \\ \varepsilon^{\frac{3n}{4}} \psi_{2n}, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{4}$$

where $\psi = [u, v, w, \Phi]$.

Therefore, the independent variables are expanded as

$$\begin{cases} n = 1 + \varepsilon^{\frac{1}{2}} n_1 + \varepsilon n_2 + \varepsilon^{\frac{3}{2}} n_3 + \cdots, \\ u = \varepsilon^{\frac{1}{2}} u_1 + \varepsilon^{\frac{3}{4}} u_2 + \varepsilon^{\frac{5}{4}} u_3 + \cdots, \\ v = \varepsilon^{\frac{1}{2}} v_1 + \varepsilon^{\frac{3}{4}} v_2 + \varepsilon^{\frac{5}{4}} v_3 + \cdots, \\ w = \varepsilon^{\frac{1}{2}} w_1 + \varepsilon^{\frac{3}{4}} w_2 + \varepsilon^{\frac{5}{4}} w_3 + \cdots, \\ \Phi = \varepsilon^{\frac{1}{2}} \Phi_1 + \varepsilon^{\frac{3}{4}} \Phi_2 + \varepsilon^{\frac{5}{4}} \Phi_3 + \cdots, \\ n_{\mathrm{e}} = \mathrm{e}^{\Phi}. \end{cases} \tag{5}$$

Substituting Eqs. (3) and (5) into Eq. (1), we can obtain

$$\varepsilon: \begin{cases} -\dfrac{\partial n_1}{\partial X} - \dfrac{\partial n_1}{\partial Y} - \dfrac{\partial n_1}{\partial Z} + \dfrac{\partial u_1}{\partial X} + \dfrac{\partial u_1}{\partial Y} + \dfrac{\partial u_1}{\partial Z} = 0, \\ -\dfrac{\partial n_1}{\partial X} - \dfrac{\partial n_1}{\partial Y} - \dfrac{\partial n_1}{\partial Z} = -\dfrac{\partial \Phi_1}{\partial X} - \dfrac{\partial \Phi_1}{\partial Y} - \dfrac{\partial \Phi_1}{\partial Z}, \end{cases} \tag{6}$$

$$\varepsilon^{\frac{3}{2}}: \begin{cases} -\dfrac{\partial n_2}{\partial X} - \dfrac{\partial n_2}{\partial Y} - \dfrac{\partial n_2}{\partial Z} + u_1 \dfrac{\partial u_1}{\partial X} + v_1 \dfrac{\partial v_1}{\partial Y} + w_1 \dfrac{\partial w_1}{\partial Z} = 0, \\ \dfrac{\partial^2 \Phi_1}{\partial X^2} + \dfrac{\partial^2 \Phi_1}{\partial Y^2} + \dfrac{\partial^2 \Phi_1}{\partial Z^2} = \dfrac{1}{6} {\Phi_1}^3 - n_3, \end{cases} \tag{7}$$

$$\varepsilon^2: \ \frac{\partial n_1}{\partial T} - \frac{\partial n_3}{\partial X} - \frac{\partial n_3}{\partial Y} - \frac{\partial n_3}{\partial Z} + \frac{\partial (n_2 u_1)}{\partial X} + \frac{\partial (n_2 v_1)}{\partial Y} + \frac{\partial (n_2 w_1)}{\partial Z} = 0. \tag{8}$$

Substituting Eqs. (6) and (7) into Eq. (8), and eliminating n_2, n_3, we have

$$\begin{aligned} &\frac{\partial \Phi_1}{\partial T} - \frac{1}{2} {\Phi_1}^2 \left(\frac{\partial \Phi_1}{\partial X} + \frac{\partial \Phi_1}{\partial Y} + \frac{\partial \Phi_1}{\partial Z} \right) \\ &+ \nabla \left(\frac{\partial^2 \Phi_1}{\partial X^2} + \frac{\partial^2 \Phi_1}{\partial Y^2} + \frac{\partial^2 \Phi_1}{\partial Z^2} \right) \\ &+ \frac{3}{2} {\Phi_1}^2 \left(\frac{\partial \Phi_1}{\partial X} + \frac{\partial \Phi_1}{\partial Y} + \frac{\partial \Phi_1}{\partial Z} \right) = 0. \end{aligned} \tag{9}$$

Letting $\Phi_1(X, Y, Z, T) = \Phi(X, Y, Z, T)$, and simplifying Eq. (9), we can finally obtain

$$\begin{aligned} &\frac{\partial \Phi}{\partial T} + \Phi^2 \left(\frac{\partial \Phi}{\partial X} + \frac{\partial \Phi}{\partial Y} + \frac{\partial \Phi}{\partial Z} \right) \\ &+ \nabla \left(\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Y^2} + \frac{\partial^2 \Phi}{\partial Z^2} \right) = 0. \end{aligned} \tag{10}$$

Equation (10) is a new equation derived from the quantum hydrodynamics model. We name it the (3+1)-dimensional generalized mKdV-ZK equation. Let us introduce some classical equations in the plasma.

The KdV equation can be rewritten as

$$u_t + \alpha u u_x + \beta u_{xxx} = 0,$$

the general form of the mKdV equation is in the form

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0,$$

and the ZK equation is

$$u_t + \gamma_1 u u_x + \gamma_2 u_{xxx} + \gamma_3 u_{xyy} = 0.$$

The KdV equation is a basic equation in the field of mathematical physics, and the mKdV equation is a generalization of the KdV equation which is widely used to describe ion-acoustic waves in plasmas. Previously, Shivamoggi obtained the ZK equation in the study of hydrodynamics. In this paper, we design a new piecewise function perturbation form, the (3+1)-dimensional generalized mKdV-ZK equation is derived for the first time, with great difference from the classical equations. There have been no such equations before, to our best knowledge. The (3+1)-dimensional generalized mKdV-ZK equation is a generalized form of the mKdV equation and the ZK equation. In addition, if Eq. (10) is (1+1)-dimensional, the equation will be the mKdV equation. The (3+1)-dimensional generalized mKdV-ZK equation represents a new research direction for quantum plasmas. The further discussion of the new equation is of great significant for research of quantum hydrodynamics.

3. Derivation of the (3+1)-dimensional generalized time-space fractional mKdV-ZK equation

In the above section, we have reached a new (3+1)-dimensional generalized integer-order mKdV-ZK equation. For further study of ion-acoustic waves in the quantum plasmas, we try to establish the (3+1)-dimensional generalized time-space fractional mKdV-ZK equation using the semi-inverse method and the fractional variational principle. At the beginning, we introduce some definitions as follows:

Definition 1[44] The modified Riemann–Liouville fractional derivative operator is defined as

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \left[\int_{\alpha}^{t} \frac{f(\tau) - f(\alpha)}{(t-\tau)^{\alpha}} \mathrm{d}\tau \right], \quad 0 \le \alpha < 1. \tag{11}$$

Definition 2[44] The Riemann–Liouville fractional derivation operator is defined as

$$D_T^{\omega} \Phi = \begin{cases} \dfrac{\partial^n \Phi}{\partial T^n}, & \omega = n, \\ \dfrac{1}{\Gamma(n-\omega)} \dfrac{\partial^n}{\partial T^n} \int_0^{T_0} (T-s)^{n-\omega-1} \Phi \, \mathrm{d}s, & n-1 \le \omega < n. \end{cases} \tag{12}$$

According to the (3+1)-dimensional generalized integer-order mKdV-ZK equation

$$\Phi_{\mathrm{T}} + \Phi^2(\Phi_X + \Phi_Y + \Phi_Z) + \nabla(\Phi_{XX} + \Phi_{YY} + \Phi_{ZZ}) = 0, \tag{13}$$

letting $\Phi = \Phi_1 + \Phi_2 + \Phi_3$, and assuming the potential function $\Phi_1(X,Y,Z,T) = A_X(X,Y,Z,T)$, $\Phi_2(X,Y,Z,T) = A_Y(X,Y,Z,T)$, $\Phi_3(X,Y,Z,T) = A_Z(X,Y,Z,T)$, we can obtain

$$\begin{aligned} &\frac{1}{3}(A_{XT} + A_{YT} + A_{ZT}) + A_X^2 A_{XX} + A_Y^2 A_{YY} \\ &+ A_Z^2 A_{ZZ} + A_{XXXX} + A_{XXXY} \\ &+ A_{XXZZ} + A_{XXYY} + A_{YYYY} + A_{YYZZ} \\ &+ A_{XXZZ} + A_{YYZZ} + A_{ZZZZ} = 0. \end{aligned} \tag{14}$$

Then, the function of the potential equation (14) has the form

$$\begin{aligned} J(A) = \iiint_V \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z \int_{T^*} \mathrm{d}T \Bigg[A \Bigg(&\frac{1}{3}a_1 A_{XT} + \frac{1}{3}a_2 A_{YT} \\ &+ \frac{1}{3}a_3 A_{ZT} + a_4 A_X^2 A_{XX} + a_5 A_Y^2 A_{YY} \\ &+ a_6 A_Z^2 A_{ZZ} + a_7 A_{XXXX} + a_8 A_{XXXY} \\ &+ a_9 A_{XXZZ} + a_{10} A_{XXYY} + a_{11} A_{YYYY} + a_{12} A_{YYZZ} \\ &+ a_{13} A_{XXZZ} + a_{14} A_{YYZZ} + a_{15} A_{ZZZZ} \Bigg) \Bigg], \end{aligned} \tag{15}$$

where b_i, $i = 1, \ldots, 15$ are Lagrangian multipliers. Integrating the above equation by parts and taking $A|_R = A_X|_R = A_Y|_R = A_Z|_R = A_T|_{T^*} = 0$, Eq. (15) can be written as

$$\begin{aligned} J(A) = \iiint_V \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z \int_{T^*} \mathrm{d}T \Bigg[&-\frac{1}{3}a_1 A_X A_T - \frac{1}{3}a_2 A_Y A_T \\ &- \frac{1}{3}a_3 A_Z A_T - \frac{1}{3}a_4 A_X^4 - \frac{1}{3}a_5 A_Y^4 - \frac{1}{3}a_6 A_Z^4 \\ &+ a_7 A_{XX}^2 + a_8 A_{XY}^2 + a_9 A_{XZ}^2 + a_{10} A_{XY}^2 + a_{11} A_{YY}^2 \\ &+ a_{12} A_{YZ}^2 + a_{13} A_{XZ}^2 + a_{14} A_{YZ}^2 + a_{15} A_{ZZ}^2 \Bigg]. \end{aligned} \tag{16}$$

Using the variation of the above function, integrating each term by parts and applying the optimum condition, the resulting form can be described as

$$\begin{aligned} &F(X,Y,Z,T,A_X,A_Y,A_Z,A_T,A_{XX},A_{XY}\cdots) \\ &= -\frac{\partial}{\partial T}\left(\frac{\partial F}{\partial A_T}\right) - \frac{\partial}{\partial X}\left(\frac{\partial F}{\partial A_X}\right) - \frac{\partial}{\partial Y}\left(\frac{\partial F}{\partial A_Y}\right) \\ &\quad - \frac{\partial}{\partial Z}\left(\frac{\partial F}{\partial A_Z}\right) + \frac{\partial^2}{\partial X^2}\left(\frac{\partial F}{\partial A_{XX}}\right) \\ &\quad + \frac{\partial^2}{\partial X \partial Y}\left(\frac{\partial F}{\partial A_{XY}}\right) + \frac{\partial^2}{\partial X \partial Z}\left(\frac{\partial F}{\partial A_{XZ}}\right) \\ &\quad + \frac{\partial^2}{\partial Y^2}\left(\frac{\partial F}{\partial A_{YY}}\right) + \frac{\partial^2}{\partial Y \partial Z}\left(\frac{\partial F}{\partial A_{YZ}}\right) + \frac{\partial^2}{\partial Z^2}\left(\frac{\partial F}{\partial A_{ZZ}}\right) \\ &= \frac{2}{3}a_1 A_{XT} + \frac{2}{3}a_2 A_{YT} + \frac{2}{3}a_3 A_{ZT} + 4a_4 A_X^2 A_{XX} \\ &\quad + 4a_5 A_Y^2 A_{YY} + 4a_6 A_Z^2 A_{ZZ} + 2a_7 A_{XXXX} \\ &\quad + 2a_8 A_{XXYY} + 2a_9 A_{XXZZ} + 2a_{10} A_{XXYY} \\ &\quad + 2a_{11} A_{YYYY} + 2a_{12} A_{YYZZ} + 2a_{13} A_{XXZZ} \\ &\quad + 2a_{14} A_{YYZZ} + 2a_{15} A_{ZZZZ}. \end{aligned} \tag{17}$$

Comparing Eq. (17) with Eq. (14), we obtain the following Lagrangian multipliers

$$\begin{aligned} &a_1 = a_2 = a_3 = \frac{1}{2}, \quad a_4 = a_5 = a_6 = \frac{1}{4}, \\ &a_i = \frac{1}{2}, \quad i = 7, 8, \ldots, 15. \end{aligned} \tag{18}$$

Then, the Lagrangian form of the (3+1)-dimensional generalized integer-order mKdV-ZK equation is

$$\begin{aligned} &L(A_T, A_X, A_Y, A_Z, A_{XX}, A_{XY}, A_{XZ}, A_{YY}, A_{YZ}, A_{ZZ}) \\ &= -\frac{1}{6}A_X A_T - \frac{1}{6}A_Y A_T - \frac{1}{6}A_Z A_T - \frac{1}{12}A_X^4 \\ &\quad - \frac{1}{12}A_Y^4 - \frac{1}{12}A_Z^4 + \frac{1}{2}A_{XX}^2 + \frac{1}{2}A_{XY}^2 \\ &\quad + \frac{1}{2}A_{XZ}^2 + \frac{1}{2}A_{XY}^2 + \frac{1}{2}A_{YY}^2 + \frac{1}{2}A_{YZ}^2 \\ &\quad + \frac{1}{2}A_{XZ}^2 + \frac{1}{2}A_{YZ}^2 + \frac{1}{2}A_{ZZ}^2. \end{aligned} \tag{19}$$

The Lagrangian form of the (3+1)-dimensional generalized time-space fractional mKdV-ZK equation has the following form

$$\begin{aligned} &F(D_T^{\omega}A, D_X^{\alpha}A, D_Y^{\beta}A, D_Z^{\gamma}A, D_X^{\alpha\alpha}A, D_X^{\alpha}D_Y^{\beta}A, \\ &D_X^{\alpha}D_Z^{\gamma}A, D_Y^{\beta\beta}A, D_Y^{\beta}D_Z^{\gamma}A, D_Z^{\gamma\gamma}A) \\ &= -\frac{1}{6}D_X^{\alpha}A\, D_T^{\omega}A - \frac{1}{6}D_Y^{\beta}A\, D_T^{\omega}A - \frac{1}{6}D_Z^{\gamma}A\, D_T^{\omega}A \\ &\quad - \frac{1}{12}(D_X^{\alpha}A)^4 - \frac{1}{12}(D_Y^{\beta}A)^4 - \frac{1}{12}(D_Z^{\gamma}A)^4 \\ &\quad + \frac{1}{2}(D_X^{\alpha\alpha}A)^2 + \frac{1}{2}(D_X^{\alpha}D_Y^{\beta}A)^2 + \frac{1}{2}(D_X^{\alpha}D_Z^{\gamma}A)^2 \\ &\quad + \frac{1}{2}(D_X^{\alpha}D_Y^{\beta}A)^2 + \frac{1}{2}(D_Y^{\beta\beta}A)^2 + \frac{1}{2}(D_Y^{\beta}D_Z^{\gamma}A)^2 \\ &\quad + \frac{1}{2}(D_X^{\alpha}D_Z^{\gamma}A)^2 + \frac{1}{2}(D_Y^{\beta}D_Z^{\gamma}A)^2 + \frac{1}{2}(D_Z^{\gamma\gamma}A)^2. \end{aligned} \tag{20}$$

Thus, the function form of the (3+1)-dimensional generalized time-space fractional mKdV-ZK equation can be obtained as

$$J_F(A) = \int_R (\mathrm{d}X)^\alpha \int_R (\mathrm{d}Y)^\beta \int_R (\mathrm{d}Z)^\gamma \int_{T^*} (\mathrm{d}T)^\omega \cdot F(D_T^\omega A, D_X^\alpha A, D_Y^\beta A, D_Z^\gamma A, D_X^{\alpha\alpha} A, D_X^\alpha D_Y^\beta A, D_X^\alpha D_Z^\gamma A, D_Y^{\beta\beta} A, D_Y^\beta D_Z^\gamma A, D_Z^{\gamma\gamma} A). \tag{21}$$

Using the Agrawal method the variation of function (21) can be rewritten as

$$\begin{aligned}\delta J_F(A) = \int_R (\mathrm{d}X)^\alpha \int_R (\mathrm{d}Y)^\beta \int_R (\mathrm{d}Z)^\gamma \int_{T^*} (\mathrm{d}T)^\omega \cdot \Bigg[& \left(\frac{\partial F}{\partial D_T^\omega A}\right) \delta D_T^\omega A + \left(\frac{\partial F}{\partial D_X^\alpha A}\right) \delta D_X^\alpha A + \left(\frac{\partial F}{\partial D_Y^\beta A}\right) \delta D_Y^\beta A \\ & + \left(\frac{\partial F}{\partial D_Z^\gamma A}\right) \delta D_Z^\gamma A + \left(\frac{\partial F}{\partial D_X^{\alpha\alpha} A}\right) \delta D_X^{\alpha\alpha} A + \left(\frac{\partial F}{\partial D_X^\alpha D_Y^\beta A}\right) \delta D_X^\alpha D_Y^\beta A + \left(\frac{\partial F}{\partial D_X^\alpha D_Z^\gamma A}\right) \delta D_X^\alpha D_Z^\gamma A \\ & + \left(\frac{\partial F}{\partial D_Y^{\beta\beta} A}\right) \delta D_Y^{\beta\beta} A + \left(\frac{\partial F}{\partial D_Y^\beta D_Z^\gamma A}\right) \delta D_Y^\beta D_Z^\gamma A + \left(\frac{\partial F}{\partial D_Z^{\gamma\gamma} A}\right) \delta D_Z^{\gamma\gamma} A \Bigg], \end{aligned} \tag{22}$$

where

$$\int_a^T (\mathrm{d}\tau)^i f(\tau) = i \int_a^T \mathrm{d}x (T-\tau)^i f(\tau). \tag{23}$$

The fractional integration by parts is given by

$$\int_a^b (\mathrm{d}\tau)^i f(x) D_z^i g(x) = \Gamma(1+i) \left[g(x) f(x)|_a^b - \int_a^b (\mathrm{d}x)^i g(x) D_z^i f(x) \right], \ f(x), g(x) \in [a,b]. \tag{24}$$

We can obtain

$$\begin{aligned}\delta J_F(A) = \int_R (\mathrm{d}X)^\alpha \int_R (\mathrm{d}Y)^\beta \int_R (\mathrm{d}Z)^\gamma \int_{T^*} (\mathrm{d}T)^\omega \cdot \Bigg[& -D_T^\omega \left(\frac{\partial F}{\partial D_T^\omega A}\right) - D_X^\alpha \left(\frac{\partial F}{\partial D_X^\alpha A}\right) - D_Y^\beta \left(\frac{\partial F}{\partial D_Y^\beta A}\right) \\ & - D_Z^\gamma \left(\frac{\partial F}{\partial D_Z^\gamma A}\right) + D_X^{\alpha\alpha} \left(\frac{\partial F}{\partial D_X^{\alpha\alpha}}\right) + D_X^\alpha D_Y^\beta \left(\frac{\partial F}{\partial D_X^\alpha D_Y^\beta A}\right) + D_X^\alpha D_Z^\gamma \left(\frac{\partial F}{\partial D_X^\alpha D_Z^\gamma A}\right) \\ & + D_Y^{\beta\beta} \left(\frac{\partial F}{\partial D_Y^{\beta\beta}}\right) + D_Y^\beta D_Z^\gamma \left(\frac{\partial F}{\partial D_Y^\beta D_Z^\gamma A}\right) + D_Z^{\gamma\gamma} \left(\frac{\partial F}{\partial D_Z^{\gamma\gamma}}\right) \Bigg]. \end{aligned} \tag{25}$$

Optimizing the variation Eq. (22), $\delta J_F(A) = 0$, the Euler–Lagrange equation of the (3+1)-dimensional generalized time-space fractional mKdV-ZK equation can be written as

$$\begin{aligned} & -D_T^\omega \left(\frac{\partial F}{\partial D_T^\omega A}\right) - D_X^\alpha \left(\frac{\partial F}{\partial D_X^\alpha A}\right) - D_Y^\beta \left(\frac{\partial F}{\partial D_Y^\beta A}\right) - D_Z^\gamma \left(\frac{\partial F}{\partial D_Z^\gamma A}\right) + D_X^{\alpha\alpha} \left(\frac{\partial F}{\partial D_X^{\alpha\alpha}}\right) \\ & + D_X^\alpha D_Y^\beta \left(\frac{\partial F}{\partial D_X^\alpha D_Y^\beta A}\right) + D_X^\alpha D_Z^\gamma \left(\frac{\partial F}{\partial D_X^\alpha D_Z^\gamma A}\right) + D_Y^{\beta\beta} \left(\frac{\partial F}{\partial D_Y^{\beta\beta}}\right) + D_Y^\beta D_Z^\gamma \left(\frac{\partial F}{\partial D_Y^\beta D_Z^\gamma A}\right) + D_Z^{\gamma\gamma} \left(\frac{\partial F}{\partial D_Z^{\gamma\gamma}}\right) = 0. \end{aligned} \tag{26}$$

Substituting Eq. (20) into Eq. (26) yields

$$\begin{aligned} & \frac{1}{3} D_X^\alpha D_T^\omega A + \frac{1}{3} D_Y^\beta D_T^\omega A + \frac{1}{3} D_Z^\gamma D_T^\omega A \\ & + (D_X^\alpha A)^2 D_X^{\alpha\alpha} A + D_X^{\alpha\alpha\alpha\alpha} A + D_X^{\alpha\alpha} D_Y^{\beta\beta} A \\ & + D_X^{\alpha\alpha} D_Z^{\gamma\gamma} A + (D_Y^\beta A)^2 D_Y^{\beta\beta} A + D_X^{\alpha\alpha} D_Y^{\beta\beta} A \\ & + D_Y^{\beta\beta\beta\beta} A + D_Y^{\beta\beta} D_Z^{\gamma\gamma} A + (D_Z^\gamma A)^2 D_Z^{\gamma\gamma} A \\ & + D_X^{\alpha\alpha} D_Z^{\gamma\gamma} A + D_Y^{\beta\beta} D_Z^{\gamma\gamma} A + D_Z^{\gamma\gamma\gamma\gamma} A = 0. \end{aligned} \tag{27}$$

Letting

$$D_X^\alpha A(X,Y,Z,T) = \Phi_1(X,Y,Z,T),$$
$$D_Y^\beta A(X,Y,Z,T) = \Phi_2(X,Y,Z,T),$$
$$D_Z^\gamma A(X,Y,Z,T) = \Phi_3(X,Y,Z,T),$$

and substituting them into Eq. (27), we have

$$\begin{aligned} & D_T^\omega \Phi + \Phi^2 \left(D_X^\alpha \Phi + D_Y^\beta \Phi + D_Z^\gamma \Phi\right) \\ & + D_X^{\alpha\alpha\alpha} \Phi + D_X^\alpha D_Y^{\beta\beta} \Phi + D_X^\alpha D_Z^{\gamma\gamma} \Phi \\ & + D_X^{\alpha\alpha} D_Y^\beta \Phi + D_Y^{\beta\beta\beta} \Phi + D_Y^\beta D_Z^{\gamma\gamma} \Phi \\ & + D_X^{\alpha\alpha} D_Z^\gamma \Phi + D_Y^{\beta\beta} D_Z^\gamma \Phi + D_Z^{\gamma\gamma\gamma} \Phi = 0. \end{aligned} \tag{28}$$

Equation (28) is the (3+1)-dimensional generalized time-space fractional mKdV-ZK equation. This new (3+1)-dimensional fractional-order equation will promote the study of fractional-order nonlinear equations and is more suitable to

describe ion-acoustic waves.

4. Conservation laws of the (3+1)-dimensional generalized time-space fractional mKdV-ZK equation

In Section 3, we have obtained the (3+1)-dimensional generalized time-space fractional mKdV-ZK equation. In this part, we consider the conservation laws[45] of the Eq. (28), which have great significance for the future research.

First, we present the (3+1)-dimensional generalized time-space fractional mKdV-ZK equation with four independent variables:

$$G(X,Y,Z,T,\Phi,D_T^{\omega}\Phi,D_X^{\alpha}\Phi,D_Y^{\beta}\Phi,D_Z^{\gamma}\Phi,\cdots)=0,$$
$$\omega,\alpha,\beta,\gamma>0. \tag{29}$$

Introduce the Lie group transformation of an one-parameter Lie group

$$\begin{cases}
\overline{X}\to X+\varepsilon\xi(X,Y,Z,T,\Phi)+o(\varepsilon^2),\\
\overline{Y}\to Y+\varepsilon\zeta(X,Y,Z,T,\Phi)+o(\varepsilon^2),\\
\overline{Z}\to Z+\varepsilon\upsilon(X,Y,Z,T,\Phi)+o(\varepsilon^2),\\
\overline{T}\to T+\varepsilon\tau(X,Y,Z,T,\Phi)+o(\varepsilon^2),\\
\overline{\Phi}\to \Phi+\varepsilon\eta(X,Y,Z,T,\Phi)+o(\varepsilon^2),\\
D_T^{\omega}\overline{\Phi}\to D_T^{\omega}\Phi+\varepsilon\eta_{\omega}^{T}(X,Y,Z,T,\Phi)+o(\varepsilon^2),\\
D_X^{\alpha}\overline{\Phi}\to D_X^{\alpha}\Phi+\varepsilon\eta_{\alpha}^{X}(X,Y,Z,T,\Phi)+o(\varepsilon^2),\\
D_Y^{\beta}\overline{\Phi}\to D_Y^{\beta}\Phi+\varepsilon\eta_{\beta}^{Y}(X,Y,Z,T,\Phi)+o(\varepsilon^2),\\
D_Z^{\gamma}\overline{\Phi}\to D_Z^{\gamma}\Phi+\varepsilon\eta_{\gamma}^{Z}(X,Y,Z,T,\Phi)+o(\varepsilon^2),\\
\cdots\cdots,
\end{cases} \tag{30}$$

where ξ, ζ, υ, τ and η are infinitesimal functions and $\varepsilon\ll 1$ means a group parameter.

The expressions of η_{ω}^{T}, η_{α}^{X}, η_{α}^{XXX}, $\eta_{\alpha,\beta}^{XYY}$, $\eta_{\alpha,\gamma}^{XZZ}\cdots$ are given as follows:

$$\begin{cases}
\eta_{\omega}^{T}=D_T^{\omega}(\eta)+\xi D_T^{\omega}(\Phi_X)-D_T^{\omega}(\xi\Phi_X)+\zeta D_T^{\omega}(\Phi_Y)-D_T^{\omega}(\zeta\Phi_Y)+\upsilon D_T^{\omega}(\Phi_Z)\\
\qquad -D_T^{\omega}(\upsilon\Phi_Z)+D_T^{\omega}(D_T(\tau)\Phi)-D_T^{\omega+1}(\tau\Phi)+\tau D_T^{\omega+1}(\Phi),\\
\eta_{\alpha}^{X}=D_X^{\alpha}(\eta)+D_X^{\alpha}(\Phi)D_X(\xi)-D_Y^{\beta}(\Phi)D_X(\zeta)-D_Z^{\gamma}(\Phi)D_X(\upsilon),\\
\eta_{\alpha}^{XXX}=D_X^{\alpha}(\eta_{\alpha}^{XX})-\Phi_{XXX}D_X^{\alpha}(\xi)-\Phi_{XXY}D_X^{\alpha}(\zeta)-\Phi_{XXZ}D_X^{\alpha}(\upsilon)-\Phi_{XXT}D_X^{\alpha}(\tau),\\
\eta_{\alpha,\beta}^{XYY}=D_X^{\alpha}(\eta_{\beta}^{YY})-\Phi_{XXY}D_X^{\alpha}(\xi)-\Phi_{XYY}D_X^{\alpha}(\zeta)-\Phi_{XYZ}D_X^{\alpha}(\upsilon)-\Phi_{XYT}D_X^{\alpha}(\tau),\\
\eta_{\alpha,\gamma}^{XZZ}=D_X^{\alpha}(\eta_{\gamma}^{ZZ})-\Phi_{XXZ}D_X^{\alpha}(\xi)-\Phi_{XYZ}D_X^{\alpha}(\zeta)-\Phi_{XZZ}D_X^{\alpha}(\upsilon)-\Phi_{XZT}D_X^{\alpha}(\tau),\\
\cdots\cdots\cdots
\end{cases} \tag{31}$$

where D_T, D_X, D_Y and D_Z represent the total derivative operator defined in the form

$$D_{X_j}=\frac{\partial}{\partial X_j}+\Phi_j\frac{\partial}{\partial\Phi}+\Phi_{jk}\frac{\partial}{\partial\Phi_k}+\cdots j,\quad k=1,\cdots,4, \tag{32}$$

obviously, $X_1=X$, $X_2=Y$, $X_3=Z$, $X_4=T$, and $\Phi_j=\frac{\partial\Phi}{\partial X_j}$, $\Phi_{jk}=\frac{\partial^2\Phi}{\partial X_j\partial X_k}$.

Apply the generalized Leibnitz rules as given by

$$D_t^{\omega}(f(t)\,g(t))=\sum_{n=0}^{\infty}\binom{\omega}{n}D_t^{\omega-n}f(t)\,D_t^{n}g(t),\ \omega>0,$$
$$\binom{\omega}{n}=\frac{(-1)^{n-1}\,\omega\,\Gamma(n-\omega)}{\Gamma(1-\omega)\,\Gamma(n+1)}. \tag{33}$$

The chain rule for a compound function is

$$\frac{\mathrm{d}^m f(g(t))}{\mathrm{d}t^m}=\sum_{k=0}^{m}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}\left[-g(t)^{k-r}\right]\frac{d^k f(g(t))}{\mathrm{d}t^k}. \tag{34}$$

We can obtain the expression of the extend symmetry operator η_{ω}^{T} as follows:

$$\begin{aligned}
\eta_{\omega}^{T}=&\frac{\partial^{\omega}\eta}{\partial T^{\omega}}-(\eta_{\Phi}-\omega D_T^{\omega}(\tau))\frac{\partial^{\omega}\Phi}{\partial T^{\omega}}-\Phi\frac{\partial^{\omega}\eta_{\Phi}}{\partial T^{\omega}}\\
&+\sum_{n=1}^{\infty}\left[\binom{\omega}{n}\frac{\partial^{\omega}\eta_{\Phi}}{\partial T^{\omega}}-\binom{\omega}{n+1}D_T^{n+1}(\tau)\right]D_T^{\omega-n}\Phi\\
&-\sum_{n=1}^{\infty}\binom{\omega}{n}\left[D_T^{n}(\xi)D_T^{\omega-n}(\Phi_X)\right.\\
&\left.+D_T^{n}(\zeta)D_T^{\omega-n}(\Phi_Y)+D_T^{n}(\upsilon)D_T^{\omega-n}(\Phi_Z)\right]+Ra,
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
Ra=&\sum_{n-2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\left[\binom{\omega}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{T^{n-\omega}}{\Gamma(n+1-\omega)}\right.\\
&\left.\times(-\Phi)^{r}\frac{\partial^{m}}{\partial T^{m}}(\Phi)^{k-r}\frac{\partial^{n-m-k}\eta}{\partial T^{n-m}\partial\Phi^{k}}\right].
\end{aligned} \tag{36}$$

In the same way, the extend infinitesimal η_{α}^{X}, η_{β}^{Y} and η_{γ}^{Z} can be given as follows:

$$\begin{aligned}
\eta_{\alpha}^{X}=&\frac{\partial^{\alpha}\eta}{\partial X^{\alpha}}+(\eta_{\Phi}-\alpha D_X^{\alpha}(\xi))\frac{\partial^{\alpha}\Phi}{\partial X^{\alpha}}-\Phi\frac{\partial^{\alpha}\eta_{\Phi}}{\partial X^{\alpha}}\\
&+\sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^{\alpha}\eta_{\Phi}}{\partial X^{\alpha}}-\binom{\alpha}{n+1}D_T^{n+1}(\xi)\right]D_Y^{\alpha-n}\Phi\\
&-\sum_{n=1}^{\infty}\binom{\alpha}{n}\left[D_X^{n}(\zeta)D_X^{\alpha-n}(\Phi_Y)+D_X^{n}(\upsilon)D_X^{\alpha-n}(\Phi_Z)\right.\\
&\left.+D_X^{n}(\tau)D_X^{\alpha-n}(\Phi_T)\right]+Rb,
\end{aligned} \tag{37}$$

where

$$Rb = \sum_{n-2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\left[\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{X^{n-\alpha}}{\Gamma(n+1-\alpha)} \times(-\Phi)^r\frac{\partial^m}{\partial X^m}(\Phi)^{k-r}\frac{\partial^{n-m-k}\eta}{\partial X^{n-m}\partial\Phi^k}\right], \tag{38}$$

$$\eta^Y_\beta = \frac{\partial^\beta\eta}{\partial Y^\beta}+(\eta_\Phi-\beta D^\beta_Y(\zeta))\frac{\partial^\beta\eta_\Phi}{\partial Y^\beta}-\Phi\frac{\partial^\beta\eta_\Phi}{\partial Y^\beta} +\sum_{n=1}^{\infty}\left[\binom{\beta}{n}\frac{\partial^\beta\eta_\Phi}{\partial Y^\beta}-\binom{\beta}{n+1}D^{n+1}_T(\zeta)\right]D^{\beta-n}_Y\Phi -\sum_{n=1}^{\infty}\binom{\beta}{n}\left[D^n_Y(\xi)D^{\beta-n}_Y(\Phi_X)+D^n_Y(\upsilon)D^{\beta-n}_Z(\Phi_Z) +D^n_Y(\tau)D^{\beta-n}_Y(\Phi_T)\right]+Rc, \tag{39}$$

where

$$Rc = \sum_{n-2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\left[\binom{\beta}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{Y^{n-\beta}}{\Gamma(n+1-\beta)} \times(-\Phi)^r\frac{\partial^m}{\partial Y^m}(\Phi)^{k-r}\frac{\partial^{n-m-k}\eta}{\partial Y^{n-m}\partial\Phi^k}\right], \tag{40}$$

$$\eta^Z_\gamma = \frac{\partial^\gamma\eta}{\partial Z^\gamma}+(\eta_\Phi-\gamma D^\gamma_Z(\upsilon))\frac{\partial^\gamma\Phi}{\partial Z^\gamma}-\Phi\frac{\partial^\gamma\eta_\Phi}{\partial Z^\gamma} +\sum_{n=1}^{\infty}\left[\binom{\gamma}{n}\frac{\partial^\gamma\eta_\Phi}{\partial Z^\gamma}-\binom{\gamma}{n+1}D^{n+1}_T(\upsilon)\right]D^{\gamma-n}_Z\Phi -\sum_{n=1}^{\infty}\binom{\gamma}{n}\left[D^n_Z(\xi)D^{\gamma-n}_Z(\Phi_X)+D^n_Z(\zeta)D^{\gamma-n}_Z(\Phi_Y) +D^n_Z(\tau)D^{\gamma-n}_Z(\Phi_T)\right]+Rd, \tag{41}$$

where

$$Rd = \sum_{n-2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\left[\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{Z^{n-\gamma}}{\Gamma(n+1-\gamma)} \times(-\Phi)^r\frac{\partial^m}{\partial Z^m}(\Phi)^{k-r}\frac{\partial^{n-m-k}\eta}{\partial Z^{n-m}\partial\Phi^k}\right]. \tag{42}$$

By the Lie symmetry theory, the infinitesimal generator has the form

$$V=\xi\frac{\partial}{\partial X}+\zeta\frac{\partial}{\partial Y}+\upsilon\frac{\partial}{\partial Z}+\tau\frac{\partial}{\partial T}+\eta\frac{\partial}{\partial\Phi}. \tag{43}$$

Under the infinitesimal transformations, the invariance of the system (29) leads to the following invariance condition

$$Pr^{(n)}V(\Delta)|_{\Delta=0}=0,\ n=1,2,3,\cdots, \tag{44}$$

where $\Delta=G(X,Y,Z,T,D^\omega_T\Phi,D^\alpha_X\Phi,\cdots)$.

Then, the invariance criterion could be written in the form

$$\eta^T_\omega+2\Phi_\eta(D^\omega_X\Phi+D^\beta_Y\Phi+D^\gamma_Z\Phi) +\Phi^2(\eta^X_\alpha+\eta^Y_\beta+\eta^Z_\gamma)+\eta^{XXX}_\alpha+\eta^{XYY}_{\alpha,\beta} +\eta^{XZZ}_{\alpha,\gamma}+\eta^{XXY}_{\alpha,\beta}+\eta^{YYY}_\beta+\eta^{YZZ}_{\beta,\gamma} +\eta^{XXZ}_{\alpha,\gamma}+\eta^{YYZ}_{\beta,\gamma}+\eta^{ZZZ}_\gamma=0. \tag{45}$$

Substituting Eqs. (31), (32), (35), (37), and (39) into Eq. (45), we can obtain

$$\xi=c_1\frac{X}{\alpha}+c_3,\quad \zeta=c_1\frac{Y}{\beta}+c_2,\quad \upsilon=2c_1\frac{Z}{\gamma}+c_4, \tau=3c_1\frac{T}{\omega}+c_5,\quad \eta=-3c_1\Phi. \tag{46}$$

Since the corresponding infinitesimal generators can be expressed as

$$V_1=\frac{\partial}{\partial T},\quad V_2=\frac{\partial}{\partial X},\quad V_3=\frac{\partial}{\partial Y},\quad V_4=\frac{\partial}{\partial Z}, V_5=\frac{X}{\alpha}\frac{\partial}{\partial X}+\frac{Y}{\beta}\frac{\partial}{\partial Y}+\frac{2Z}{\gamma}\frac{\partial}{\partial Z}+\frac{3T}{\omega}\frac{\partial}{\partial T}-3\Phi\frac{\partial}{\partial\Phi}. \tag{47}$$

With the help of Lie group analysis, we acquire the conservation laws of Eq. (28) defined by the following form

$$D_T(C^T)+D_X(C^X)+D_Y(C^Y)+D_Z(C^Z)=0, \tag{48}$$

where C^T, C^X, C^Y and C^Z denote the conserved vectors.

A formal Lagrangian for the (3+1)-dimensional generalized time-space fractional mKdV-ZK equation can be expressed as

$$\mathcal{L} = u(X,Y,Z,T)[D^\omega_T\Phi+\Phi^2(D^\alpha_X\Phi+D^\beta_Y\Phi+D^\gamma_Z\Phi) +D^{\alpha\alpha\alpha}_X\Phi+D^\alpha_XD^{\beta\beta}_Y\Phi+D^\alpha_XD^{\gamma\gamma}_Z\Phi +D^{\alpha\alpha}_XD^\beta_Y\Phi+D^{\beta\beta\beta}_Y\Phi+D^\beta_YD^{\gamma\gamma}_Z\Phi+D^{\alpha\alpha}_XD^\gamma_Z\Phi +D^{\beta\beta}_YD^\gamma_Z\Phi+D^{\gamma\gamma\gamma}_Z\Phi]=0, \tag{49}$$

where $u(X,Y,Z,T)$ is a new dependent variable.

The adjoint equation of Eq. (28) is given by

$$Q^*=\frac{\delta\mathcal{L}}{\delta\Phi}=0, \tag{50}$$

where the expression of the Ruler-lagrange operator $\frac{\delta}{\delta\Phi}$ is defined as

$$\frac{\delta}{\delta\Phi} = \frac{\partial}{\partial\Phi}+(D^\omega_T)^*\frac{\partial}{\partial D^\omega_T\Phi}+(D^\alpha_X)^*\frac{\partial}{\partial D^\alpha_X\Phi} +(D^\beta_Y)^*\frac{\partial}{\partial D^\beta_Y\Phi}+(D^\gamma_Z)^*\frac{\partial}{\partial D^\gamma_Z\Phi} +\sum_{k=1}^{\infty}(-1)^kD_{i1}D_{i2}\cdots D_{ik}\frac{\partial}{\partial\Phi_{i1\,i2\cdots ik}}, \tag{51}$$

where $(D^\omega_T)^*$, $(D^\alpha_X)^*$, $(D^\beta_Y)^*$ and $(D^\gamma_Z)^*$ mean the adjoint operators of D^ω_T, D^α_X, D^β_Y and D^γ_Z.

Therefore, we obtain the adjoint equation of Eq. (50) as follows:

$$Q^* = (D^\omega_T)^*u+\Phi^2\left[(D^\alpha_X)^*u+(D^\beta_Y)^*u+(D^\gamma_Z)^*u\right] -(D^{\alpha\alpha\alpha}_X)^*u-(D^\alpha_XD^{\beta\beta}_Y)^*u-(D^\alpha_XD^{\gamma\gamma}_Z)^*u -(D^{\alpha\alpha}_XD^\beta_Y)^*u-(D^{\beta\beta\beta}_Y)^*u-(D^\beta_YD^\gamma_Z\gamma)^*u -(D^{\alpha\alpha}_XD^\gamma_Z)^*u-(D^{\beta\beta}_YD^\gamma_Z)^*u-(D^{\gamma\gamma\gamma}_Z)^*u=0. \tag{52}$$

In addition, $W=\eta-\tau\Phi_T-\xi\Phi_X-\zeta\Phi_Y-\upsilon\Phi_Z$ is the Lie characteristic function, we have

$$W_1=\Phi_X,\quad W_2=\Phi_Y,\quad W_3=\Phi_Z,\quad W_4=\Phi_T,$$

$$W_5 = 3\Phi\frac{\partial}{\partial\Phi} - \frac{X}{\omega}\frac{\partial}{\partial X} - \frac{Y}{\beta}\frac{\partial}{\partial Y} - \frac{2Z}{\gamma}\frac{\partial}{\partial Z} - \frac{3T}{\omega}\frac{\partial}{\partial T}. \quad (53)$$

Using the Riemann–Liouville fractional derivative, the components of the conserved vectors have the forms

$$\begin{aligned}
C^T &= \tau\mathcal{L} + \sum_{k=0}^{n-1}(-1)^k D_T^{\omega-1-k}(W) D_T^k \frac{\partial\mathcal{L}}{\partial(D_T^\omega\Phi)} \\
&\quad - (-1)^n J\left(W, D_T^n \frac{\partial\mathcal{L}}{\partial(D_T^\omega\Phi)}\right), \\
C^X &= \xi\mathcal{L} + W\left(\frac{\partial\mathcal{L}}{\partial D_X^\alpha\Phi} + D_X^{\alpha\alpha}\frac{\partial\mathcal{L}}{\partial D_X^{\alpha\alpha\alpha}\Phi}\right. \\
&\quad \left. + D_Y^{\beta\beta}\frac{\partial\mathcal{L}}{\partial D_X^\alpha D_Y^{\beta\beta}\Phi} + D_Z^{\gamma\gamma}\frac{\partial\mathcal{L}}{\partial D_X^\alpha D_Z^{\gamma\gamma}\Phi}\right) \\
&\quad - D_X^\alpha(W)\left[D_X^\alpha\frac{\partial\mathcal{L}}{\partial D_X^{\alpha\alpha\alpha}\Phi}\right] - D_Y^\beta(W)\left[D_Y^\beta\frac{\partial\mathcal{L}}{\partial D_X^\alpha D_Y^{\beta\beta}\Phi}\right] \\
&\quad - D_Z^\gamma(W)\left[D_Z^\gamma\frac{\partial\mathcal{L}}{\partial D_X^\alpha D_Z^{\gamma\gamma}\Phi}\right] \\
&\quad + D_X^{\alpha\alpha}(W)\left[\frac{\partial\mathcal{L}}{\partial D_X^{\alpha\alpha\alpha}}\right] + D_Y^{\beta\beta}(W)\left[\frac{\partial\mathcal{L}}{\partial D_X^\alpha D_Y^{\beta\beta}\Phi}\right] \\
&\quad + D_Z^{\gamma\gamma}\left[\frac{\partial\mathcal{L}}{\partial D_X^\alpha D_Z^{\gamma\gamma}\Phi}\right], \\
C^Y &= \zeta\mathcal{L} + \sum_{k=0}^{m-1}(-1)^k D_Y^{\beta-1-k}(W) D_Y^k \frac{\partial\mathcal{L}}{\partial(D_Y^\beta\Phi)} \\
&\quad - (-1)^n J\left(W, D_Y^m \frac{\partial\mathcal{L}}{\partial(D_Y^\beta\Phi)}\right), \\
C^Z &= \upsilon\mathcal{L} + \sum_{k=0}^{s-1}(-1)^k D_Z^{\gamma-1-k}(W) D_Z^k \frac{\partial\mathcal{L}}{\partial(D_Z^\gamma\Phi)} \\
&\quad - (-1)^n J\left(W, D_Z^s \frac{\partial\mathcal{L}}{\partial(D_Z^\gamma\Phi)}\right), \quad (54)
\end{aligned}$$

where $n = [\omega]+1$, $m = [\beta]+1$, $s = [\gamma]+1$, and

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T \frac{f(\tau,x,y)g(\mu,x,y)}{(\mu-\tau)^{\alpha+1-n}}\,\mathrm{d}\mu\,\mathrm{d}\tau.$$

5. Non-resonant multiwave solutions for the (3+1)-dimensional generalized mKdV-ZK equation

Based on the multi-exponential function method and the linear superposition principle,[46] here we propose a new Hirota bilinear equation and construct the multiwave solution of the (3+1)-dimensional generalized mKdV-ZK equation. First, the equation transformed into a bilinear equation which is expressed in the form of D-operator using the variable transformation. Then, expanding the auxiliary function f to the power series of small parameter ε, the dispersion relation of the (3+1)-dimensional generalized mKdV-ZK equation is obtained by comparing the same power series of parameter ε. Finally, we solve the linear equation of function f and get the specific form of function f. Meanwhile, the exact solutions of the equation can be obtained with the help of dispersion relation and variable transformation.

We assume that Φ is a solution to the nonlinear evolution equation, and take the transformation of the relevant variables $\Phi = 2(\ln f)_X$, where $f = f(X,Y,Z,T)$ is a auxiliary function of X, Y, Z and T. We can obtain

$$\begin{aligned}
&(D_T + D_X^3 + D_X D_Y^2 + D_X D_Z^2 + D_X^2 D_Y + D_Y^3 \\
&+ D_Y D_Z^2 + D_X^2 D_Z + D_Y^2 D_Z + D_Z^3) f\cdot f = 0. \quad (55)
\end{aligned}$$

5.1. Single wave solution

We assume

$$f = 1 + \varepsilon \mathrm{e}^{\theta}, \quad \theta = kX + lY + mZ - wT. \quad (56)$$

Substitute Eq. (56) into the bilinear Eq. (55), where ε, k, l, m and w are constants and ε is small parameter. Rearrange the power of small parameter ε, take the coefficient of each power of ε and set it to 0. We can obtain the following dispersion relation

$$w = (k+l+m)(k^2+l^2+m^2), \quad (57)$$

substitute Eqs. (56) and (57) into the transformation formula $\Phi = 2(\ln f)_X$, the single wave solution can be written as

$$\Phi = \frac{2\varepsilon k \mathrm{e}^{kX+lY+mZ-wT}}{1+\mathrm{e}^{kX+lY+mZ-wT}}. \quad (58)$$

5.2. Two-wave solution

We assume

$$\begin{aligned}
f &= 1 + \varepsilon_1 \mathrm{e}^{\theta_1} + \varepsilon_2 \mathrm{e}^{\theta_2} + \varepsilon_1\varepsilon_2 a_{12}\mathrm{e}^{\theta_1+\theta_2}, \\
\theta_i &= k_i X + l_i Y + m_i Z - w_i T, \quad i = 1,2, \quad (59)
\end{aligned}$$

and substitute Eq. (59) into the bilinear Eq. (55), the two-wave solution is derived as

$$\Phi = \frac{2\left[k_1\varepsilon_1\mathrm{e}^{\theta_1} + k_2\varepsilon_2\mathrm{e}^{\theta_2} + a_{12}(k_1+k_2)\varepsilon_1\varepsilon_2\mathrm{e}^{\theta_1+\theta_2}\right]}{1+\varepsilon_1\mathrm{e}^{\theta_1}+\varepsilon_2\mathrm{e}^{\theta_2}+a_{12}\varepsilon_1\varepsilon_2\mathrm{e}^{\theta_1+\theta_2}}, \quad (60)$$

where

$$\begin{aligned}
w_i &= (k_i+l_i+m_i)(k_i^2+l_i^2+m_i^2), \quad i = 1,2, \\
a_{12} &= \frac{(w_1-w_2) - [(k_1-k_2)+(l_1-l_2)+(m_1-m_2)][(k_1-k_2)^2+(l_1-l_2)^2+(m_1-m_2)^2]}{[(k_1-k_2)+(l_1-l_2)+(m_1-m_2)][(k_1-k_2)^2+(l_1-l_2)^2+(m_1-m_2)^2] - (w_1-w_2)}.
\end{aligned}$$

5.3. Three-wave solution

We assume

$$f = 1+\varepsilon_1 e^{\theta_1}+\varepsilon_2 e^{\theta_2}+\varepsilon_3 e^{\theta_3}+\varepsilon_1\varepsilon_2 a_{12}e^{\theta_1+\theta_2}+\varepsilon_1\varepsilon_3 a_{13}e^{\theta_1+\theta_3}+\varepsilon_2\varepsilon_3 a_{23}e^{\theta_2+\theta_3}+\varepsilon_1\varepsilon_2\varepsilon_3 a_{123}e^{\theta_1+\theta_2+\theta_3},$$
$$\theta_i = k_i X + l_i Y + m_i Z - w_i T, \quad i = 1,2, \tag{61}$$

substitute Eq. (61) into the bilinear Eq. (55), the three-wave solution is derived as

$$\Phi = \frac{2\sum_{i=1}^{3} k_i\varepsilon_i e^{\theta_i}+\sum_{i\le i<j\le 3}(k_i+k_j)\varepsilon_i\varepsilon_j a_{ij}e^{\theta_i+\theta_j}+(k_1+k_2+k_3)\varepsilon_1\varepsilon_2\varepsilon_3 a_{123}e^{\theta_1+\theta_2+\theta_3}}{1+\sum_{i=1}^{3}\varepsilon_i e^{\theta_i}+\sum_{i\le i<j\le 3}\varepsilon_i\varepsilon_j a_{ij}e^{\theta_i+\theta_j}+\varepsilon_1\varepsilon_2\varepsilon_3 a_{123}e^{\theta_1+\theta_2+\theta_3}}, \tag{62}$$

where

$$w_i = (k_i+l_i+m_i)(k_i^2+l_i^2+m_i^2), \quad i = 1,2,3, \quad a_{123} = a_{12}a_{13}a_{23},$$
$$a_{ij} = \frac{(w_i-w_j)-[(k_i-k_j)+(l_i-l_j)+(m_i-m_j)][(k_i-k_j)^2+(l_i-l_j)^2+(m_i-m_j)^2]}{[(k_i-k_j)+(l_i-l_j)+(m_i-m_j)][(k_i-k_j)^2+(l_i-l_j)^2+(m_i-m_j)^2]-(w_i-w_j)}.$$

6. Discussions on non-resonant multiwave solutions to the (3+1)-dimensional generalized mKdV-ZK equation

In the above section, the (3+1)-dimensional generalized mKdV-ZK equation is transformed into the Hirota bilinear equation with the variable transformation $\Phi = 2(\ln f)_X$, then non-resonant multiwave solutions of the equation are given using the multi-exponential function method. In the following we discuss the motion characteristics of ion-acoustic waves by changing the parameters. Figure 1 shows the propagation of the single wave on the X–Y, X–Z and Y–Z planes with $T = 1$. We find that the single wave does not change in shape and amplitude in the propagation process.

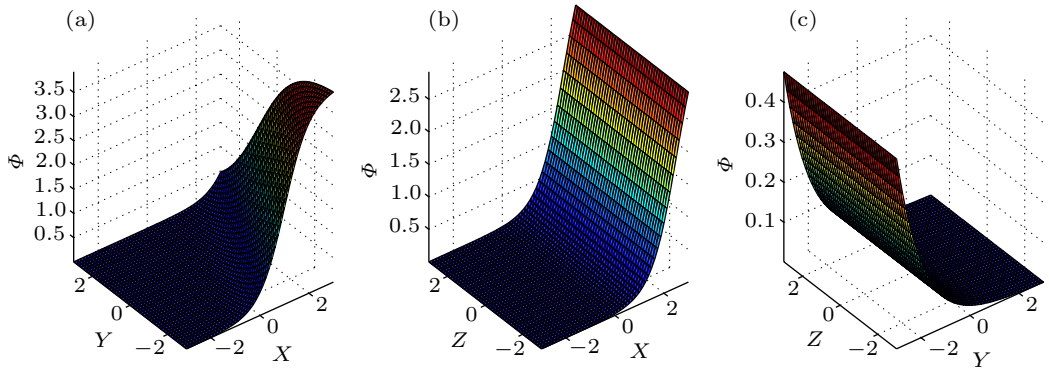


Fig. 1. The profile of the single-wave solution (58) with $\varepsilon = 1$, $k = 2$, $l = -1$, $m = 0$ and $T = 1$.

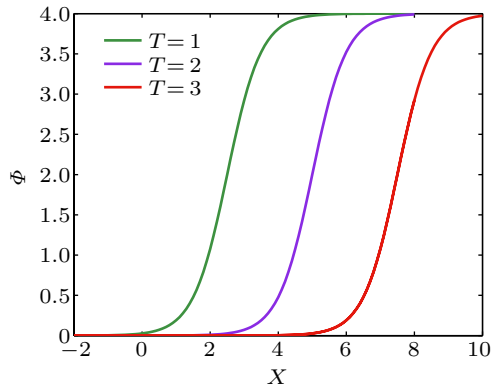


Fig. 2. The wave amplitude Φ is presented with $\varepsilon = 1$, $k = -2$, $l = 1$, $m = 0$.

Figure 2 reflects the motion characteristics of a single wave more intuitively. Therefore, we conclude that the single wave moves along the X-axis with constant amplitude and velocity.

Figure 3 shows the curve propagation of the two waves in the X–T, Y–T and Z–T planes by selecting appropriate parameters, where two waves remain parallel during the motion process. Figure 4 shows that the amplitude Φ depends on the value of k_1, and we find that the amplitude of two waves attains an extreme value with $k_1 = 1.35$. Meanwhile, the propagation trajectory of the two waves is invariable with other parameters changing.

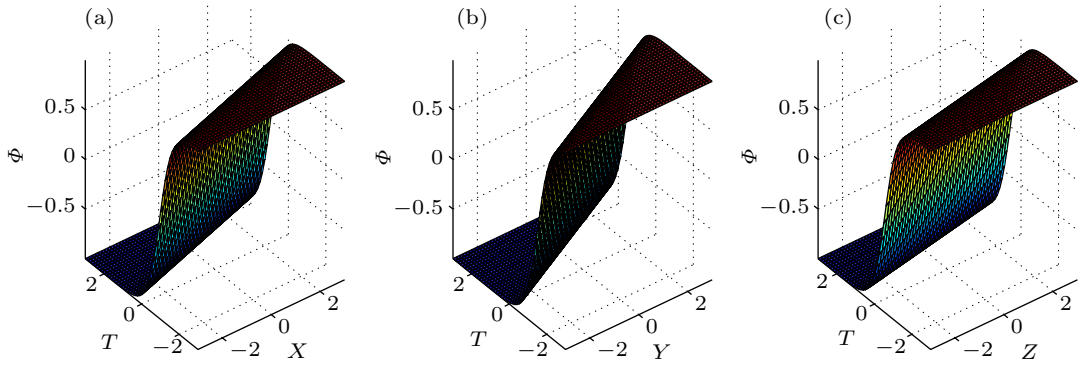


Fig. 3. The profile of the two-wave solution (60) with $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $k_1 = 0.5$, $k_2 = -0.5$, $l_1 = 0.8$, $l_2 = -0.8$, $m_1 = 0.3$ and $m_2 = -0.3$.

Figure 5 shows the characteristics of the three waves on the X–T plane with the appropriate parameters selected. Different from the single wave and the two waves, we choose the parameters $\varepsilon_1 = 1$, $\varepsilon_2 = 2$, $\varepsilon_3 = -1.8$. With the increase of parameter k_1, we find that the shape and amplitude of three waves are exactly the same before $T = 0$. However, there is a positive correlation between amplitude and value of parameter k_1 after $T = 0$.

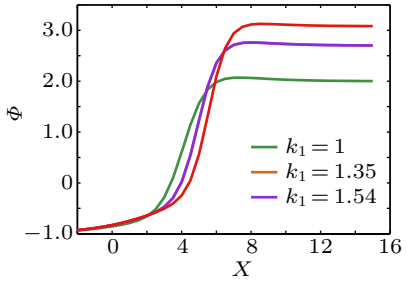


Fig. 4. The wave amplitude Φ is presented with $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $k_2 = -0.5$, $l_1 = 0.8$, $l_2 = -0.8$, $m_1 = 0.3$, $m_2 = -0.3$, corresponding $k_1 = 1$, $k_1 = 1.35$ and $k_1 = 1.54$.

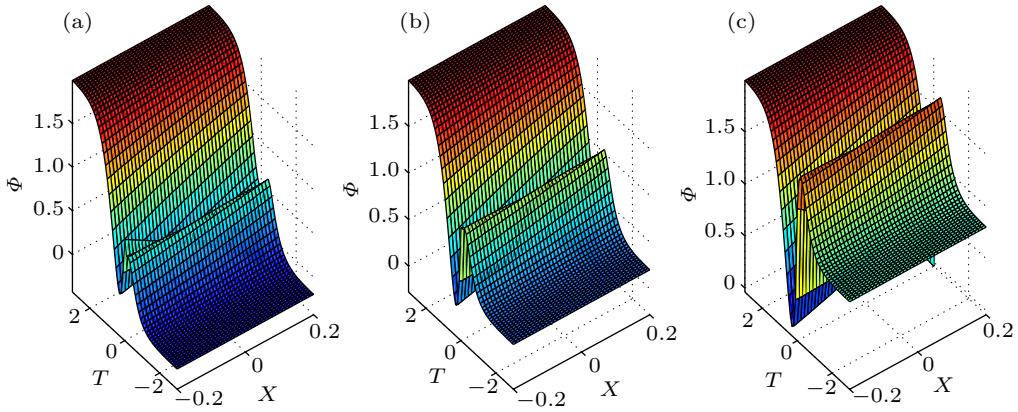


Fig. 5. The profile of the three-wave solution (62) with $k_2 = 1$, $k_3 = -2$, $l_1 = 1.3$, $l_2 = -2$, $l_3 = 0.3$, $m_1 = 1.5$, $m_2 = 2$, $m_3 = 2$, corresponding $k_1 = 1.9$, $k_1 = 2.1$ and $k_1 = 2.4$.

7. Conclusions

In summary, we have proposed a piecewise perturbation for the first time. Especially, the (3+1)-dimensional generalized mKdV-ZK equation is obtained using the perturbation method and multi-scale analysis. It is remarked that we have never seen the similar equation in quantum plasmas. The new model is a generalization of the mKdV equation and the ZK equation, which is of great significant for research of ion acoustic waves in quantum plasmas. In order to further discuss the quantum hydrodynamic, the (3+1)-dimensional generalized integer-order mKdV-ZK equation is transformed into the space-time fractional equation using the semi-inverse method and the fractional variational method. It is more suitable to designate propagation of ion-acoustic waves in actual cases than the integer order model. Moreover, we show the conservation laws of the (3+1)-dimensional generalized space-time fractional mKdV-ZK equation with the help of Lie Group analysis. Based on the multi-exponential function method and bilinear equation, the multiwave solutions are found. The influence mechanisms of different parameters on the multiwave solutions are discussed, and it is demonstrated that the variable parameter k_1 affects the wave amplitude. The curve propagation of multiwave is studied to deeply understand quantum plasmas.

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