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# A note on the definition of gravitational energy for quadratic curvature gravity via topological regularization*

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Within the framework of four-dimensional quadratic curvature gravities in the appearance of a negative cosmological constant, a definition for the gravitational energy of solutions with anti-de Sitter (AdS) asymptotics was put forward by Giribet *et al.* [*Phys. Rev. D* **98** 044046 (2018)]. This was achieved by adding proper topological invariant terms to the gravity action to render the variation problem well-posed. We prove that the definition via the procedure of topological regularization can be covered by our previous work [*Int. J. Mod. Phys. A* **35** 2050102 (2020)] in four dimensions. Motivated by this, we further generalize the results to generic diffeomorphism invariant theories of gravity in arbitrary even dimensions.

**Keywords:** gravitational energy, higher-derivative gravity, asymptotically AdS gravity

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## 1. Introduction

Recently, the authors of Ref. [1] have put forward a novel definition of the gravitational energy for the most general four-dimensional quadratic curvature gravity theory under asymptotically anti-de Sitter (AdS) boundary conditions, whose Lagrangian takes the form

$$\begin{aligned}\mathcal{L}_{\rm QG} &= \sqrt{-g}L_{\rm QG},\\ L_{\rm QG} &= R-2\Lambda+\alpha R^{\mu\nu}R_{\mu\nu}+\beta R^2,\end{aligned}\tag{1}$$

where $\Lambda$ is the conventional negative cosmological constant, $\alpha$ and $\beta$ are coupling constant parameters. Their main idea is to follow the procedure proposed in the case of general relativity[2] to add a Gauss–Bonnet term (i.e., a topological invariant term in four dimensions) with some weight factor to the gravity action, for the sake of guaranteeing that the total action exhibits well-posed variational principle when evaluated on the four-dimensional maximally symmetric solution, that is, the global $AdS_4$ spacetime. Adding the term yields the modified Iyer–Wald prepotential $q^{\mu\nu}_{\rm top}$ given in Ref. [3]. However, there exists a typo in $q^{\mu\nu}_{\rm top}$ given in Ref. [1], where the plus $+$ between the two terms should be minus $-$, i.e.,

$$\begin{aligned}q^{\mu\nu}_{\rm top} &= \frac{1}{2}\left(\nabla^\rho\xi^\sigma\right)\Big[\delta^{\mu\nu}_{\rho\sigma}+2\beta R\delta^{\mu\nu}_{\rho\sigma}+4\alpha R^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]}\\ &\quad+\frac{\ell^2}{4}(1+2\Lambda(\alpha+4\beta))R^{\theta\omega}_{\ \ \gamma\lambda}\delta^{\gamma\lambda\mu\nu}_{\theta\omega\rho\sigma}\Big]\\ &\quad-2\xi^\sigma\nabla^\rho\left(\beta R\delta^{\mu\nu}_{\rho\sigma}+2\alpha R^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]}\right).\end{aligned}\tag{2}$$

As a convention, hereafter a pair of square brackets on $m$ indices can refer to the anti-symmetrization over those indices with the common factor of $(m!)^{-1}$, and the generalized Kronecker delta $\delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m}$ is given by $\delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m}=m!\delta^{[\mu_1}_{[\nu_1}\cdots\delta^{\mu_m]}_{\nu_m]}$. For example, $\delta^{\mu\nu}_{\rho\sigma}=\delta^\mu_\rho\delta^\nu_\sigma-\delta^\nu_\rho\delta^\mu_\sigma$, and $R^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]}=\left(R^\mu_\rho\delta^\nu_\sigma-R^\nu_\rho\delta^\mu_\sigma-R^\mu_\sigma\delta^\nu_\rho+R^\nu_\sigma\delta^\mu_\rho\right)/4$. In addition, the radius $\ell$ of the $D$-dimensional AdS solution is associated with the negative cosmological constant $\Lambda$ via

$$\ell^2=-\frac{(D-1)(D-2)}{2\Lambda},\tag{3}$$

while $\xi^\mu$ denotes a Killing vector field. Based on the prepotential (2), the gravitational energy $Q_{\rm top}$ for the four-dimensional quadratic curvature gravities was proposed via the surface integral of $q^{\mu\nu}_{\rm top}$ in a subregion, which is given by a $(D-1)$-dimensional hypersurface $\Sigma$ with the boundary $\partial\Sigma$, namely,

$$Q_{\rm top}=\frac{1}{8\pi}\int_\Sigma q^{\mu\nu}_{\rm top}\,{\rm d}\Sigma_{\mu\nu}.\tag{4}$$

It has been testified[1] that $Q_{\rm top}$ is consistent with the one obtained using the well-known covariant phase space method.[3–5] What's more, $Q_{\rm top}$ yields the gravitational energy and angular momentum of the solutions of black holes and gravitational waves in perfect agreement with the ones via the so-called Abbott–Deser–Tekin (ADT) formalism.[6–9]

On the other hand, quite recently, in Ref. [10], within the framework of a general $D$-dimensional gravity theory described by the Lagrangian

$$\mathcal{L}_{\rm Riem}=\sqrt{-g}L_{\rm Riem}\left(g^{\alpha\beta},R_{\mu\nu\rho\sigma}\right),\tag{5}$$

a Komar-like formula for the conserved charges was defined

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by

$$\mathcal{Q}_{\rm Riem}=\frac{1}{8\pi}\int_{\partial\Sigma}\mathcal{K}^{\mu\nu}d\Sigma_{\mu\nu} \tag{6}$$

in terms of the 2-form potential $\mathcal{K}^{\mu\nu}$ (it is expressed in the following equation), provided that this gravity theory allows the existence of the negative cosmological constant $\Lambda$ and possesses aymptotically AdS boundary conditions. To the Lagrangian $\mathcal{L}_{\rm Riem}$, it is worth noting that derivatives with respect to the metric tensor $g_{\mu\nu}$ are involved only in the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$, not anywhere else.

The purpose of this paper is to demonstrate that the modified prepotential $q^{\mu\nu}_{\rm top}$ is identified with the 2-form $\mathcal{K}^{\mu\nu}$ in the context of the most general four-dimensional quadratic curvature gravity theory. As a consequence, the gravitational energy $Q_{\rm top}=\mathcal{Q}_{\rm Riem}$ in four dimensions. What's more, inspired with the consistence of both the formulas in four dimensions, we are going to extend the method of topological regularization to the generic gravity theories described by the Lagrangian (5) in arbitrary even dimensions.

## 2. A comparison of the potential $q^{\mu\nu}_{\rm top}$ with $\mathcal{K}^{\mu\nu}$ and a generalization of $q^{\mu\nu}_{\rm top}$ in any even dimension

In the present section, we demonstrate explicitly that the prepotential $q^{\mu\nu}_{\rm top}$ for the four-dimensional quadratic curvature gravity theories is completely consistent with $\mathcal{K}^{\mu\nu}$ in four dimensions. Furthermore, $q^{\mu\nu}_{\rm top}$ will be generalized to generic gravity theories described by the Lagrangian (5) in arbitrary even dimensions.

As a warmup, we follow the work[10] to present the concrete expression of the 2-form potential $\mathcal{K}^{\mu\nu}$ in the formula (6). It is read off as

$$\mathcal{K}^{\mu\nu}=\mathcal{P}^{\mu\nu}{}_{\rho\sigma}\nabla^{\rho}\xi^{\sigma}-2\xi^{\sigma}\nabla^{\rho}\mathcal{P}^{\mu\nu}{}_{\rho\sigma}, \tag{7}$$

with the tensor $\mathcal{P}^{\mu\nu\rho\sigma}$ defined through

$$\mathcal{P}^{\mu\nu}{}_{\rho\sigma}=P^{\mu\nu}_{R\rho\sigma}-\frac{k}{4(D-3)\hat{\Lambda}}R^{\alpha\beta}{}_{\gamma\lambda}\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}+\frac{k(D-4)}{2}\delta^{\mu\nu}_{\rho\sigma} \tag{8}$$

in terms of the tensor $P^{\mu\nu\rho\sigma}_{R}$ in the form

$$P^{\mu\nu\rho\sigma}_{R}=\frac{\partial L_{\rm Riem}}{\partial R_{\mu\nu\rho\sigma}}. \tag{9}$$

In Eq. (8), both the constant parameters $\hat{\Lambda}$ and $k$ can be expressed as

$$\hat{\Lambda}=-\frac{1}{\ell^2},\quad \mathcal{P}^{\mu\nu}_{\rho\sigma}(g_{\alpha\beta}\to\bar{g}_{\alpha\beta})=0. \tag{10}$$

Alternatively, the parameter $k$ can be solved from the equation $P^{\mu\nu}_{R\rho\sigma}(g_{\alpha\beta}\to\bar{g}_{\alpha\beta})=k\delta^{\mu\nu}_{\rho\sigma}$, where the background metric $\bar{g}_{\alpha\beta}$ is of the maximally symmetric AdS spacetime with the Riemann curvature tensor $\bar{R}^{\mu\nu}_{\rho\sigma}(\bar{g}_{\alpha\beta})=\hat{\Lambda}\delta^{\mu\nu}_{\rho\sigma}$. In addition, with the help of Eq. (8), the potential $\mathcal{K}^{\mu\nu}$ can be further expressed as

$$\begin{aligned}\mathcal{K}^{\mu\nu}&=K^{\mu\nu}_{R}-\frac{6k}{(D-3)\hat{\Lambda}}R^{[\rho\sigma}_{\rho\sigma}\nabla^{\mu}\xi^{\nu]}+k(D-4)\nabla^{[\mu}\xi^{\nu]},\\ K^{\mu\nu}_{R}&=P^{\mu\nu}_{R\rho\sigma}\nabla^{\rho}\xi^{\sigma}-2\xi^{\sigma}\nabla^{\rho}P^{\mu\nu}_{R\rho\sigma},\end{aligned} \tag{11}$$

where the 2-form $K^{\mu\nu}_{R}$ is one half of the Noether potential obtained via the covariant phase space method.[3–5] It should be pointed out that it has been illustrated[10] that the perturbation of the potential $\mathcal{K}^{\mu\nu}$ on the background AdS spacetime coincides with the result via the covariant phase space method, as well as that through the (off-shell) ADT approach.[6–9,11]

Now, we concentrate on the most general four-dimensional quadratic curvature gravity with the Lagrangian (1). According to the definition (9), the rank-4 tensor $P^{\mu\nu\rho\sigma}_{\rm QG}$ corresponding to such a Lagrangian reads

$$P^{\mu\nu}_{{\rm QG}\rho\sigma}=\frac{1}{2}\Big(\delta^{\mu\nu}_{\rho\sigma}+2\beta R\delta^{\mu\nu}_{\rho\sigma}+4\alpha R^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]}\Big). \tag{12}$$

By substituting the metric $\bar{g}_{\mu\nu}$ of the background AdS$_4$ into $P^{\mu\nu}_{{\rm QG}\rho\sigma}$ and making use of $\bar{R}^{\mu\nu}_{\rho\sigma}=\hat{\Lambda}\delta^{\mu\nu}_{\rho\sigma}$, we arrive at

$$P^{\mu\nu}_{{\rm QG}\rho\sigma}(g_{\alpha\beta}\to\bar{g}_{\alpha\beta})=k_{\rm QG}\delta^{\mu\nu}_{\rho\sigma}, \tag{13}$$

where the constant parameter $k_{\rm QG}$ is given by

$$\begin{aligned}k_{\rm QG}&=\frac{1}{2}\big[1+2(D-1)(D\beta+\alpha)\hat{\Lambda}\big]\\&=\frac{1}{2}\big(1+2\Lambda(\alpha+4\beta)\big).\end{aligned} \tag{14}$$

It assists the general forth-rank tensor $\mathcal{P}^{\mu\nu}{}_{\rho\sigma}$ given by Eq. (8) to turn into

$$\mathcal{P}^{\mu\nu}_{{\rm QG}\rho\sigma}=P^{\mu\nu}_{{\rm QG}\rho\sigma}+\frac{\ell^2\big(1+2\Lambda(\alpha+4\beta)\big)}{8}R^{\alpha\beta}{}_{\gamma\lambda}\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}. \tag{15}$$

Correspondingly, the potential $\mathcal{K}^{\mu\nu}_{\rm QG}$ for the four-dimensional quadratic curvature gravity theories is expressed as

$$\begin{aligned}\mathcal{K}^{\mu\nu}_{\rm QG}&=\mathcal{P}^{\mu\nu}_{{\rm QG}\rho\sigma}\nabla^{\rho}\xi^{\sigma}-2\xi^{\sigma}\nabla^{\rho}P^{\mu\nu}_{{\rm QG}\rho\sigma}\\&=q^{\mu\nu}_{\rm top}.\end{aligned} \tag{16}$$

This further implies that the formula for the gravitational energy via the topological regularization scheme coincides with the formulation of the conserved charges proposed in Ref. [10]. Compared to the (off-shell) ADT potential $q^{\mu\nu}_{\rm ADT}$,[8,9,11] we have

$$\begin{aligned}q^{\mu\nu}_{\rm ADT}&=\delta\big(P^{\mu\nu\rho\sigma}_{\rm QG}\nabla_{\rho}\xi_{\sigma}\big)-2\bar{\xi}^{\sigma}\bar{\nabla}^{\rho}\delta P^{\mu\nu}_{{\rm QG}\rho\sigma}+k_{\rm QG}h\bar{\nabla}^{[\mu}\bar{\xi}^{\nu]}\\&\quad-2k_{\rm QG}\bar{\xi}^{[\mu}\delta^{\nu]\lambda}_{\rho\sigma}\bar{\nabla}^{\sigma}h^{\rho}_{\lambda},\end{aligned} \tag{17}$$

where $h_{\mu\nu}=g_{\mu\nu}-\bar{g}_{\mu\nu}$. One can see that the perturbation of $\mathcal{K}^{\mu\nu}_{\rm QG}$ on the AdS$_4$ background, given by

$$\delta\mathcal{K}^{\mu\nu}_{\rm QG}=q^{\mu\nu}_{\rm ADT}+\frac{\ell^2k_{\rm QG}}{2}\bar{\nabla}_{\gamma}\Big[\delta^{\lambda\gamma\mu\nu}_{\alpha\beta\rho\sigma}\Big(\bar{\nabla}^{\alpha}h^{\beta}_{\lambda}\Big)\bar{\nabla}^{\rho}\bar{\xi}^{\sigma}\Big], \tag{18}$$

is equivalent with $q^{\mu\nu}_{\rm ADT}$ since the difference between them is just the divergence of a 3-form. Thus, as shown in Ref. [1], the formula (4) naturally produces the same conserved charges as the ADT method.

Next, under the guidance from the equivalence between the potentials $q^{\mu\nu}_{\rm top}$ and $\mathcal{K}^{\mu\nu}$ in four dimensions, it is possible for us to extend straightforwardly the procedure of topological regularization to $(2n+2)$-dimensional gravity theories, where the integer $n\geq 1$. Specifically, we take into consideration of the conserved quantities for the asymptotically AdS$_{(2n+2)}$ solutions within the context of the gravities described by the Lagrangian (5) in the appearance of the negative cosmological constant. By following Ref. [10], we introduce a Lovelock-type tensor $P^{\mu\nu\rho\sigma}_{(2n+2)}$ in $2(n+1)$ dimensions in the form[12]

$$P^{\mu\nu}_{(2n+2)\rho\sigma}=\frac{1}{4^n}R^{\alpha_1\beta_1}_{\gamma_1\lambda_1}\cdots R^{\alpha_n\beta_n}_{\gamma_n\lambda_n}\delta^{\gamma_1\lambda_1\cdots\gamma_n\lambda_n\mu\nu}_{\alpha_1\beta_1\cdots\alpha_n\beta_n\rho\sigma}, \tag{19}$$

which inherits the symmetries of the Riemann curvature tensor and is divergence-free, that is, $\nabla_\mu P^{\mu\nu\rho\sigma}_{(2n+2)}=0$. Apparently, the value of $P^{\mu\nu}_{(2n+2)\rho\sigma}$ on the AdS background spacetime is given by

$$\bar{P}^{\mu\nu}_{(2n+2)\rho\sigma}=P^{\mu\nu}_{(2n+2)\rho\sigma}\left(g_{\alpha\beta}\to\bar{g}_{\alpha\beta}\right)=\frac{(2n)!\hat{\Lambda}^n}{2^n}\delta^{\mu\nu}_{\rho\sigma}. \tag{20}$$

In algebraic computations the perturbation of the 2-form $P^{\mu\nu\rho\sigma}_{(2n+2)}\nabla_\rho\xi_\sigma$ on the AdS$_{(2n+2)}$ spacetimes gives rise to[10]

$$\begin{aligned}&\frac{2^n}{(2n)!\hat{\Lambda}^n}\delta\left(P^{\mu\nu}_{(2n+2)\rho\sigma}\nabla^\rho\xi^\sigma\right)\\&=\frac{1}{4(2n-1)\hat{\Lambda}}\delta\left(R^{\alpha\beta}_{\gamma\lambda}\delta^{\gamma\lambda\mu\nu}_{\alpha\beta\rho\sigma}\nabla^\rho\xi^\sigma\right)\\&\quad-(n-1)\delta\left(\delta^{\mu\nu}_{\rho\sigma}\nabla^\rho\xi^\sigma\right).\end{aligned} \tag{21}$$

As a consequence, in $2(n+1)$ dimensions, the tensor $\mathcal{P}^{\mu\nu\rho\sigma}$ associated with the Lagrangian (5) can be alternatively defined as

$$\mathcal{P}^{\mu\nu}_{(2n+2)\rho\sigma}=P^{\mu\nu}_{R\rho\sigma}-\frac{2^nk}{(2n)!\hat{\Lambda}^n}P^{\mu\nu}_{(2n+2)\rho\sigma}. \tag{22}$$

One can check that the tensor $\mathcal{P}^{\mu\nu\rho\sigma}_{(2n+2)}$ vanishes on the AdS$_{(2n+2)}$ spacetimes by making use of Eq. (20). Furthermore, the relevant potential $\mathcal{K}^{\mu\nu}$ in Eq. (11) becomes

$$\mathcal{K}^{\mu\nu}_{(2n+2)}=K^{\mu\nu}_R-\frac{2^nk}{(2n)!\hat{\Lambda}^n}P^{\mu\nu\rho\sigma}_{(2n+2)}\nabla_\rho\xi_\sigma. \tag{23}$$

In the requirement that the generalized prepotential $q^{\mu\nu}_{\rm top(2n+2)}$ for the $2(n+1)$-dimensional Lagrangian (5) is identified with $\mathcal{K}^{\mu\nu}_{(2n+2)}$, namely,

$$q^{\mu\nu}_{\rm top(2n+2)}=\mathcal{K}^{\mu\nu}_{(2n+2)}, \tag{24}$$

the supplemental topologically invariant term to the $(2n+2)$-dimensional Lagrangian could be proposed as follows:

$$\mathcal{E}_{(2n+2)}=\gamma_{(2n+2)}\sqrt{-g}P^{\mu\nu}_{(2n+2)\rho\sigma}R^{\rho\sigma}{}_{\mu\nu}, \tag{25}$$

where the coupling constant $\gamma_{(2n+2)}$ is determined by

$$\mathcal{P}^{\mu\nu\rho\sigma}_{(2n+2)}=\frac{\partial L_{\rm Riem}}{\partial R_{\mu\nu\rho\sigma}}+\frac{\partial\left(\mathcal{E}_{(2n+2)}/\sqrt{-g}\right)}{\partial R_{\mu\nu\rho\sigma}}, \tag{26}$$

giving rise to

$$\gamma_{(2n+2)}=-\frac{2^nk}{(2n)!(n+1)\hat{\Lambda}^n}. \tag{27}$$

As an example, in the context of the most general four-dimensional quadratic curvature gravity, $\gamma_4=\ell^2k_{\rm QG}/2$ is just the coupling constant $\gamma$ presented in Ref. [1], and $\mathcal{E}_{(4)}/(\gamma_4\sqrt{-g})$ becomes the Gauss–Bonnet term. What's more, in the case of $(2n+2)$-dimensional Einstein gravity, the constant $k=1/2$ and $\gamma_{(2n+2)}$ coincide with the coupling constant $\alpha_{n+1}$ given in Ref. [13]. Indeed, making use of Eq. (27), one can follow the procedure in Ref. [1] to check that the surface term from the variation of the Lagrangian $\mathcal{L}_{\rm Riem}$ supplemented with the topological term $\mathcal{E}_{(2n+2)}$ vanishes on the AdS$_{(2n+2)}$ spaces, rendering the variation of the Lagrangian well-behaved.

## 3. Summary

We have demonstrated that the definition for the gravitational energy of asymptotically AdS solutions via topological regularization is consistent with the one presented in Ref. [10] in the framework of four-dimensional quadratic curvature gravity theories with a negative cosmological constant. What is more, we have generalized the results in four dimensions to the generic gravity theories with AdS asymptotics in arbitrary even dimensions. The Lagrangian describing such gravities is supposed to be a functional built from the curvature tensor.

## References


[1] Giribet G, Miskovic O, Olea R and Rivera-Betancour D 2018 *Phys. Rev. D* **98** 044046

[2] Aros R, Contreras M, Olea R, Troncoso R and Zanelli J 2000 *Phys. Rev. Lett.* **84** 1647

[3] Iyer V and Wald R M 1994 *Phys. Rev. D* **50** 846

[4] Lee J and Wald R M 1990 *J. Math. Phys.* **31** 725

[5] Wald R M and Zoupas A 2000 *Phys. Rev. D* **61** 084027

[6] Abbott L F and Deser S 1982 *Nucl. Phys. B* **195** 76

[7] Abbott L F and Deser S 1982 *Phys. Lett. B* **116** 259

[8] Deser S and Tekin B 2002 *Phys. Rev. Lett.* **89** 101101

[9] Deser S and Tekin B 2003 *Phys. Rev. D* **67** 084009

[10] Peng J J and Liu H F 2020 *Int. J. Mod. Phys. A* **35** 2050102

[11] Kim W, Kulkarni S and Yi S H 2013 *Phys. Rev. Lett.* **111** 081101

[12] Lovelock D 1971 *J. Math. Phys.* **12** 498

[13] Aros R, Contreras M, Olea R, Troncoso R and Zanelli J 2000 *Phys. Rev. D* **62** 044002