

A two-dimensional quantum walk driven by a single two-side coin

Quan Lin(林泉), Hao Qin(秦豪), Kun-Kun Wang(王坤坤), Lei Xiao(肖磊), Peng Xue(薛鹏)

Citation: Chin. Phys. B . 2020, 29(11): 110303 . **doi:** 10.1088/1674-1056/abaee8
Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

The entanglement of deterministic aperiodic quantum walks

Ting-Ting Liu(刘婷婷), Ya-Yun Hu(胡亚运), Jing Zhao(赵静), Ming Zhong(钟鸣), Pei-Qing Tong(童培庆)

Chin. Phys. B . 2018, 27(12): 120305 . **doi:** 10.1088/1674-1056/27/12/120305

Localization of quantum walks on finite graphs

Yang-Yi Hu(胡杨熠), Ping-Xing Chen(陈平形)

Chin. Phys. B . 2016, 25(12): 120303 . **doi:** 10.1088/1674-1056/25/12/120303

A quantum walk in phase space with resonator-assisted double quantum dots

Zhi-Hao Bian(边志浩), Hao Qin(秦豪), Xiang Zhan(詹翔), Jian Li(李剑), Peng Xue(薛鹏)

Chin. Phys. B . 2016, 25(2): 020307 . **doi:** 10.1088/1674-1056/25/2/020307

Implementation of a one-dimensional quantum walk in both position and phase spaces

Qin Hao, Xue Peng

Chin. Phys. B . 2014, 23(1): 010301 . **doi:** 10.1088/1674-1056/23/1/010301

Non-Markovian decoherent quantum walks

Xue Peng, Zhang Yong-Sheng

Chin. Phys. B . 2013, 22(7): 070302 . **doi:** 10.1088/1674-1056/22/7/070302

SPECIAL TOPIC — Quantum computation and quantum simulation

A two-dimensional quantum walk driven by a single two-side coin*

Quan Lin(林泉)[1,†], Hao Qin(秦豪)[2,†] Kun-Kun Wang(王坤坤)[1], Lei Xiao(肖磊)[1], and Peng Xue(薛鹏)[1,‡]

[1] *Beijing Computational Science Research Center, Beijing* 100084, *China*
[2] *Department of Physics, Southeast University, Nanjing* 211189, *China*

(Received 30 June 2020; revised manuscript received 4 August 2020; accepted manuscript online 13 August 2020)

We study a two-dimensional quantum walk with only one walker alternatively walking along the horizontal and vertical directions driven by a single two-side coin. We find the analytical expressions of the first two moments of the walker's position distribution in the long-time limit, which indicates that the variance of the position distribution grows quadratically with walking steps, showing a ballistic spreading typically for quantum walks. Besides, we analyze the correlation by calculating the quantum mutual information and the measurement-induced disturbance respectively as the outcome of the walk in one dimension is correlated to the other with the coin as a bridge. It is shown that the quantum correlation between walker spaces increases gradually with the walking steps.

Keywords: quantum walks, quantum mutual information, measurement-induced disturbance

PACS: 03.67.−a, 03.67.Ac, 42.50.Dv **DOI:** 10.1088/1674-1056/abaee8

1. Introduction

Quantum walks (QWs) as the quantum version of classical random walks (CRWs) have been extensively studied in recent years due to striking nonclassical properties, and have found applications in quantum algorithms,[1–6] universal quantum computing,[7,8] quantum simulations,[9–16] and quantum measurements.[17,18] The simplest one-dimensional discrete-time QW consists of a single walker, usually being a quantum particle, who takes a step towards left or right on a line according to the outcome of flipping a two-side quantum coin, usually being the internal state of the quantum particle.[19–33] A higher-dimensional QW has also been proposed and investigated.[34–41] With the number of dimensions increasing, QWs with higher-dimensional walker and multiple-side coin exhibit richer properties[42–45] and can be used in quantum simulations of many-body systems.[46–48]

In this paper, we study a two-dimensional discrete-time QW with a single walker alternatively walking along the horizontal and vertical directions driven by a two-side coin, which shows a ballistic spreading of probability of the walker on a two-dimensional lattice. We show the analytical expressions of the first two moments of the walker's position distribution in the long-time limit and the variance of the position distribution growing linearly with the walking steps. Moreover, as the coin plays a role of bridge, the state of the walker in one dimension is correlated to that in the other dimension. We calculate quantum mutual information and measurement-induced disturbance respectively in order to analyze the variation of the classical and quantum correlations during QWs. It is shown that quantum correlations between two dimensions of the walker-space increase gradually with the walking steps. Thus, quantum correlations are created in QW dynamics.

The structure of this paper is organized as follows. In Section 2, we introduce a two-dimensional discrete-time QW driven by a two-side coin and then derive the analytical expressions of the first two moments of the walker's position distribution in the long-time limit. In Section 3, we calculate the quantum mutual information and measurement-induced disturbance between the two dimensions of the walker-space and analyze the numerical results to characterize the variation of both the classical and quantum correlations. Finally, in Section 4, we summarize the paper and give a conclusion.

2. A two-dimensional discrete-time quantum walk

We introduce a two-dimensional discrete-time QW,[49] including a walker and a two-side coin shown in Fig. 1. For each course, the walker first takes a step along the x-axis (left or right) according to the coin state after the first coin flipping, and then the coin is flipped again to lead the walker's movement along the y-axis (up or down). After a long evolution time, the walker exhibits a ballistic spreading of position distribution on a two-dimensional lattice. The initial state of the walker can be chosen as $|x_0\rangle \otimes |y_0\rangle$, defining the position of the walker on the two-dimensional lattice. The initial state of the coin, $|\phi_0\rangle$, usually takes the form of the superposition of the two sides of the coin, $|\phi_0\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β satisfy $|\alpha|^2 + |\beta|^2 = 1$.

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11674056 and U1930402) and the startup fund from Beijing Computational Science Research Center. KKW acknowledges support from the Project funded by China Postdoctoral Science Foundation (Grant No. 2019M660016).

†These authors contributed equally.

‡Corresponding author. E-mail: gnep.eux@gmail.com

© 2020 Chinese Physical Society and IOP Publishing Ltd

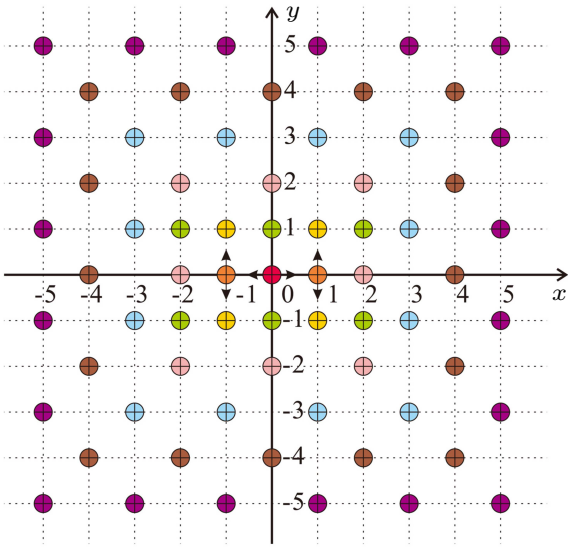


Fig. 1. Schematic for the two-dimensional QW. At the beginning of the walk, the single walker, starts from the position $(x_0 = 0, y_0 = 0)$ with a two-side quantum coin in his hand. Then the walker flips the coin and takes a step along the x axis to the left or right if the coin turns out to be head or tail. Hence, the walker is possible to appear at positions $(-1,0)$ and $(1,0)$, i.e., the orange points. After this, the walker flips the coin again and the outcome of the coin flipping decides the walker's movement along the y axis, up or down. Therefore, we may find the walker at positions $(-1,-1)$, $(-1,1)$, $(1,-1)$, and $(1,1)$, i.e., the yellow points in the figure with certain probabilities. We consider a step along the x axis and a step along the y axis respectively of the walker as a single course of our walk. During the second course, the walker possibly gets to the positions of the green points after the first coin flipping and finally arrives at the positions of the pink points and also the initial red point, which are 9 possible positions in total, after the second coin flipping. We omit the intermediate positions the walker may reach from the third course of the walk in the figure. It can be seen that after the n-th course the walk shows a spreading of position distribution including $(n+1)^2$ possible points on the two-dimensional lattice.

The step operators for the two dimensions take the form

$$\begin{aligned} S_x &= S \otimes \mathbb{1}_{\mathrm{w}} \otimes P_0 + S^{\dagger} \otimes \mathbb{1}_{\mathrm{w}} \otimes P_1, \\ S_y &= \mathbb{1}_{\mathrm{w}} \otimes S \otimes P_0 + \mathbb{1}_{\mathrm{w}} \otimes S^{\dagger} \otimes P_1, \end{aligned} \tag{1}$$

where $\mathbb{1}_{\mathrm{w}}$ is the identity operator applied on the walker, S, $S^{\dagger}$ are the conditional position shift operators on the walk's position state,

$$S\,|x(y)\rangle = |x(y)+1\rangle\,, \quad S^{\dagger}\,|x(y)\rangle = |x(y)-1\rangle\,, \quad x, y \in \mathbb{Z},$$

and P_0, P_1 are the projectors applied on the coin,

$$P_0 = |0\rangle\langle 0|\,, \quad P_1 = |1\rangle\langle 1|\,.$$

The coin operator coherently manipulates the coin parameter, leaving the position unchanged. In this paper, we take a Hadamard coin $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}/\sqrt{2}$. The evolution operator of the system for a single course of the QW becomes

$$U = S_y\,(\mathbb{1}_{\mathrm{w}} \otimes \mathbb{1}_{\mathrm{w}} \otimes H)\,S_x\,(\mathbb{1}_{\mathrm{w}} \otimes \mathbb{1}_{\mathrm{w}} \otimes H)\,. \tag{2}$$

In order to get an analytical result of the position distribution and the moments of the walker's position distribution, we perform a Fourier transformation transferring the walker state from the position space to the momentum space,[19]

$$|k\rangle = \sum_x \mathrm{e}^{\mathrm{i}kx}\,|x\rangle. \tag{3}$$

The state $|k\rangle$ is the eigenstate of S, $S^{\dagger}$ and satisfies the relations

$$S\,|k\rangle = \mathrm{e}^{-\mathrm{i}k}\,|k\rangle\,, \quad S^{\dagger}\,|k\rangle = \mathrm{e}^{\mathrm{i}k}\,|k\rangle\,.$$

The inverse Fourier transformation works as

$$|x\rangle = \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\,\mathrm{e}^{-\mathrm{i}kx}\,|k\rangle\,, \text{ and } |0\rangle = \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\,|k\rangle \text{ for } x = 0. \tag{4}$$

The evolution operator in the momentum space becomes

$$U\left(|k_x\rangle \otimes |k_y\rangle \otimes |\phi_0\rangle\right) = |k_x\rangle \otimes |k_y\rangle \otimes U_{k_y} U_{k_x}\,|\phi_0\rangle\,,$$

where $U_{k_{x(y)}}$ is a unitary operator applied on the coin state and takes the form

$$U_{k_{x(y)}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}k_{x(y)}} & \mathrm{e}^{-\mathrm{i}k_{x(y)}} \\ \mathrm{e}^{\mathrm{i}k_{x(y)}} & -\mathrm{e}^{\mathrm{i}k_{x(y)}} \end{pmatrix}. \tag{5}$$

By defining

$$U_k = U_{k_y} U_{k_x},$$

we get

$$U\left(|k_x\rangle \otimes |k_y\rangle \otimes |\phi_0\rangle\right) = |k_x\rangle \otimes |k_y\rangle \otimes U_k\,|\phi_0\rangle\,,$$

and the final state of the system of the walker and coin in the momentum space can be expressed as

$$\begin{aligned} \rho(t) \;=\; & \int_{-\pi}^{\pi} \frac{\mathrm{d}k_x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}k'_x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}k_y}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}k'_y}{2\pi}\, |k_x\rangle \langle k'_x| \\ & \otimes |k_y\rangle \langle k'_y| \otimes (U_k)^t\, |\phi_0\rangle \langle \phi_0|\, \left(U^{\dagger}_{k'}\right)^t \end{aligned} \tag{6}$$

with the initial state of the system of the walker and coin

$$\begin{aligned} \rho(0) \;=\; & \int_{-\pi}^{\pi} \frac{\mathrm{d}k_x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}k'_x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}k_y}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}k'_y}{2\pi}\, |k_x\rangle \langle k'_x| \\ & \otimes |k_y\rangle \langle k'_y| \otimes |\phi_0\rangle \langle \phi_0|\,. \end{aligned}$$

Then we trace out the walk in the y direction and obtain the reduced density matrix of the walker in the x direction

$$\begin{aligned} \rho_x(t) \;=\; & \int_{-\pi}^{\pi} \frac{\mathrm{d}k_x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}k'_x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}k_y}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}k'_y}{2\pi}\, |k_x\rangle \langle k'_x| \\ & \times \sum_y \mathrm{e}^{-\mathrm{i}y(k_y - k'_y)}\, \langle \phi_0|\, \left(U^{\dagger}_{k'_x} U^{\dagger}_{k'_y}\right)^t \left(U_{k_y} U_{k_x}\right)^t |\phi_0\rangle\,. \end{aligned} \tag{7}$$

Similarly, we can also trace out the walk in the x direction and get the reduced density matrix $\rho_y(t)$. Due to the balanced initial state we choose, the results of the position distribution in both directions are identical. Therefore, in the following we only derive the analytical expressions of the walker in the x direction.

Using the formula[19]

$$\frac{1}{2\pi} \sum_y \left(y^m\, \mathrm{e}^{-\mathrm{i}y(k_y - k'_y)}\right) = (-\mathrm{i})^m \delta^{(m)}(k_y - k'_y)$$

for further calculation, we get

$$\begin{aligned}\rho_x(t) &= \int_{-\pi}^{\pi}\frac{\mathrm{d}k_x}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k'_x}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k_y}{2\pi}|k_x\rangle\langle k'_x| \\ &\quad\times\langle\phi_0|\left(U_{k'_x}^{\dagger}U_{k_y}^{\dagger}\right)^t\left(U_{k_y}U_{k_x}\right)^t|\phi_0\rangle \\ &= \int_{-\pi}^{\pi}\frac{\mathrm{d}k_x}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k'_x}{2\pi}|k_x\rangle\langle k'_x|\times\langle\phi_0|\left(U_{k'_x}^{\dagger}\right)^t\left(U_{k_x}\right)^t|\phi_0\rangle.\end{aligned}$$

From this, the probability of the walker appearing at position x after t steps is

$$\begin{aligned}P(x;t) &= \mathrm{Tr}\Big(\big(|x\rangle\langle x|\big)\rho_x(t)\big)\Big) \\ &= \int_{-\pi}^{\pi}\frac{\mathrm{d}k_x}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k'_x}{2\pi}\mathrm{e}^{-\mathrm{i}x(k_x-k'_x)} \\ &\quad\times\langle\phi_0|\left(U_{k'_x}^{\dagger}\right)^t\left(U_{k_x}\right)^t|\phi_0\rangle. \qquad (8)\end{aligned}$$

Thus, the moment of the walker's position distribution for the QW is

$$\begin{aligned}\langle x^m\rangle_t &= \sum_x x^m P(x;t) \\ &= \int_{-\pi}^{\pi}\frac{\mathrm{d}k_x}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k'_x}{2\pi}\sum_x x^m\mathrm{e}^{-\mathrm{i}x(k_x-k'_x)} \\ &\quad\times\langle\phi_0|\left(U_{k'_x}^{\dagger}\right)^t\left(U_{k_x}\right)^t|\phi_0\rangle \\ &= \frac{(-\mathrm{i})^m}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k_x\langle\phi_0|\left(U_{k_x}^{\dagger}\right)^t\left[\frac{\mathrm{d}^m}{\mathrm{d}k_x^m}\left(U_{k_x}\right)^t\right]|\phi_0\rangle. \qquad (9)\end{aligned}$$

Here we need to use another relation[19]

$$\frac{\mathrm{d}U_{k_x}}{\mathrm{d}k_x} = -\mathrm{i}(P_0-P_1)U_{k_x} = -\mathrm{i}ZU_{k_x},$$

where $Z = P_0-P_1 = \mathbb{1}_{\mathrm{c}}-2P_1 = 2P_0-\mathbb{1}_{\mathrm{c}} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $\mathbb{1}_{\mathrm{c}}$ is the identity operator for the coin. Thus, we can get the expressions of the first and the second moments of the walker's position distribution for the walk respectively as follows:

$$\begin{aligned}\langle x\rangle_t &= \frac{(-\mathrm{i})}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k_x\langle\phi_0|\left(U_{k_x}^{\dagger}\right)^t\left[\frac{\mathrm{d}}{\mathrm{d}k_x}\left(U_{k_x}\right)^t\right]|\phi_0\rangle \\ &= \frac{(-1)}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k_x\sum_{j=1}^{t}\langle\phi_0|\left(U_{k_x}^{\dagger}\right)^j Z\left(U_{k_x}\right)^j|\phi_0\rangle, \qquad (10)\\ \langle x^2\rangle_t &= \frac{(-\mathrm{i})^2}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k_x\langle\phi_0|\left(U_{k_x}^{\dagger}\right)^t\left[\frac{\mathrm{d}^2}{\mathrm{d}k_x^2}\left(U_{k_x}\right)^t\right]|\phi_0\rangle \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k_x\left(\sum_{j=1}^{t}\sum_{j'=1}^{j}+\sum_{j=1}^{t}\sum_{j'=1}^{j-1}\right) \\ &\quad\times\langle\phi_0|\left(U_{k_x}^{\dagger}\right)^j Z\left(U_{k_x}\right)^{j-j'}Z\left(U_{k_x}\right)^j|\phi_0\rangle. \qquad (11)\end{aligned}$$

We directly take the solution from literature,[50] and obtain the eigenvectors $|\varphi_{k_xn}\rangle$ of U_{k_x} corresponding to eigenvalues $\mathrm{e}^{\mathrm{i}\theta_{k_xn}}$, i.e., $U_{k_x}|\varphi_{k_xn}\rangle = \mathrm{e}^{\mathrm{i}\theta_{k_xn}}|\varphi_{k_xn}\rangle$,

$$\begin{aligned}|\varphi_{k_x1}\rangle &= \frac{1}{\sqrt{2}}\left(1+\cos^2k_x+\cos k_x\sqrt{1+\cos^2k_x}\right)^{-\frac{1}{2}} \\ &\quad\times\begin{pmatrix}\mathrm{e}^{-\mathrm{i}k_x}\\ -\sqrt{2}\mathrm{e}^{\mathrm{i}\omega_{k_x}}-\mathrm{e}^{-\mathrm{i}k_x}\end{pmatrix},\\ |\varphi_{k_x2}\rangle &= \frac{1}{\sqrt{2}}\left(1+\cos^2k_x-\cos k_x\sqrt{1+\cos^2k_x}\right)^{-\frac{1}{2}} \\ &\quad\times\begin{pmatrix}\mathrm{e}^{-\mathrm{i}k_x}\\ \sqrt{2}\mathrm{e}^{-\mathrm{i}\omega_{k_x}}-\mathrm{e}^{-\mathrm{i}k_x}\end{pmatrix}. \qquad (12)\end{aligned}$$

The corresponding eigenvalues are $\mathrm{e}^{\mathrm{i}(\pi+\omega_{k_x})}$ and $\mathrm{e}^{-\mathrm{i}\omega_{k_x}}$, respectively, where $\theta_{k_x1} = \pi+\omega_{k_x}$, $\theta_{k_x1} = -\omega_{k_x}$, and ω_{k_x} satisfy

$$\sin\omega_{k_x} = \frac{1}{\sqrt{2}}\sin k_x, \quad -\frac{\pi}{2} < \omega_{k_x} < \frac{\pi}{2}.$$

Then, the initial state of the coin can be expanded as

$$|\phi_0\rangle = \sum_{n=1}^{2}C_{k_xn}|\varphi_{k_xn}\rangle.$$

For the first step of the walk, we have

$$U_{k_x}|\phi_0\rangle = \sum_{n=1}^{2}C_{k_xn}\mathrm{e}^{\mathrm{i}\theta_{k_xn}}|\varphi_{k_xn}\rangle.$$

Similarly, the state of the system after t steps evolves to

$$\left(U_{k_x}\right)^t|\phi_0\rangle = \sum_{n=1}^{2}C_{k_xn}\mathrm{e}^{\mathrm{i}\theta_{k_xn}t}|\varphi_{k_xn}\rangle.$$

Substituting this relation into Eqs. (10) and (11) respectively, we get

$$\begin{aligned}\langle x\rangle_t &= \frac{(-1)}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k_x\sum_{j=1}^{t}\langle\phi_0|\left(U_{k_x}^{\dagger}\right)^j Z\left(U_{k_x}\right)^j|\phi_0\rangle \\ &= -t+\frac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{d}k_x\sum_{n,n'}C_{k_xn}^{*}C_{k_xn'}\langle\varphi_{k_xn}|P_1|\varphi_{k_xn'}\rangle \\ &\quad\times\sum_{j=1}^{t}\mathrm{e}^{\mathrm{i}(\theta_{k_xn'}-\theta_{k_xn})j}, \qquad (13)\\ \langle x^2\rangle_t &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k_x\left(\sum_{j=1}^{t}\sum_{j'=1}^{j}+\sum_{j=1}^{t}\sum_{j'=1}^{j-1}\right) \\ &\quad\times\langle\phi_0|\left(U_{k_x}^{\dagger}\right)^j Z\left(U_{k_x}\right)^{j-j'}Z\left(U_{k_x}\right)^j|\phi_0\rangle \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k_x\sum_{n,n',n''}C_{k_xn}^{*}C_{k_xn'}\langle\varphi_{k_xn}|2P_0-\mathbb{1}_{\mathrm{c}}|\varphi_{k_xn''}\rangle \\ &\quad\times\langle\varphi_{k_xn''}|\mathbb{1}_{\mathrm{c}}-2P_1|\varphi_{k_xn'}\rangle \\ &\quad\times\left(\sum_{j=1}^{t}\sum_{j'=1}^{j}+\sum_{j=1}^{t}\sum_{j'=1}^{j-1}\right)\mathrm{e}^{\mathrm{i}(\theta_{k_xn''}-\theta_{k_xn})j}\mathrm{e}^{\mathrm{i}(\theta_{k_xn'}-\theta_{k_xn''})j'}. \qquad (14)\end{aligned}$$

Assuming that U_{k_x} is nondegenerated, most of the terms in Eqs. (13) and (14) are oscillatory and approach zero in average in the long-time limit. So we get the first moment of the walker's position distribution

$$\langle x\rangle_t = C_1t+\text{oscillatory terms}, \qquad (15)$$

where $C_1 = -1+\frac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{d}k_x\sum_n|C_{k_xn}|^2\langle\varphi_{k_xn}|P_1|\varphi_{k_xn}\rangle$, and the second moment

$$\langle x^2\rangle_t = C_2t^2+\text{oscillatory terms}+O(t), \qquad (16)$$

where $C_2 = 1 - \frac{2}{\pi}\int_{-\pi}^{\pi} \mathrm{d}k_x \sum_n |C_{k_x n}|^2 \langle \varphi_{k_x n}| P_0 |\varphi_{k_x n}\rangle \langle \varphi_{k_x n}| P_1 |\varphi_{k_x n}\rangle$. According to Eq. (15) and Eq. (16), the variance of the position distributions of the walks, $\sigma^2 = \langle x^2 \rangle_t - \langle x \rangle_t^2$, grows quadratically with time in the long-time limit, which is a typical behavior for QWs. In addition, by considering that U_{k_x} is degenerated, equations (15) and (16) will have some cross terms, which do not change the results of variance qualitatively.

For comparison, we perform a modification to the QW that we have introduced in order to get its corresponding two-dimensional CRW, which includes a classical walker with a classical two-side coin in his hand. During each single step of the CRW, the walker flips the coin twice in turn to decide the walk along the x axis (right or left) and along the y axis (up or down), respectively. With the walking steps increasing, the walker exhibits a diffusive spreading of the position distribution on the two-dimensional lattice, which is shown in Fig. 2. On the contrary, the QW exhibits a ballistic spreading, which is shown in Fig. 3. In fact, the different initial coin states lead to different position distributions, but do not change the quantum nature of the walk, which is usually characterized via the variance σ^2, or the standard deviation $\sigma = \sqrt{\langle x^2 \rangle_t - \langle x \rangle_t^2}$, of the position distribution. Here we choose the initial coin state $|\phi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \mathrm{i}\,|1\rangle)$, which is an equal superposition and relatively fair to compare to the classical case. Figures 2(b) and 3(b) show the variance of the position distribution of the two-dimensional CRW and that of QW for the first 10 steps, respectively. Obviously, the variance of CRW grows linearly with steps, while that of QW grows quadratically with steps. Thus, we observe the quadratic enhancement of position spreading for the QW vs. CRW.

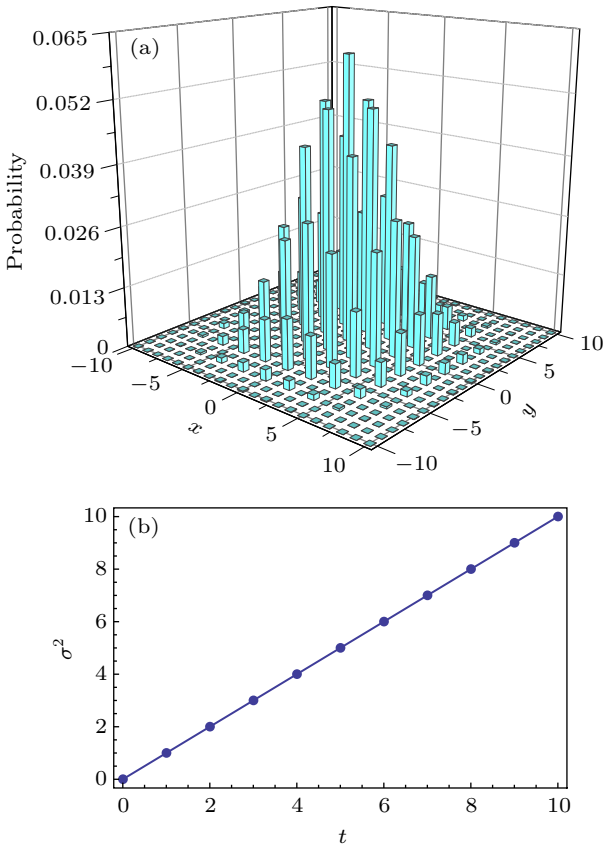


Fig. 2. (a) The position distribution of the two-dimensional CRW after the 10th step. (b) The variance of the position distributions of the two-dimensional CRW for the fist 10 steps.

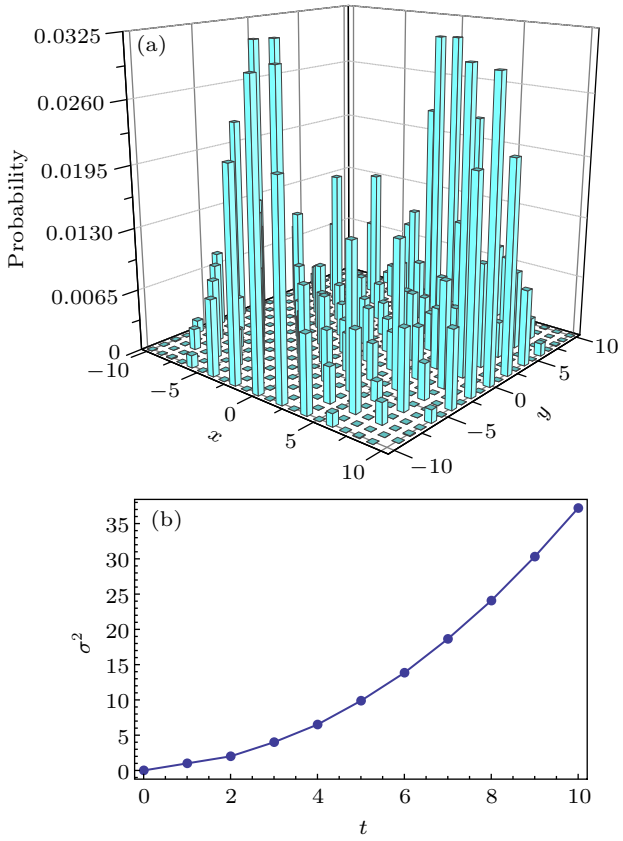


Fig. 3. (a) The position distribution of the two-dimensional QW after the 10th step. (b) The variance of the position distributions of the two-dimensional QW for the fist 10 steps.

In Refs. [35,36], Franco *et al.* give a detailed proof of the equivalence between a two-dimensional QW with a four-dimensional coin and the alternate one driven by a single-qubit coin studied here. Due to the equivalence, the distribution of the walker, the variance of the distribution, and the correlations between the walker spaces are the same for the two cases.

3. Analysis of the correlations in the QW

For the two-dimensional CRW, the classical coin does not connect the two directions of the walker space. Whereas, in QW, it is not the case. The walk in the x direction is correlated to that in the y direction because of the coin as a bridge. Therefore, it is necessary to pay attention to the variation of the correlation between the two dimensions of the walker space. Here we use the quantum mutual information and measurement-induced disturbance, respectively, to characterize the quantum correlations and the total correlations in the QW.

In order to calculate the quantum mutual information, we first trace out the coin from the final state of the QW $\rho(t)$ to get

the reduced density matrix $\rho_{\rm w}(t)$. Furthermore, we trace out the walk in the y direction from $\rho_{\rm w}(t)$ to obtain $\rho_x(t)$, which only contains information about the walk in the x direction. Similarly, we can also obtain the reduced density matrix for the y direction $\rho_y(t)$. Thus, the quantum mutual information I between the two dimensions is

$$I[\rho_{\rm w}(t)] = S[\rho_x(t)] + S[\rho_y(t)] - S[\rho_{\rm w}(t)], \tag{17}$$

where $S[\rho] = -{\rm Tr}(\rho \ln \rho)$ is the von Neumann entropy in quantum information theory. The quantum mutual information is used as a measure of the total correlation,[51] including the classical and the quantum correlations, between the two dimensions, which is depicted in Fig. 4. It is obvious to see that the quantum mutual information grows monotonically with the number of walking steps. The amount of the quantum mutual information inherently increases with the gradually expanding dimension of the walker's position and is bounded by $2\log_2 d$ approximately, where d denotes the dimension of the walker space and has the relation with the number of steps $d = 2t+1$. For comparison, we also calculate the quantum mutual information of the maximal entangled state, $|\psi_{\rm m}\rangle = \frac{1}{\sqrt{d}} \sum_{a=-t}^{t} |aa\rangle$, with increasing dimension of the walker state, which is also shown in Fig. 4(a) (dots). Besides, we plot the corresponding ln–ln plot to focus on the growth rate of the quantum mutual information for the two cases in Fig. 4(b). Although the exact value of the quantum mutual information in the QW is always smaller than that of the maximal entangled states, the growth rate is a little larger.

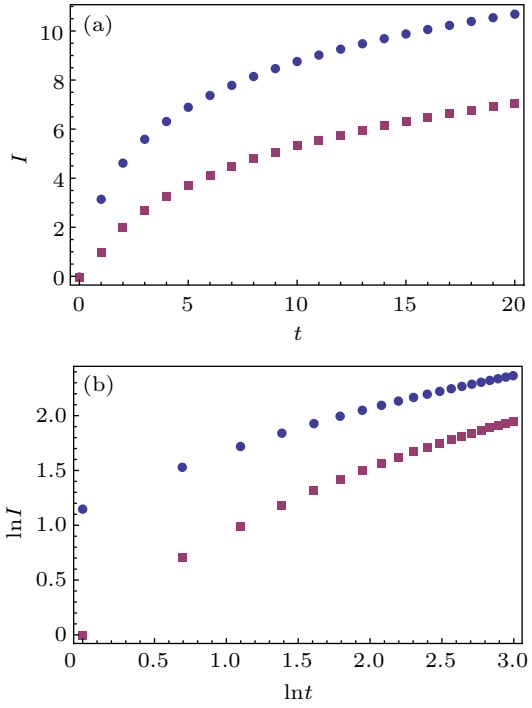


Fig. 4. (a) The quantum mutual information of two-dimensional QW (squares) and of the maximal entangled state (dots). (b) The ln–ln plot of the quantum mutual information of two-dimensional QW (squares) and of the maximal entangled state (dots).

In addition to the quantum mutual information, which quantifies the total correlations between the two dimensions, the quantum correlation therein is what we concern more about. Quantum discord is prevalently used to characterize pure quantum correlation of a bipartite system.[52] However, the evaluation of quantum discord contains a process of optimization which demands tremendous calculation workload and even intractable, especially when the dimension of the walker's position increases with the number of the steps in our case. Therefore, we choose the measurement-induced disturbance instead, which is operational and shows a loose upper bound on quantum discord, to characterize the quantum correlation in the QW. Although measurement-induced disturbance may risk overestimating nonclassicality because of lack of optimization over local measurement, it gives a loose upper bound on quantum discord and is appropriate for characterization of the quantum correlation in QWs. Measurement-induced disturbance is a quantum–classical dichotomy in order to quantify statistical correlations in bipartite states and is first introduced in Ref. [53]. This is based on the idea that while in the classical description of nature measurements can be carried out without disturbance, in the quantum description, generic measurements often disturb the system and the disturbance can be exploited to quantify the quantumness of correlations therein. These measures of quantum correlations are used to provide intuition about the quantum advantage of certain quantum computation models. Thus, it is a proper measure of quantum correlations in QWs.

After the measurement by $\{\Pi_i^x\}$ and $\{\Pi_j^y\}$ which are complete projective measurements (i.e., von Neumann measurements), the reduced state $\rho_{\rm w}(t)$ becomes

$$\Pi[\rho_{\rm w}(t)] = \sum_{i,j} \Pi_i^x \otimes \Pi_j^y \rho_{\rm w}(t) \Pi_i^x \otimes \Pi_j^y. \tag{18}$$

The induced state $\Pi[\rho_{\rm w}(t)]$ is classical[54] and thus its quantum mutual information, $I\big[\Pi[\rho_{\rm w}(t)]\big]$, quantifies the classical correlations in $\rho_{\rm w}(t)$ as well as in $\Pi[\rho_{\rm w}(t)]$. Naturally, a measure for quantum correlation is given by the measurement-induced disturbance Q, defined as

$$Q[\rho_{\rm w}(t)] = I[\rho_{\rm w}(t)] - I\big[\Pi[\rho_{\rm w}(t)]\big], \tag{19}$$

which indicates the difference between the total and classical correlations of $\rho_{\rm w}(t)$. Here the measurement $\Pi = \{\Pi_i^x \otimes \Pi_j^y\}$ is induced by the spectral resolutions of the reduced states, $\rho_x(t) = \sum_i p_i^x(t)\Pi_i^x$ and $\rho_y(t) = \sum_j p_j^y(t)\Pi_j^y$. We plot the measurement-induced disturbance between the two dimensions in the QW in Fig. 5(a). For comparison, we plot the measurement-induced disturbance of the maximal entangled state as well and find that the measurement-induced disturbance grows monotonically with the number of steps, which is qualitatively similar to the quantum mutual information. We

find that the measurement-induced disturbance in the QW becomes larger than that of the maximal entangled states after the fifth step. Entanglement and quantum correlations have often been regarded as synonymous for several years under the dichotomy for entanglement and separability.[55] In fact, the quantum correlations are more general than entanglement.[54] Thus, the maximal entangled states which contain the maximal amount of entanglement may not account for the most quantum correlations of a quantum system. Therefore, the larger value of the measurement-induced disturbance in the QW we get is reasonable to some extent. Furthermore, we give the ln–ln plot in Fig. 5(b). Obviously, the growth rate of the measurement-induced disturbance in the QW is larger than that of the maximal entangled states, which can be viewed as the genuine increasing of the quantum correlation in the QW in spite of the inherent growth with the expanding dimension of the walker. The evolution of the single coin connecting the walk between the two dimensions contributes to the increasing of the quantum correlation in the QW.

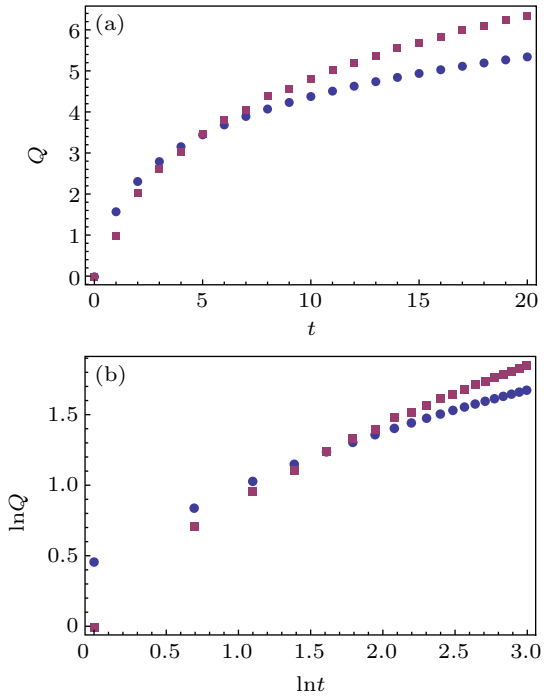


Fig. 5. (a) The measurement-induced disturbance of two-dimensional QW (squares) and of the maximal entangled state (dots). (b) The ln–ln plot of the measurement-induced disturbance of two-dimensional QW (squares) and of the maximal entangled state (dots).

4. Conclusion

In summary, we study a two-dimensional QW, including only one walker alternatively walking along the horizontal and the vertical directions driven by a single two-side coin. First, we find the analytical expressions of the first two moments of the walker's position distribution in the long-time limit, which indicate a quadratic growth of the variance of the position distribution with the number of steps. Then, for comparison, we show the numerical simulations of both the two-dimensional QW and its corresponding two-dimensional CRW and plot the position distributions and variances for both. We observe a ballistic spreading and the quadratic growth of variance of the QW, while a diffusive spreading and a linear growth of variance for the CRW. Besides, we analyze the correlations between the two dimensions in the QW via the quantum mutual information and measurement-induced disturbance, respectively characterizing the total and the quantum correlations. The quantum mutual information and measurement-induced disturbance both grow monotonically with the number of steps. Due to the inherent growth with the expanding dimension of the walker, we calculate the quantum mutual information and measurement-induced disturbance of the maximal entangled state in our system as well in order to capture the genuine increasing of the correlations in the QW and find that the evolution of the single coin connecting the walk between the two dimensions contributes to the increasing of the quantum correlation in the QW.

References

[1] Dür W, Raussendorf R, Kendon V M and Briegel H J 2002 *Phys. Rev. A* **66** 052319
[2] Ambainis A 2003 *Int. J. Quantum Inf.* **1** 507
[3] Childs A M, Cleve R, Deotto E, Farhi E, Gutmann S and Spielman D A 2003 *Proc. 35th ACM Symposium on Theory of Computing* pp. 59–68
[4] Shenvi N, Kempe J and Whaley K B 2003 *Phys. Rev. A* **67** 052307
[5] Kempe J 2003 *Contempo. Phys.* **44** 307
[6] Izaac J A, Zhan X, Bian Z H, Wang K K, Li J, Wang J B and Xue P 2017 *Phys. Rev. A* **95** 032318
[7] Childs A M 2009 *Phys. Rev. Lett.* **102** 180501
[8] Childs A M, Gosset D and Webb Z 2013 *Science* **339** 791
[9] Xue P, Zhang R, Qin H, Zhan X, Bian Z H, Li J and Sanders B C 2015 *Phys. Rev. Lett.* **114** 140502
[10] Kitagawa T, Rudner M S, Berg E and Demler E 2010 *Phys. Rev. A* **82** 033429
[11] Oliveira A C, Portugal R and Donangelo R 2006 *Phys. Rev. A* **74** 012312
[12] Xue P and Sanders B C 2013 *Phys. Rev. A* **87** 022334
[13] Zhang R, Xue P and Twamley J 2014 *Phys. Rev. A* **89** 042317
[14] Zhan X, Qin H, Bian Z H, Li J and Xue P 2014 *Phys. Rev. A* **90** 012331
[15] Xiao L, Zhan X, Bian Z H, Wang K K, Zhang X, Wang X P, Li J, Mochizuki K, Kim D, Kawakami N, Yi W, Obuse H, Sanders B C and Xue P 2017 *Nat. Phys.* **13** 1117
[16] Xiao L, Deng T S, Wang K K, Zhu G Y, Wang Z, Yi W and Xue P 2020 *Nat. Phys.* **16** 761
[17] Kurzyński P and Wójcik A 2013 *Phys. Rev. Lett.* **110** 200404
[18] Bian Z H, Li J, Qin H, Zhan X, Zhang R, Sanders B C and Xue P 2015 *Phys. Rev. Lett.* **114** 203602
[19] Brun T A, Carteret H A and Ambainis A 2003 *Phys. Rev. Lett.* **91** 130602
[20] Cardano F, Massa F, Qassim H, Karimi E, Slussarenko S, Paparo D, de Lisio C, Sciarrino F, Santamato E, Boyd R W and Marrucci L 2015 *Sci. Adv.* **1** e1500087
[21] Defienne H, Barbieri M, Walmsley I A, Smith B J and Gigan S 2016 *Sci. Adv.* **1** e1501054
[22] Wang K K, Qiu X, Xiao L, Zhan X, Bian Z H, Yi W and Xue P 2019 *Phys. Rev. Lett.* **122** 020501
[23] Du J, Li H, Xu X, Shi M, Wu J, Zhou X and Han R 2003 *Phys. Rev. A* **67** 042316
[24] Ryan C A, Laforest M, Boileau J C and Laflamme R 2005 *Phys. Rev. A* **72** 062317

[25] Broome M A, Fedrizzi A, Lanyon B P, Kassal I, Aspuru-Guzik A and White A G 2010 *Phys. Rev. Lett.* **104** 153602
[26] Schreiber A, Cassemiro K N, Potocek V, Gabris A, Mosley P J, Andersson E, Jex I and Silberhorn C 2010 *Phys. Rev. Lett.* **104** 050502
[27] Kitagawa T, Broome M A, Fedrizzi A, Rudner M S, Berg E, Kassal I, Aspuru-Guzik A, Demler E and White A G 2012 *Nat. Commun.* **3** 882
[28] Qin H and Xue P 2016 *Chin. Phys. B* **25** 010501
[29] Bian Z H, Qin H, Zhan Z, Li J and Xue P 2016 *Chin. Phys. B* **25** 020307
[30] Luo H, Zhan Z, Zhang R and Xue P 2016 *Chin. Phys. B* **25** 110304
[31] Zhang R, Xu Y and Xue P 2015 *Chin. Phys. B* **24** 010303
[32] Xue P, Qin H, Zhan Z, Bian Z H and Li J 2014 *Chin. Phys. B* **23** 110307
[33] Qin H and Xue P 2014 *Chin. Phys. B* **23** 010301
[34] Mackay T D, Bartlett S D, Stephenson L T and Sanders B C 2002 *J. Phys. A* **35** 2745
[35] Franco C Di, Gettrick M Mc and Busch Th 2011 *Phys. Rev. Lett.* **106** 080502
[36] Franco C Di, Gettrick M Mc, Machida T and Busch Th 2011 *Phys. Rev. A* **84** 042337
[37] Brun T A, Carteret H A and Ambainis A 2003 *Phys. Rev. A* **67** 052317
[38] Stefanak M, Barnett S M, Kollar B, Kiss T and Jex I 2011 *New J. Phys.* **13** 033029
[39] Rohde P P, Schreiber A, Stefanak M, Jex I and Silberhorn C 2001 *New J. Phys.* **13** 013001
[40] Ahlbrecht A, Vogts H, Werner A H and Werner R F 2011 *J. Math. Phys.* **52** 042201
[41] Berry S D and Wang J B 2011 *Phys. Rev. A* **83** 042317
[42] Zahringer F, Kirchmair G, Gerritsma R, Solano E, Blatt R and Roos C F 2010 *Phys. Rev. Lett.* **104** 100503
[43] Peruzzo A, Lobino M, Matthews J C F, Matsuda N, Politi A, Poulios K, Zhou X, Lahini Y, Ismail N, Worhoff K, Bromberg Y, Silberberg Y, Thompson M G and OBrien J L 2010 *Science* **329** 1500
[44] Schreiber A, Gabris A, Rohde P P, Laiho K, Stefanak M, Potocek V, Hamilton C, Jex I and Silberhorn C 2012 *Science* **336** 55
[45] Crespi A, Osellame R, Ramponi R, Giovannetti V, Fazio R, Sansoni L, Nicola F D, Sciarrino F and Mataloni P 2013 *Nat. Photon.* **7** 322
[46] Mohseni M, Rebentrost P, Lloyd S and Aspuru-Guzik A 2008 *J. Chem. Phys.* **129** 174106
[47] Sension R J 2007 *Nature* **446** 740
[48] Wang B, Chen T and Zhang X 2018 *Phys. Rev. Lett.* **121** 100501
[49] DErrico A, Cardano F, Maffei M, Dauphin A, Barboza R, Esposito C, Piccirillo B, Lewenstein M, Massignan P and Marrucci L 2020 *Optica* **7** 108
[50] Nayak A and Vishwanath A 2000 e-print quant-ph/0010117
[51] Groisman B, Popescu S and Winter A 2005 *Phys. Rev. A* **72** 032317
[52] Ollivier H and Zurek W H 2001 *Phys. Rev. Lett.* **88** 017901
[53] Luo S 2008 *Phys. Rev. A* **77** 022301
[54] Modi K, Paterek T, Son W, Vedral V and Williamson M 2010 *Phys. Rev. Lett.* **104** 080501
[55] Vedral V, Plenio M B, Rippin M A and Knight P L 1997 *Phys. Rev. Lett.* **78** 2275