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Chaotic state as an output of vacuum state evolving in diffusion channel and generation of displaced chaotic state for quantum controlling*

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We show that chaotic state can be produced as an output of vacuum state evolving in diffusion channel, while displaced chaotic state is output of a coherent state evolving in diffusion channel. We also introduce the thermo vaccum state for the displaced chaotic state and evaluate the average photon number. The displaced chaotic state may be used exhibiting quantum controlling.

Keywords: displaced chaotic state, quantum controlling

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1. Introduction

Recently, quantum controlling has been paid much attention to physicists. Usually, quantum controlling can be implemented by gradually inputing photons into system or reservoir, and increasing photon number can be realized by passing an initial coherent state through a diffusion channel.[1–3] In this paper we shall show that a displaced chaotic state can be generated in this way, this state exhibits photon addition.

For describing light fields physicists usually employ density operators,[3–5] for example, the density operator of chaotic light is expressed by

$$\rho_{\rm c} = \left(1-{\rm e}^{f}\right){\rm e}^{fa^{\dagger}a}, \quad f=-\beta\hbar\omega, \\ \beta = 1/\left(KT\right), \tag{1}$$

where K is the Boltzman constant, T denotes temperature, a and $a^{\dagger}$ represent photon annihilator and creator respectively, $\left[a,a^{\dagger}\right]=1$. From $\rho_{\rm c}$ we are led to

$$\mathrm{tr}\left(\rho_{\rm c}a^{\dagger}a\right)=\frac{1}{{\rm e}^{\beta\hbar\omega}-1}, \tag{2}$$

indicating Planck energy formula. In this paper we shall study the properties of the density operator of displaced chaotic light[6,7]

$$\rho_{\rm d}=C{\rm e}^{\lambda a^{\dagger}}\,{\rm e}^{fa^{\dagger}a}{\rm e}^{\lambda^{*}a}, \tag{3}$$

here C is the normalization constant to be determined shortly later by $\mathrm{tr}\,\rho_{\rm d}=1$. Obviously, by noting the normally ordered form of ${\rm e}^{fa^{\dagger}a}$

$${\rm e}^{fa^{\dagger}a}=:\exp\left[\left({\rm e}^{f}-1\right)a^{\dagger}a\right]:, \tag{4}$$

we see that when $f\to-\infty$, $T\to 0$, ${\rm e}^{fa^{\dagger}a}$ is replaced by the pure vacuum state $:{\rm e}^{-a^{\dagger}a}:=|0\rangle\langle 0|$, then $\rho_{\rm d}\to C{\rm e}^{\lambda a^{\dagger}}|0\rangle\langle 0|{\rm e}^{\lambda^{*}a}$ stands for a pure coherent state. Hence $\rho_{\rm d}$ can be treated as an intermediate quantum state between the chaotic state ($\lambda=0$) and the coherent state. In this paper we examine how a displaced chaotic light can be produced when a coherent light undergoes a diffusion process. We also show that chaotic state can be produced as an output of vacuum state evolving in diffusion channel. We also introduce the thermo vacuum state for the displaced chaotic state and evaluate the average photon number, which indicates that the displaced chaotic state may be used exhibiting quantum controlling.

2. The normalization of ρ

Using Eq. (4) and the coherent state representation,[8–14] we have

$$\int\frac{{\rm d}^{2}\alpha}{\pi}|\alpha\rangle\langle\alpha|=1, \tag{5}$$

where

$$|\alpha\rangle=\exp\left[-\frac{1}{2}|\alpha|^{2}+\alpha a^{\dagger}\right]|0\rangle, \\ a|\alpha\rangle=\alpha|\alpha\rangle \tag{6}$$

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is a coherent state, we evaluate

$$\begin{aligned}1 &= \mathrm{Tr}\rho_{\mathrm{d}} = \mathrm{Tr}\left[\int \frac{\mathrm{d}^2\alpha}{\pi}|\alpha\rangle\langle\alpha|\rho_{\mathrm{d}}\right]\\ &= C\int \frac{\mathrm{d}^2\alpha}{\pi}\langle\alpha|\,\mathrm{e}^{\lambda a^\dagger}\mathrm{e}^{fa^\dagger a}\mathrm{e}^{\lambda^* a}|\alpha\rangle\\ &= C\int \frac{\mathrm{d}^2\alpha}{\pi}\mathrm{e}^{\lambda\alpha^*+\lambda^*\alpha}\langle\alpha|:\exp\left[\left(\mathrm{e}^f-1\right)a^\dagger a\right]:|\alpha\rangle\\ &= C\int \frac{\mathrm{d}^2\alpha}{\pi}\mathrm{e}^{\lambda\alpha^*+\lambda^*\alpha}\exp\left[-\left(1-\mathrm{e}^f\right)|\alpha|^2\right]\\ &= \frac{C}{1-\mathrm{e}^f}\exp\left[\frac{|\lambda|^2}{1-\mathrm{e}^f}\right], \end{aligned} \tag{7}$$

so the normalization of ρ_{d} is

$$C = \left(1-\mathrm{e}^f\right)\exp\left[\frac{-|\lambda|^2}{1-\mathrm{e}^f}\right], \tag{8}$$

$$\rho_{\mathrm{d}} = \left(1-\mathrm{e}^f\right)\exp\left(\frac{-|\lambda|^2}{1-\mathrm{e}^f}\right)\mathrm{e}^{\lambda a^\dagger}\mathrm{e}^{fa^\dagger a}\mathrm{e}^{\lambda^* a}. \tag{9}$$

In the following we shall show that a displaced chaotic light can be produced by passing a pure coherent state through a diffusion channel, in another word, for such a channel when the input is described by ρ_{c}, then the output is ρ_{d}. We also calculate the variation of ρ_{d} in an amplitude dissipation channel. The work is arranged as follows: in Section 2 we present the master equation for diffusion channel from the classical diffusion equation in a direct way. In Section 3 we derive both the normally ordered and anti-normally ordered forms of the density operator for the displaced chaotic state. In Section 4 we derive the thermo vacuum state for ρ_{d} and calculate the photon number in it.

3. Master equation for diffusion channel

Stemming from the classical diffusion equation of distribution function $P(z,t)$

$$\frac{\partial P(z,t)}{\partial t} = -\kappa\frac{\partial^2 P(z,t)}{\partial z\partial z^*}, \tag{10}$$

where κ is the diffusion rate, we want to derive master equation for diffusion channel for quantum optical field. Considering $P(z,t)$ as P-representation of $\rho(t)$ in the coherent state representation

$$\rho(t) = \int \frac{\mathrm{d}^2 z}{\pi}P(z,t)|z\rangle\langle z|, \tag{11}$$

where

$$|z\rangle = \exp\left[za^\dagger - z^* a\right]0\rangle. \tag{12}$$

Using the method of integration within ordered product (IWOP) of operators we know

$$|z\rangle\langle z| =: \mathrm{e}^{-|z|^2+za^\dagger+z^*a-a^\dagger a}:, \tag{13}$$

$$\int \frac{\mathrm{d}^2 z}{\pi}|z\rangle\langle z| = \int \frac{\mathrm{d}^2 z}{\pi}:\mathrm{e}^{-|z|^2+za^\dagger+z^*a-a^\dagger a}:=1. \tag{14}$$

Differentiating the two sides of Eq. (11) with respect to time t, and using the classical diffusion equation we have

$$\begin{aligned}\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} &= \int \frac{\mathrm{d}^2 z}{\pi}\frac{\partial P(z,t)}{\partial t}|z\rangle\langle z|\\ &= -\kappa\int \frac{\mathrm{d}^2 z}{\pi}\frac{\partial^2 P(z,t)}{\partial z\partial z^*}|z\rangle\langle z| \qquad (15)\\ &= -\kappa\int \frac{\mathrm{d}^2 z}{\pi}P(z,t)\frac{\partial^2}{\partial z\partial z^*}|z\rangle\langle z|. \qquad (16)\end{aligned}$$

As a result of Eq. (13) we have

$$\begin{aligned}a^\dagger|z\rangle\langle z| &=: a^\dagger\mathrm{e}^{-|z|^2+za^\dagger+z^*a-a^\dagger a}:\\ &= \left(z^*+\frac{\partial}{\partial z}\right):\mathrm{e}^{-|z|^2+za^\dagger+z^*a-a^\dagger a}:\\ &= \left(z^*+\frac{\partial}{\partial z}\right)|z\rangle\langle z|,\\ |z\rangle\langle z|a &= \left(z+\frac{\partial}{\partial z^*}\right)|z\rangle\langle z|. \end{aligned} \tag{17}$$

It then follows

$$\begin{aligned}-\frac{\partial^2}{\partial z\partial z^*}|z\rangle\langle z| &= z\left(z^*+\frac{\partial}{\partial z}\right)|z\rangle\langle z| - \left(z^*+\frac{\partial}{\partial z}\right)\left(z+\frac{\partial}{\partial z^*}\right)|z\rangle\langle z| - |z|^2|z\rangle\langle z| + \left(z+\frac{\partial}{\partial z^*}\right)(z^*|z\rangle\langle z|)\\ &= za^\dagger|z\rangle\langle z| - \left(z^*+\frac{\partial}{\partial z}\right)|z\rangle\langle z|a - |z|^2|z\rangle\langle z| + \left(z+\frac{\partial}{\partial z^*}\right)|z\rangle\langle z|a^\dagger\\ &= a^\dagger a|z\rangle\langle z| - a^\dagger|z\rangle\langle z|a - a|z\rangle\langle z|a^\dagger + |z\rangle\langle z|aa^\dagger. \end{aligned} \tag{18}$$

Substituting Eq. (18) into Eq. (16) yields

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \kappa\int \frac{\mathrm{d}^2 z}{\pi}P(z,t)(a^\dagger a|z\rangle\langle z| - a^\dagger|z\rangle\langle z|a - a|z\rangle\langle z|a^\dagger + |z\rangle\langle z|aa^\dagger), \tag{19}$$

which gives

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \kappa(a^\dagger a\rho - a^\dagger\rho a - a\rho a^\dagger + \rho aa^\dagger). \tag{20}$$

It describes two linear processes, acting simultaneously, namely, the damping, expressed by $a\rho a^\dagger$, and the exciting, expressed by $a^\dagger\rho a$, coexist.

4. Solving the master equation by entangled state representation

By introducing the entangled state

$$|\eta\rangle = \exp\left(-\frac{1}{2}|\eta|^2 + \eta a^\dagger - \eta^* \tilde{a}^\dagger + a^\dagger \tilde{a}^\dagger\right)|0\tilde{0}\rangle, \tag{21}$$

where the fictitious mode $\tilde{a}^\dagger$ and $\tilde{a}$ obey $\left[\tilde{a}, \tilde{a}^\dagger\right] = 1$, and $\tilde{a}|\tilde{0}\rangle = 0$, we have

$$\langle\eta|(a^\dagger - \tilde{a}) = \eta^*\langle\eta|, \qquad \langle\eta|(a - \tilde{a}^\dagger) = \eta\langle\eta|, \tag{22}$$

and the completeness

$$\int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta| = \int \frac{\mathrm{d}^2\eta}{\pi} : \exp[-|\eta|^2 + \eta(a^\dagger - \tilde{a}) + \eta^*(a - \tilde{a}^\dagger) - a^\dagger a - \tilde{a}^\dagger\tilde{a}] := 1. \tag{23}$$

Let $|\eta = 0\rangle \equiv |I\rangle$, we see

$$a|I\rangle = \tilde{a}^\dagger|I\rangle, \quad a^\dagger|I\rangle = \tilde{a}|I\rangle, \quad (a^\dagger a)^n|I\rangle = (\tilde{a}^\dagger\tilde{a})^n|I\rangle. \tag{24}$$

By operating both sides of Eq. (20) on the state $|I\rangle$, and letting $|\rho\rangle = \rho|I\rangle$, using Eq. (24) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\rho(t)\rangle &= -\kappa(a^\dagger a\rho - a^\dagger\rho a - a\rho a^\dagger + \rho a a^\dagger)|I\rangle \\ &= -\kappa(a^\dagger - \tilde{a})(a - \tilde{a}^\dagger)|\rho(t)\rangle,\end{aligned} \tag{25}$$

its formal solution is

$$|\rho(t)\rangle = \exp[-\kappa t(a^\dagger - \tilde{a})(a - \tilde{a}^\dagger)]|\rho_0\rangle. \tag{26}$$

It then follows from Eqs. (26) and (22) that

$$\begin{aligned}\langle\eta|\rho\rangle &= \langle\eta|\exp[-\kappa t(a^\dagger - \tilde{a})(a - \tilde{a}^\dagger)]|\rho_0\rangle \\ &= \mathrm{e}^{-\kappa t|\eta|^2}\langle\eta|\rho_0\rangle,\end{aligned} \tag{27}$$

then using the completeness relation of $|\eta\rangle$ and

$$: \exp[f(a^\dagger a + \tilde{a}^\dagger\tilde{a})] : = (f+1)^{a^\dagger a + \tilde{a}^\dagger\tilde{a}}, \tag{28}$$

as well as Eqs. (23) and (24) we derive

$$\begin{aligned}|\rho(t)\rangle &= \int \frac{\mathrm{d}^2\eta}{\pi}\mathrm{e}^{-\kappa t|\eta|^2}|\eta\rangle\langle\eta|\rho_0\rangle \\ &= \int \frac{\mathrm{d}^2\eta}{\pi} : \exp[-(1+\kappa t)|\eta|^2 + \eta(a^\dagger - \tilde{a}) + \eta^*(a - \tilde{a}^\dagger) + a^\dagger\tilde{a}^\dagger + a\tilde{a} - a^\dagger a - \tilde{a}^\dagger\tilde{a}] : |\rho_0\rangle \\ &= \frac{1}{1+\kappa t} : \exp\left[\frac{\kappa t}{1+\kappa t}(a^\dagger\tilde{a}^\dagger + a\tilde{a} - a^\dagger a - \tilde{a}^\dagger\tilde{a})\right] : |\rho_0\rangle \\ &= \frac{1}{1+\kappa t}\exp\left(\frac{\kappa t}{1+\kappa t}a^\dagger\tilde{a}^\dagger\right)\left(\frac{1}{1+\kappa t}\right)^{a^\dagger a + \tilde{a}^\dagger\tilde{a}}\exp\left(\frac{\kappa t}{1+\kappa t}a\tilde{a}\right)|\rho_0\rangle.\end{aligned} \tag{29}$$

Due to

$$\exp\left(\frac{\kappa t}{1+\kappa t}a\tilde{a}\right)|\rho_0\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\kappa t}{1+\kappa t}a\right)^n\rho_0 a^{\dagger n}|I\rangle, \tag{30}$$

thus equation (29) becomes

$$\begin{aligned}|\rho(t)\rangle &= \exp\left(\frac{\kappa t}{1+\kappa t}a^\dagger\tilde{a}^\dagger\right)\left(\frac{1}{1+\kappa t}\right)^{a^\dagger a + 1}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\kappa t}{1+\kappa t}a\right)^n\rho_0 a^{\dagger n}\left(\frac{1}{1+\kappa t}\right)^{a^\dagger a}|I\rangle \\ &= \sum_{m,n=0}^{\infty}\frac{1}{m!n!}\frac{(\kappa t)^{m+n}}{(\kappa t+1)^{m+n+1}}a^{\dagger m}\left(\frac{1}{1+\kappa t}\right)^{a^\dagger a}a^n\rho_0 a^{\dagger n}\left(\frac{1}{1+\kappa t}\right)^{a^\dagger a}a^m|I\rangle,\end{aligned} \tag{31}$$

and $\rho(t)$ is

$$\rho(t) = \sum_{m,n=0}^{\infty}\frac{1}{m!n!}\frac{(\kappa t)^{m+n}}{(\kappa t+1)^{m+n+1}}a^{\dagger m}\left(\frac{1}{1+\kappa t}\right)^{a^\dagger a}a^n\rho_0 a^{\dagger n}\left(\frac{1}{1+\kappa t}\right)^{a^\dagger a}a^m \equiv \sum_{m,n=0}^{\infty}M_{m,n}\rho_0 M_{m,n}^\dagger. \tag{32}$$

where $M_{m,n}$ is

$$M_{m,n} = \sqrt{\frac{1}{m!n!}\frac{(\kappa t)^{m+n}}{(\kappa t+1)^{m+n+1}}}a^{\dagger m}\left(\frac{1}{1+\kappa t}\right)^{a^\dagger a}a^n, \tag{33}$$

satisfying $\sum_{m,n=0}^{\infty}M_{m,n}^\dagger M_{m,n} = 1$, which is trace-conservative.

5. Generation of the displaced chaotic state

We now show that the displaced chaotic state can be generated by a coherent state passing through a diffusion channel. Taking $\rho_0 = |z\rangle\langle z|$ in Eq. (32), a pure coherent state, and using

$$\exp\left(a^\dagger a \ln\frac{1}{1+\kappa t}\right)|z\rangle = \exp\left(\frac{-|z|^2}{2}\right)\exp\left(za^\dagger\frac{1}{1+\kappa t}\right)|0\rangle$$

as well as the summation method within normal ordering symbol, we derive

$$\begin{aligned}
\rho_{|z\rangle\langle z|}(t) &= \sum_{m,n=0}^{\infty}\frac{1}{m!n!}\frac{(\kappa t)^{m+n}}{(\kappa t+1)^{m+n+1}}a^{\dagger m}\left(\frac{1}{1+\kappa t}\right)^{a^\dagger a}a^n|z\rangle\langle z|a^{\dagger n}\left(\frac{1}{1+\kappa t}\right)^{a^\dagger a}a^m\\
&= \sum_{m}^{\infty}\frac{1}{m!}\frac{(\kappa t)^m}{(\kappa t+1)^{m+1}}a^{\dagger m}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!}\frac{(\kappa t)^n}{(\kappa t+1)^n}\exp\left[a^\dagger a\ln\left(\frac{1}{1+\kappa t}\right)\right]|z\rangle\langle z|\exp\left[a^\dagger a\ln\left(\frac{1}{1+\kappa t}\right)\right]a^m\\
&= \sum_{m}^{\infty}\frac{1}{m!}\frac{(\kappa t)^m}{(\kappa t+1)^{m+1}}a^{\dagger m}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!}\frac{(\kappa t)^n}{(\kappa t+1)^n}\exp\left(-|z|^2\right)\exp\left[za^\dagger\frac{1}{1+\kappa t}\right]|0\rangle\langle 0|\exp\left[z^*a\frac{1}{1+\kappa t}\right]a^m\\
&= \sum_{m}^{\infty}\frac{1}{m!}\frac{(\kappa t)^m}{(\kappa t+1)^{m+1}}a^{\dagger m}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!}\frac{(\kappa t)^n}{(\kappa t+1)^n}\exp\left(-|z|^2\right)\exp\left[za^\dagger\frac{1}{1+\kappa t}\right]:\exp\left(-a^\dagger a\right):\exp\left[z^*a\frac{1}{1+\kappa t}\right]a^m\\
&= \exp\left[\frac{\kappa t}{\kappa t+1}|z|^2\right]\sum_{m}^{\infty}\frac{1}{m!}\frac{(\kappa t)^m}{(\kappa t+1)^{m+1}}a^{\dagger m}\exp\left(-|z|^2\right):\exp\left[\frac{za^\dagger}{1+\kappa t}+\frac{z^*a}{1+\kappa t}-a^\dagger a\right]:a^m\\
&= \frac{1}{1+\kappa t}:\exp\left(\frac{\kappa t}{1+\kappa t}a^\dagger a+\frac{z}{1+\kappa t}a^\dagger+\frac{z^*}{1+\kappa t}a-a^\dagger a-\frac{|z|^2}{1+\kappa t}\right):
\end{aligned}\tag{34}$$

or

$$\rho_{|z\rangle\langle z|}(t) = \frac{1}{1+\kappa t}\exp\left(-\frac{|z|^2}{1+\kappa t}\right)\exp\left(\frac{za^\dagger}{1+\kappa t}\right)\exp\left[a^\dagger a\ln\left(\frac{\kappa t}{1+\kappa t}\right)\right]\exp\left(\frac{z^*a}{1+\kappa t}\right),\tag{35}$$

which satisfies $\mathrm{tr}\,\rho_{|z\rangle\langle z|}(t)=1$. Comparing it with the standard form in Eq. (9)

$$\rho_{\mathrm{d}} = \left(1-\mathrm{e}^f\right)\exp\left(\frac{-|\lambda|^2}{1-\mathrm{e}^f}\right)\mathrm{e}^{\lambda a^\dagger}\mathrm{e}^{fa^\dagger a}\mathrm{e}^{\lambda^* a},\tag{36}$$

we see the correspondence

$$\begin{aligned}
&\lambda\to\frac{z}{1+\kappa t}, && f\to\ln\frac{\kappa t}{1+\kappa t},\\
&1-\mathrm{e}^f\to\frac{1}{1+\kappa t}, && \frac{-|\lambda|^2}{1-\mathrm{e}^f}\to\frac{-|z|^2}{1+\kappa t},
\end{aligned}\tag{37}$$

so $\rho_{|z\rangle\langle z|}(t)$ just belongs to the displaced chaotic states, when $t=0$,

$$\begin{aligned}
\rho_{|z\rangle\langle z|}(t=0) &= \exp\left(-|z|^2\right)\exp\left(za^\dagger\right)\\
&\times\lim_{t=0}:\exp\left(\frac{-1}{1+\kappa t}a^\dagger a\right):\exp\left(z^*a\right)=|z\rangle\langle z|.
\end{aligned}$$

6. Anti-normally ordered form of $\rho_{|z\rangle\langle z|}(t)$

By using the overlap of two coherent states

$$\langle z'\,|z\rangle = e^{-|z|^2/2-|z'|^2/2+z'^*z},\tag{38}$$

we have

$$\langle -z'|\rho\,(t)\,|z'\rangle = \mathrm{e}^{-|z'|^2}\int\frac{\mathrm{d}^2z}{\pi}P(z,t)\,\mathrm{e}^{-|z|^2-z'^*z+z^*z'}.\tag{39}$$

Since $(z^*z'-z'^*z)$ is pure-imaginary, we may consider the right-hand side as the Fourier transform of $P(z)\,\mathrm{e}^{-|z|^2}$, so its inverse is

$$P(z,t)\,\mathrm{e}^{-|z|^2} = \int\frac{\mathrm{d}^2z'}{\pi}\langle -z'|\rho\,(t)\,|z'\rangle\,\mathrm{e}^{|z'|^2+z'^*z-z^*z'},\tag{40}$$

hence

$$\begin{aligned}
\rho\,(t) &= \int\frac{\mathrm{d}^2z}{\pi}P(z,t)|z\rangle\langle z| = \int\frac{\mathrm{d}^2z}{\pi}\mathrm{e}^{-|z|^2}\int\frac{\mathrm{d}^2z'}{\pi}\langle -z'|\rho\,(t)\,|z'\rangle\,\mathrm{e}^{|z'|^2+z'^*z-z^*z'}\vdots\,\delta\left(z-a\right)\delta\left(z^*-a^\dagger\right)\vdots\\
&= \int\frac{\mathrm{d}^2\beta}{\pi}\vdots\langle-\beta|\rho|\beta\rangle\exp[|\beta|^2+\beta^*a-\beta a^\dagger+a^\dagger a]\vdots,
\end{aligned}\tag{41}$$

where the symbol $\vdots\,\vdots$ denotes anti-normal ordering, this is the operator identity of transverting density operator into its anti-normally ordered form. Substituting Eq. (35) into Eq. (40) and performing the integration leads to the anti-normally ordered form of $\rho_{|z\rangle\langle z|}(t)$

$$\rho_{|z\rangle\langle z|}(t)=\frac{1}{1+\kappa t}\int\frac{\mathrm{d}^2\beta}{\pi}:\exp\left[\frac{-\kappa t}{1+\kappa t}|\beta|^2+\beta^*(a-\frac{z}{1+\kappa t})+\beta(\frac{z^*}{1+\kappa t}-a^\dagger)-\frac{|z|^2}{1+\kappa t}+a^\dagger a\right]\vdots=\frac{1}{\kappa t}:\exp\left[\frac{-1}{\kappa t}(z-a)(z^*-a^\dagger)\right]\vdots,\tag{42}$$

its P-representation is just

$$P_{|z\rangle\langle z|}(\alpha,t)=\frac{1}{\kappa t}\exp\left[-\frac{1}{\kappa t}|z-\alpha|^2\right], \tag{43}$$

indeed satisfying Eq. (10).

7. Evolution of vacuum state in diffusion channel

In particular, when the initial state is a vacuum state $|0\rangle\langle 0|$, then after passing through a diffusion channel, equation (35) becomes to the following chaotic state,

$$\rho_{|0\rangle\langle 0|}(t)=\frac{1}{1+\kappa t}\exp\left(a^\dagger a\ln\frac{\kappa t}{1+\kappa t}\right). \tag{44}$$

By comparison with the standard formula of $\rho_{\rm c}$ in Eq. (1), we can identify

$$1-\mathrm{e}^f\to\frac{1}{1+\kappa t},\quad f\to\ln\frac{\kappa t}{1+\kappa t}, \tag{45}$$

let

$$\ln\frac{\kappa t}{1+\kappa t}\equiv-\beta\hbar\omega, \tag{46}$$

according to Eq. (2) we see that the photon number in this chaotic state is

$$\mathrm{tr}\left[\rho_{|0\rangle\langle 0|}(t)a^\dagger a\right]=\kappa t, \tag{47}$$

where κ is the diffusion constant, t denotes diffusion time. Therefore we know that chaotic state can be produced as an output of vacuum state evolving in diffusion channel.

8. Thermo-vacuum state for the displaced chaotic state

We now search for the thermo vacuum state of $\rho_{|z\rangle\langle z|}(t)$ in Eq. (35), by introducing a tilde-mode operator and coherent state $\left|\tilde{\beta}\right\rangle$,

$$[\tilde{a},\tilde{a}^\dagger]=1,\quad \tilde{a}\left|\tilde{0}\right\rangle=0,\quad \left|\tilde{\beta}\right\rangle=e^{-|\beta|^2/2+\beta\tilde{a}^\dagger}\left|\tilde{0}\right\rangle, \tag{48}$$

we deduce

$$\begin{aligned}
&\exp\left(\frac{z^*a}{\kappa t}\right)\exp\left[a^\dagger a\ln\left(\frac{\kappa t}{1+\kappa t}\right)\right]\exp\left(\frac{za^\dagger}{\kappa t}\right)=\exp\left(\frac{z^*a}{\kappa t}\right):\exp\left(\frac{\kappa t}{1+\kappa t}a^\dagger a-a^\dagger a\right):\exp\left(\frac{za^\dagger}{\kappa t}\right)\\
&=\exp\left(\frac{z^*a}{\kappa t}\right)\int\frac{\mathrm{d}^2\beta}{\pi}:\exp\left(-|\beta|^2+\beta^*\sqrt{\frac{\kappa t}{1+\kappa t}}a^\dagger+\beta\sqrt{\frac{\kappa t}{1+\kappa t}}a-a^\dagger a\right):\exp\left(\frac{za^\dagger}{\kappa t}\right)\\
&=\int\frac{\mathrm{d}^2\beta}{\pi}\langle\tilde{\beta}|\,\tilde{0}\rangle\exp\left(\frac{z^*\beta^*}{\kappa t}\sqrt{\frac{\kappa t}{1+\kappa t}}\right)\left|\beta^*\sqrt{\frac{\kappa t}{1+\kappa t}}\right\rangle\left\langle\beta^*\sqrt{\frac{\kappa t}{1+\kappa t}}\right|\exp\left(\frac{z\beta}{\kappa t}\sqrt{\frac{\kappa t}{1+\kappa t}}\right)\langle\tilde{0}|\,\tilde{\beta}\rangle\\
&=\int\frac{\mathrm{d}^2\beta}{\pi}\langle\tilde{\beta}|\exp\left(\frac{z^*\beta^*}{\kappa t}\sqrt{\frac{\kappa t}{1+\kappa t}}+\beta^*\sqrt{\frac{\kappa t}{1+\kappa t}}a^\dagger\right)|0,\tilde{0}\rangle\langle 0,\tilde{0}|\exp\left(\frac{z\beta}{\kappa t}\sqrt{\frac{\kappa t}{1+\kappa t}}+\beta\sqrt{\frac{\kappa t}{1+\kappa t}}a\right)\left|\tilde{\beta}\right\rangle\\
&=\int\frac{\mathrm{d}^2\beta}{\pi}\langle\tilde{\beta}|\exp\left(z^*\tilde{a}^\dagger\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+\sqrt{\frac{\kappa t}{1+\kappa t}}\tilde{a}^\dagger a^\dagger\right)|0,\tilde{0}\rangle\langle 0,\tilde{0}|\exp\left(z\tilde{a}\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+\sqrt{\frac{\kappa t}{1+\kappa t}}\tilde{a}a\right)\left|\tilde{\beta}\right\rangle\\
&=t\tilde{r}\int\frac{\mathrm{d}^2\beta}{\pi}\left|\tilde{\beta}\right\rangle\langle\tilde{\beta}|\exp\left(z^*\tilde{a}^\dagger\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+\sqrt{\frac{\kappa t}{1+\kappa t}}\tilde{a}^\dagger a^\dagger\right)|0,\tilde{0}\rangle\langle 0,\tilde{0}|\exp\left(z\tilde{a}\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+\sqrt{\frac{\kappa t}{1+\kappa t}}\tilde{a}a\right),
\end{aligned} \tag{49}$$

from which we know that the thermo-vacuum state is

$$\frac{1}{\sqrt{1+\kappa t}}\exp\left(-\frac{|z|^2}{2\kappa t}\right)\exp\left(z^*\tilde{a}^\dagger\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+\sqrt{\frac{\kappa t}{1+\kappa t}}\tilde{a}^\dagger a^\dagger\right)|0,\tilde{0}\rangle\equiv|\psi\rangle. \tag{50}$$

It is in form a two-mode squeezed state, and is normalized, *e.g.*,

$$\begin{aligned}
\langle\psi\,|\psi\rangle&=\frac{1}{1+\kappa t}\exp\left(-\frac{|z|^2}{\kappa t}\right)\langle 0,\tilde{0}|\exp\left(z\tilde{a}\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+\sqrt{\frac{\kappa t}{1+\kappa t}}\tilde{a}a\right)\exp\left(z^*\tilde{a}^\dagger\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+\sqrt{\frac{\kappa t}{1+\kappa t}}\tilde{a}^\dagger a^\dagger\right)|0,\tilde{0}\rangle\\
&=\frac{1}{1+\kappa t}\exp\left(-\frac{|z|^2}{\kappa t}\right)\int\frac{\mathrm{d}^2z_1'\,\mathrm{d}^2z_2'}{\pi^2}\langle 0,\tilde{0}|\exp\left(z\tilde{a}\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+\sqrt{\frac{\kappa t}{1+\kappa t}}\tilde{a}a\right)|z_1',\tilde{z}_2'\rangle\langle z_1',\tilde{z}_2'|\\
&\quad\times\exp\left(z^*\tilde{a}^\dagger\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+\sqrt{\frac{\kappa t}{1+\kappa t}}\tilde{a}^\dagger a^\dagger\right)|0,\tilde{0}\rangle=\frac{1}{1+\kappa t}\exp\left(-\frac{|z|^2}{\kappa t}\right)\\
&\quad\times\int\frac{\mathrm{d}^2z_1'\,\mathrm{d}^2z_2'}{\pi^2}\exp\left(-|z_1'|^2-|z_2'|^2+zz_2'\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+\sqrt{\frac{\kappa t}{1+\kappa t}}z_2'z_1'+z^*z_2'^*\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+\sqrt{\frac{\kappa t}{1+\kappa t}}z_2'^*z_1'^*\right)\\
&=\frac{1}{1+\kappa t}\exp\left(-\frac{|z|^2}{\kappa t}\right)\int\frac{\mathrm{d}^2z_2'}{\pi}\exp\left(-\frac{1}{1+\kappa t}|z_2'|^2+zz_2'\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+z^*z_2'^*\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}\right)=1.
\end{aligned} \tag{51}$$

Using the thermo-vacuum state we can directly calculate the photon number variation in the diffusion process via the following steps. Firstly, we consider

$$\begin{aligned}\langle\psi\,\tilde{a}\tilde{a}^{\dagger}\,|\psi\rangle &= \frac{1}{1+\kappa t}\exp\left(-\frac{|z|^2}{\kappa t}\right)\langle 0,\tilde{0}|\exp\left(z\tilde{a}\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+\sqrt{\frac{\kappa t}{1+\kappa t}}\tilde{a}a\right)\int\frac{\mathrm{d}^2z_1'\,\mathrm{d}^2z_2'}{\pi^2}\,\tilde{a}|z_1',\tilde{z}_2'\rangle\langle z_1',\tilde{z}_2'|\tilde{a}^{\dagger}\\ &\quad\times\exp\left(z^*\tilde{a}^{\dagger}\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+\sqrt{\frac{\kappa t}{1+\kappa t}}\tilde{a}^{\dagger}a^{\dagger}\right)|0,\tilde{0}\rangle\\ &= \frac{1}{1+\kappa t}\exp\left(-\frac{|z|^2}{\kappa t}\right)\int\frac{\mathrm{d}^2z_2'}{\pi}\exp\left(-\frac{1}{1+\kappa t}|z_2'|^2+zz_2'\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}+z^*z_2'^*\sqrt{\frac{1}{\kappa t\,(1+\kappa t)}}\right)|z_2'|^2\\ &= (1+\kappa t)\exp\left(-\frac{|z|^2}{\kappa t}\right)\int\frac{\mathrm{d}^2z_2'}{\pi}\exp\left(-|z_2'|^2+zz_2'\sqrt{\frac{1}{\kappa t}}+z^*z_2'^*\sqrt{\frac{1}{\kappa t}}\right)|z_2'|^2\\ &= \exp\left(-\frac{|z|^2}{\kappa t}\right)\left(-\frac{\partial}{\partial f}\right)\int\frac{\mathrm{d}^2z_2'}{\pi}\exp\left(-f|z_2'|^2+zz_2'\sqrt{\frac{1}{\kappa t}}+z^*z_2'^*\sqrt{\frac{1}{\kappa t}}\right)\Bigg|_{f=1}\\ &= -(1+\kappa t)\exp\left(-\frac{|z|^2}{\kappa t}\right)\frac{\partial}{\partial f}\left[\frac{1}{f}\exp\frac{|z|^2}{f\kappa t}\right]|_{f=1}=-\exp\left(-\frac{|z|^2}{\kappa t}\right)\left(-\frac{1}{f^2}\exp\frac{|z|^2}{f\kappa t}-\frac{1}{f^3}\frac{|z|^2}{\kappa t}\exp\frac{|z|^2}{f\kappa t}\right)\Bigg|_{f=1}\\ &= (1+\kappa t)\left(1+\frac{|z|^2}{\kappa t}\right),\end{aligned}\tag{52}$$

then according to

$$a\,|\psi\rangle=\sqrt{\frac{\kappa t}{1+\kappa t}}\tilde{a}^{\dagger}\,|\psi\rangle\,,\tag{53}$$

we obtain

$$\langle\psi\,a^{\dagger}a\,|\psi\rangle=\frac{\kappa t}{1+\kappa t}\,\langle\psi\,\tilde{a}\tilde{a}^{\dagger}\,|\psi\rangle=\kappa t\left(1+\frac{|z|^2}{\kappa t}\right)=|z|^2+\kappa t.\tag{54}$$

Comparing with the initial photon number

$$\mathrm{tr}\left[a^{\dagger}a\rho_{|z\rangle\langle z|}(0)\right]=|z|^2\,,$$

we see that the photon number is increases by κt, by adjusting the diffusion coefficient κ, we may realize quantum controlling. Obviously, when κ is small, the meaning of diffusion is clear.

In summary, we have shown that a chaotic state can be produced as an output of vacuum state evolving in diffusion channel, while displaced chaotic state is output of a coherent state evolving in diffusion channel. The displaced chaotic state may be used exhibiting quantum controlling.

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