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Protecting the entanglement of two-qubit over quantum channels with memory via weak measurement and quantum measurement reversal*

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Based on the quantum technique of the weak measurement and quantum measurement reversal (WMR), we propose a scheme to protect entanglement for an entangled two-qubit pure state from four typical quantum noise channels with memory, *i.e.*, the amplitude damping channel, the phase damping channel, the bit flip channel, and the depolarizing channel. For a given initial state $|\psi\rangle = a|00\rangle + d|11\rangle$, it is found that the WMR operation indeed helps to protect entanglement from the above four quantum channels with memory, and the protection effect of WMR scheme is better when the coefficient a is small. For the other initial state $|\phi\rangle = b|01\rangle + c|10\rangle$, the effect of the protection scheme is the same regardless of the coefficient b and the WMR operation can protect entanglement in the amplitude damping channel with memory. Moreover, the protection of entanglement in quantum noise channels without memory in contrast to the results of the channels with memory is more effective. For $|\psi\rangle$ or $|\phi\rangle$, we also find that the memory parameters play a significant role in the suppression of entanglement sudden death and the initial entanglement can be drastically amplified. Another more important result is that the relationship between the concurrence, the memory parameter, the weak measurement strength, and quantum measurement reversal strength is found through calculation and discussion. It provides a strong basis for the system to maintain maximum entanglement in the nosie channel.

Keywords: quantum entanglement, weak measurement and quantum measurement reversal, quantum channel with memory, concurrence

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1. Introduction

Quantum entanglement[1,2] plays a central role in quantum information processing (QIP)[3] and has attracted considerable attention. However, quantum entanglement is very fragile and prone to the quantum nosie channel due to any realistic systems inevitably interacting with their external channels and, in certain cases, entanglement sudden death (ESD).[4–7] The decoherence is the most limiting factor for the applications of entanglement in many tasks, such as teleportation,[8] dense coding,[9] quantum cryptography,[10] remote state preparation,[11,12] quantum remote control,[13] and distributed quantum computation.[14] This means that quantum entanglement is usually very difficult to be created, maintained, and manipulated in quantum systems. Therefore, it is very necessary and significant to create and preserve the quantum entanglement in the fields of QIP.

In this context, it is an extremely fundamental task to protect entanglement from the noise channel. There are basically two types of quantum noise channels, without memory and with memory. Whether the channel has memory or not, the entanglement will disappear gradually with decoherence, so it is very important to protect entanglement. An interesting quantum technique, which is the weak measurement theory, proposed by Aharonov, Albert, and Vaidman,[15,16] has intrigued many researchers. The weak measurement and quantum measurement reversal (WMR) can protect the entanglement of a single-qubit[17–19] and even circumvent ESD of two qubits,[20–25] a hybrid qubit-qutrit system,[26] and a two-qutrit system.[27] The WMR is also applied to protect and amplify quantum coherence.[28,29] Early researches focus on the assumption that the channel acts as a memoryless configuration, which acts identically and independently on the qubits. However, in many realistic scenarios, the Kraus operators of the noise channel map cannot be expressed in a tenor product form, which means that the noise channel has memory or is correlated among consecutive uses, so memory or correlated noise channels appear to be more reasonable and significant in quantum information theory. In response, memory or correlated amplitude damping channels[30,31] and quantum coherence over noise channels with memory[32] have been investigated. While the application of WMR in various quantum noise channels with memory has not been studied.

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Motivated by the above consideration, we study the role of WMR in the protection of entanglement for the two-qubit system in the amplitude damping (AD) channel, the phase damping (PD) channel, the bit flip (BF) channel, and the depolarizing (Dep.) channel with memory. The memory effects are characterized by a memory parameter, which ranges from 0 to 1. The evolution of entanglement under the correlated channel are investigated with and without the assistance of WMR. We derive the exact expressions of the entanglement and find that the WMR operation indeed helps to protect entanglement from different correlated decoherence channels for two initial Bell-like states. What is more significant is that the WMR scheme and improving memory parameters can not only effectively circumvent ESD but also amplify the initial entanglement, which is rather significant in QIP.

This paper is structured as follows: In Section 2, we introduce the evolution process of the noise channel, the concurrence, and the WMR theory of the scheme. In Section 3, we first review four noise channels with memory and give the detailed scheme of entanglement protection based on the WMR. Finally, we summarize the conclusions in Section 4.

2. Basic theory

2.1. Noise channel evolution process and initial states

In what follows, we focus on the noise channels with memory for two consecutive uses. The Kraus operators for two consecutive uses of a channel with memory are[33]

$$E_{i,j} = \sqrt{P_i[(1-\eta)P_j + \eta\delta_{i,j}]}A_i \otimes A_j, \tag{1}$$

where $0 \leq \eta \leq 1$ is the memory coefficient of channel. The limiting cases $\eta = 0$ and $\eta = 1$ correspond to channels without and with full memory, respectively.

Mathematically, the evolved state of two consecutive uses with time-correlated Markov noise is described by[34]

$$\varepsilon(\rho) = (1-\eta)\sum_{i,j} E^{\mathrm{u}}_{i,j}\rho E^{\mathrm{u}+}_{i,j} + \eta\sum_{k} E^{\mathrm{c}}_{k,k}\rho E^{\mathrm{c}+}_{k,k}. \tag{2}$$

It is evident from the above expression that the same operation with probability η specified by $E^{\mathrm{c}}_{k,k}$ is correlated noise, while the probability $1-\eta$ corresponding to $E^{\mathrm{u}}_{i,j}$ is uncorrelated noise. In the following, we make use of Eq. (2) to study the effect of memory parameter of various quantum channels on quantum entanglement of two-qubit states. For simplicity, in this paper, we first consider two Bell-like states

$$|\psi\rangle = a|00\rangle + d|11\rangle, \tag{3}$$

$$|\phi\rangle = b|01\rangle + c|10\rangle, \tag{4}$$

where the superposition coefficients are complex and satisfy normalization $|a|^2 + |d|^2 = 1$ and $|b|^2 + |c|^2 = 1$.

2.2. Concurrence

For the two-qubit system, a faithful measure of the entanglement is the concurrence proposed by Wootters *et al.*,[35] which is defined as

$$C(\rho_{AB}) = \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\}, \tag{5}$$

where λ_i $(i = 1,2,3,4)$ are the non-negative square roots of the moduli of the eigenvalues of $M = \rho\left(\sigma_y^A \otimes \sigma_y^B\right)\rho^*\left(\sigma_y^A \otimes \sigma_y^B\right)$, with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ and ρ^* denoting the complex conjugate of ρ. σ_y^A and σ_y^B are Pauli matrices for subsystems A and B.

In special cases, the matrix ρ_{AB} will be simplified to X-class state, as follows:[36]

$$\rho_{AB} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{6}$$

Then, the concurrence can be expressed as

$$C(\rho_{AB}) = 2\max\{0, |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}, |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}\}. \tag{7}$$

Obviously, $C = 0$ means the entanglement is zero, and $C = 1$ stands for the maximal entanglement.

2.3. Weak measurement and quantum measurement reversal

To suppress the entanglement decay, we perform the WMR.[37,38] For the qubit case, the weak measurement (WM) can be written as a non-unitary quantum operation

$$M_{WM} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \tag{8}$$

where $0 \leq p \leq 1$ is the WM strength for the qubit. The quantum measurement reversal (QMR) for the qubit is also a non-unitary quantum operation that can be written as

$$M_{\mathrm{QMR}} = \begin{pmatrix} \sqrt{1-q} & 0 \\ 0 & 1 \end{pmatrix}, \tag{9}$$

where $0 \leq q \leq 1$ is the strength of QMR.

The final state after the sequential pre-WM, quantum noise channel with memory and a post-QMR is

$$\begin{aligned}\rho_{\mathrm{WMR}} = {} & (M_{\mathrm{QMR}} \otimes M_{\mathrm{QMR}})\{\varepsilon[(M_{\mathrm{WM}} \otimes M_{\mathrm{WM}})\rho_0 \\ & \times (M_{\mathrm{WM}} \otimes M_{\mathrm{WM}})^+]\}(M_{\mathrm{QMR}} \otimes M_{\mathrm{QMR}})^+.\end{aligned} \tag{10}$$

3. Quantum entanglement in four noise channels with memory

3.1. Amplitude damping channel with memory

The AD channel, which is used to characterize spontaneous emission, describes the energy dissipation from a quantum system. The Kraus operators for a single-qubit system are given by[34]

$$A_0 = \begin{pmatrix} \sqrt{1-r} & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{r} & 0 \end{pmatrix}, \tag{11}$$

where $r = 1 - e^{-\gamma t}$. We consider two consecutive uses of the AD channel, the channel without memory for which the channel acts independently over each channel input, and the Kraus operators of channel without memory are given by

$$E^{u}_{i,j} = A_i \otimes A_j, \quad (i,j = 0,1). \tag{12}$$

However, for the channel with memory, the Kraus operators cannot be constructed by $A_i \otimes A_i$, since the quantum channels with memory where the noise over successive channel uses exhibits some correlation, the explicit expression of the Kraus operators $E^{c}_{k,k}$ is determined by solving the correlated Lindblad equation,[34] which gives the following formalism:

$$E^{c}_{00} = \begin{pmatrix} \sqrt{1-r} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$E^{c}_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{r} & 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

3.1.1. Entanglement for the initial state $|\psi\rangle$

First, we consider the case that the initial pure state is $|\psi\rangle$. According to Eq. (2), the corresponding time-dependent density matrix of the AD channel with memory in the standard product basis $\{|ij\rangle, (i,j = 0,1)\}$ is

$$\begin{aligned} \rho^{A\psi}_{11} &= |a|^2 + (\bar{\eta} r^2 + \eta r)|d|^2, \\ \rho^{A\psi}_{22} &= \rho^{A\psi}_{33} = \bar{\eta} r \bar{r} |d|^2, \\ \rho^{A\psi}_{44} &= (\bar{\eta} \bar{r}^2 + \eta \bar{r})|d|^2, \\ \rho^{A\psi}_{14} &= \rho^{A\psi *}_{41} = (\bar{\eta}\bar{r} + \eta\sqrt{\bar{r}}) a d^*, \end{aligned} \tag{14}$$

where we have used the notation $\bar{o} \equiv 1 - o$ with $o = r, \eta, p, q$.

According to the definition of the concurrence, the concurrence $C^{A\psi}_{d}$ of the quantum state $\rho^{A\psi}$ is calculated to be

$$C^{A\psi}_{d} = \max\{0, \Delta^{A\psi}_{d} = 2[(\bar{\eta}\bar{r} + \eta\sqrt{\bar{r}})|ad| - \bar{\eta} r \bar{r} |d|^2]\}. \tag{15}$$

The behaviors of concurrence $C^{A\psi}_{d}$ as function of dimensionless parameters decoherent strength r and memory parameter η for two initial entangled states are illustrated in Fig. 1 with $|a| = 1/3$ and $|b| = 1/3$. It is noted that for the AD memoryless channel ($\eta = 0$) in Fig. 1(a), the concurrence reduces to $C^{A\psi}_{d} = \max\{0, 2\bar{r}|d|(|a| - r|d|)\}$, which degrades monotonically as r increases. Furthermore, it is essential to point out that $C^{A\psi}_{d}$ experiences ESD. While considering memory effects in correlated AD channel, the results are more interesting. The concurrence $C^{A\psi}_{d}$ increases with the increase of memory parameter η, which seems that the memory effect has a twofold effect on the entanglement. One suppresses the entanglement degradation, while the other induces the entanglement revival after a short period of ESD. This result is similar to the entanglement dynamics in non-Markovian environments, where memory effect also induces the entanglement revival.[39] When the memory effect is strong enough, the phenomenon of ESD is completely eliminated.

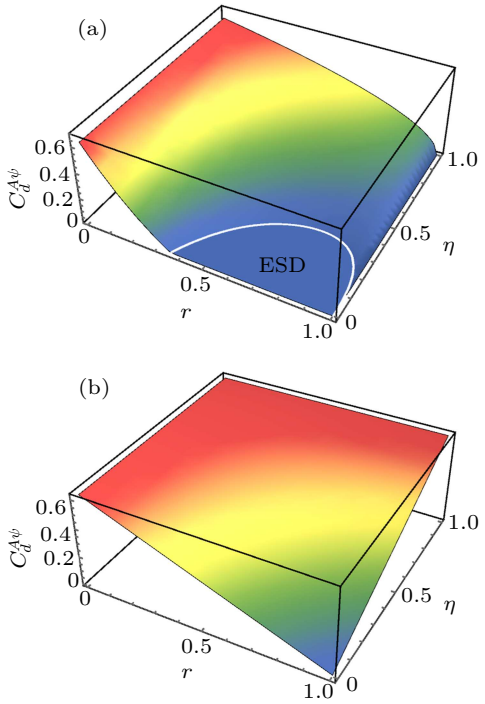


Fig. 1. In the AD channel with memory, the concurrence $C_d^{A\psi(\phi)}$ is the function of decoherent strength r and the memory parameter η for two initial entangled states $|\psi\rangle = a|00\rangle + d|11\rangle$ and $|\phi\rangle = b|01\rangle + c|10\rangle$. We assume (a) $|a| = 1/3$, (b) $|b| = 1/3$.

After the sequential pre-WM, AD channel with memory and a post-QMR , the concurrence of density matrix $\rho_r^{A\psi}$ is

$$C^{A\psi}_{r} = \max\left\{0, \Delta^{A\psi}_{r} = \frac{2\bar{q}(|X^{A\psi}| - V^{A\psi})}{U^{A\psi}\bar{q}^2 + 2V^{A\psi}\bar{q} + W^{A\psi}}\right\}. \tag{16}$$

where $X^{A\psi} = (\bar{\eta}\bar{r} + \eta\sqrt{\bar{r}})\bar{p} a d^*$, $U^{A\psi} = |a|^2 + (\bar{\eta} r^2 + \eta r)\bar{p}^2 |d|^2$, $V^{A\psi} = \bar{\eta} r \bar{r} \bar{p}^2 |d|^2$, and $W^{A\psi} = (\bar{\eta}\bar{r}^2 + \eta\bar{r})\bar{p}^2|d|^2$. The probability of success is $P^{A\psi} = U^{A\psi}\bar{q}^2 + 2V^{A\psi}\bar{q} + W^{A\psi}$.

In order to make sure that $\Delta_r^{A\psi}$ reaches its maximal value, we need to optimize the strength q of QMR. Using the fact that

$$a + b \geq 2\sqrt{ab}, \quad (a, b > 0), \tag{17}$$

we have the following inequality:

$$\Delta^{A\psi}_{r} \leq \frac{|X^{A\psi}| - V^{A\psi}}{V^{A\psi} + \sqrt{U^{A\psi}W^{A\psi}}}, \tag{18}$$

where the equality sign stands if and only if

$$\bar{q} = \sqrt{\frac{W^{A\psi}}{U^{A\psi}}}. \tag{19}$$

Substituting the optimal strength of QMR, *i.e.*, Eq. (19), into the expression of $\Delta_r^{A\psi}$ in Eq. (16), we can get the maximal concurrence.

To quantify the effect of the WMR, we define $\chi^{A\psi} = C^{A\psi}_{r}/C^{A\psi}_{d}$, where $\chi^{A\psi} > 1 (\leq 1)$ means that the WMR does

(not) work. In Fig. 2, it is clear to see that $C_r^{A\Psi}$ is always larger than $C_d^{A\Psi}$, which means that the WMR indeed helps to protect entanglement from correlated AD decoherence. We found that the protection effect of the WMR scheme decreases with the increase of coefficient a until the coefficient is $a=\sqrt{2}/2$, and then the protection effect increases with a increasing, but the effect is still not as good as that of a with smaller value. Especially for the optimal initial state, although the system has more entanglement than other states, the protection effect of WMR is much weaker. In view of the strong protection effect of WMR when a is taken as smaller value, the coefficients a in subsequent discussions are all taken as $|a|=1/3$.

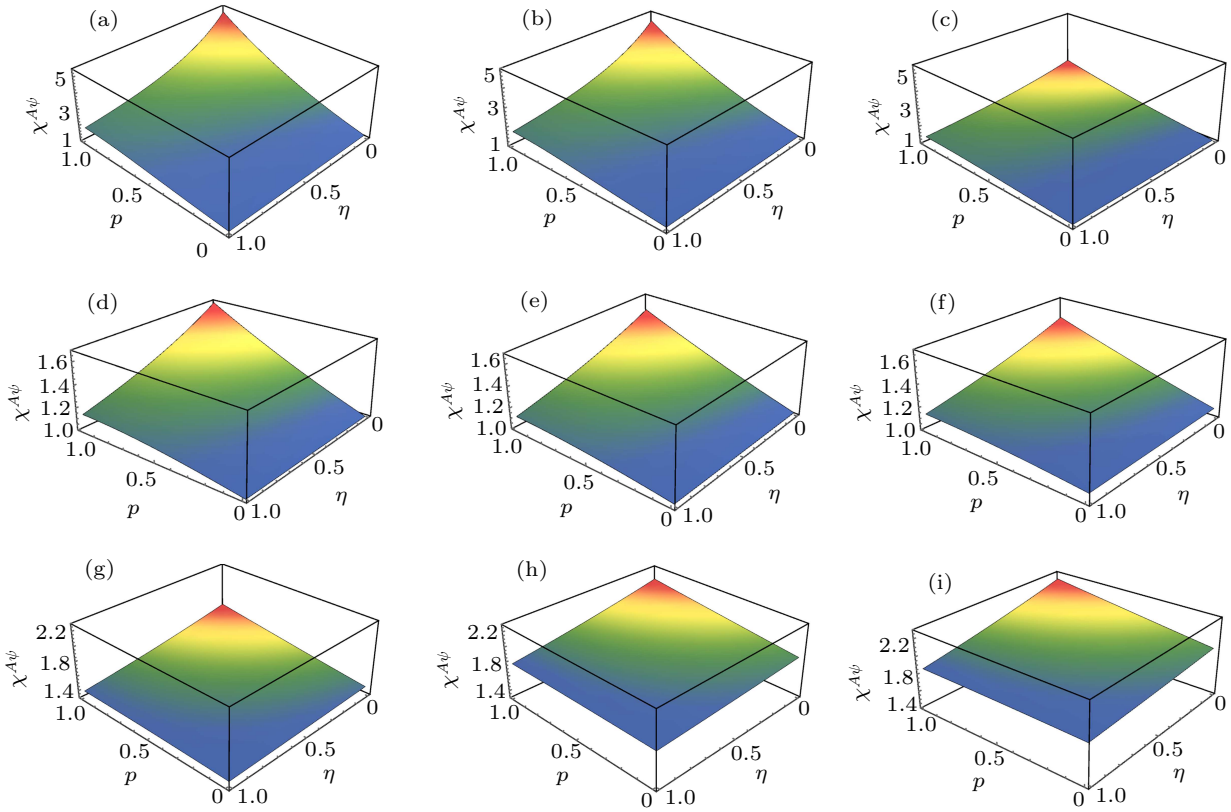


Fig. 2. In the AD channel with memory, $\chi^{A\Psi}$ is a function of the weak measurement strength p and the memory parameter η for the initially entangled states $|\psi\rangle=a|00\rangle+d|11\rangle$. We assume $r=0.2$, (a) $a=\sqrt{0.1}$; (b) $a=1/3$; (c) $a=\sqrt{0.2}$; (d) $a=\sqrt{0.4}$; (e) $a=\sqrt{2}/2$; (f) $a=\sqrt{0.6}$; (g) $a=\sqrt{0.8}$; (h) $a=\sqrt{8}/3$; (i) $a=\sqrt{0.9}$.

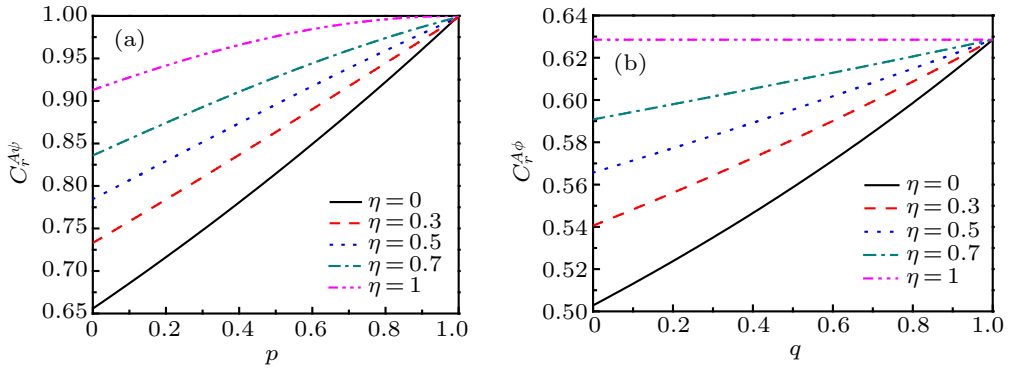


Fig. 3. In the AD channel with memory, the concurrence $C_r^{A\Psi(\phi)}$ is a function of WM strength p (q) for the two initially entangled states $|\psi\rangle=a|00\rangle+d|11\rangle$ and $|\phi\rangle=b|01\rangle+c|10\rangle$. We assume $r=0.2$, (a) $|a|=1/3$, (b) $|b|=1/3$.

In Fig. 3(a), we also investigate how the concurrence $C_r^{A\Psi}$ changes with the WM strength p. One can obviously see that $C_r^{A\Psi}$ increases significantly with the increase of p for the fixed r. Moreover, it is clear to see that the stronger η is, the larger the concurrence $C_r^{A\Psi}$ is. Therefore, if we want to get larger concurrence, η should be bigger and p can approach 1.

3.1.2. Entanglement for the initial state $|\phi\rangle$

Next, we consider the case of the initial pure state is $|\phi\rangle$. According to Eq. (2), the corresponding time-dependent density matrix of the AD channel with memory in the standard product basis $\{|ij\rangle,\ (i,j=0,1)\}$ is

$$\rho_{11}^{A\phi}=\bar{\eta}r,\quad \rho_{22}^{A\phi}=\bar{\eta}\bar{r}|b|^2+\eta|b|^2,$$

$$\rho_{33}^{A\phi} = \bar{\eta}\bar{r}|c|^2 + \eta|c|^2, \quad \rho_{23}^{A\varphi} = \rho_{32}^{A\varphi *} = (\bar{\eta}\bar{r}+\eta)bc^*. \quad (20)$$

The concurrence $C_d^{A\phi}$ of $\rho^{A\phi}$ is calculated to be

$$C_d^{A\phi} = \max\{0, \Delta_d^{A\phi} = 2(\bar{\eta}\bar{r}+\eta)|bc|\}. \quad (21)$$

The concurrence $C_d^{A\phi}$ decreases with the increasing of decoherent strength r and increases with memory parameter η increasing in Fig. 1(b). In the case of $\eta = 1$, the concurrence $C_d^{A\phi}$ is unaffected by the AD decoherence.

After the sequential pre-WM, AD channel with memory and a post-QMR , the concurrence of the density matrix $\rho_r^{A\phi}$ is

$$C_r^{A\phi} = \max\left\{0, \Delta_r^{A\phi} = \frac{2(\bar{\eta}\bar{r}+\eta)|bc|}{\bar{\eta}r\bar{q}+(\bar{\eta}\bar{r}+\eta)}\right\}. \quad (22)$$

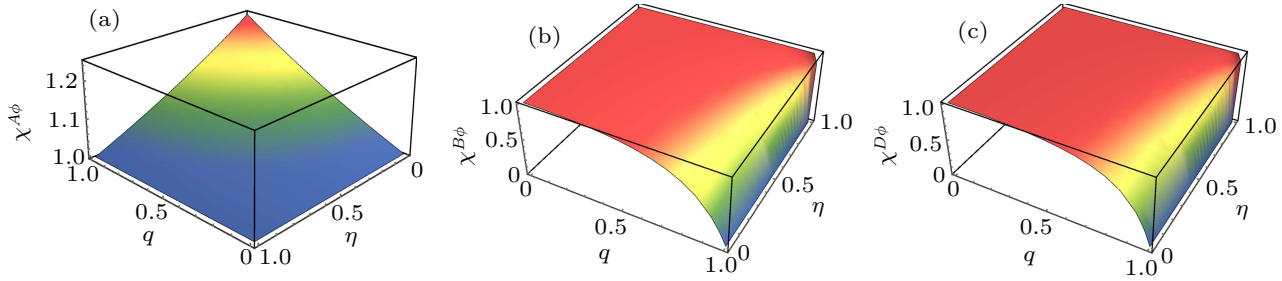


Fig. 4. In the quantum channels with memory, $\chi^{A(B,D)\phi}$ is a function of the weak measurement strength p and the memory parameter η for the initially entangled state $|\phi\rangle = b|01\rangle + c|10\rangle$. We assume $r = 0.2$ and $|b| = 1/3$.

As shown in Fig. 4(a), it is easy to see that $\chi^{A\phi} = C_r^{A\phi}/C_d^{A\phi}$ is always larger than 1. This means that WMR can improve entanglement from the correlated AD channel for the initial state $|\phi\rangle$, but it is not as good as the initial state $|\psi\rangle$. $\chi^{A\phi} = C_r^{A\phi}/C_d^{A\phi}$ is independent of the coefficients b and c. So the effect of the optimal protection scheme is the same regardless of the coefficients. Here, the probability of success is $P^{A\phi} = \bar{\eta}r\bar{p}\bar{q}^2 + (\bar{\eta}\bar{r}+\eta)\bar{p}\bar{q}$. We find that $C_r^{A\phi}$ is independent of the WM strength p. So in Fig. 3(b), we give the relationship between the concurrence $C_r^{A\phi}$ and QMR strength q. It is obvious that $C_r^{A\phi}$ can be enhanced by increasing q and η.

3.2. Phase damping channel with memory

In this section, we go on to study the quantum entanglement of two qubits over the PD channel with memory. The Kraus operators for a single qubit are defined as

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-r} \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{r} \end{pmatrix}. \quad (23)$$

By unitary combination of E_0 and E_3, a new set of operation elements of PD channel can be given in terms of the Pauli operators $\sigma_0 = I$ and σ_3,

$$A_i = \sqrt{P_i}\sigma_i, \quad (24)$$

where $P_0 = (1+\sqrt{1-r})/2$ and $P_3 = (1-\sqrt{1-r})/2$.

Similarly, if we consider two consecutive uses of PD channel, there are no correlations between the two uses of the channel without memory, and the Kraus operators are given by[40]

$$E_{i,j}^{\mathrm{u}} = \sqrt{P_i P_j}\sigma_i \otimes \sigma_j. \quad (25)$$

In the presence of memory, the two uses of the channel exhibit some correlation, and the Kraus operators of this channel with memory are given by

$$E_{k,k}^{\mathrm{c}} = \sqrt{P_k}\sigma_k \otimes \sigma_k, \quad (26)$$

where $i, j, k = (0, 3)$.

3.2.1. Entanglement for the initial state $|\psi\rangle$

According to Eq. (2), the concurrence $C_d^{P\psi}$ of the corresponding time-dependent density matrix $\rho^{P\psi}$ is

$$C_d^{P\psi} = \max\{0, \Delta_d^{P\psi} = 2[\bar{\eta}(P_0 - P_3)^2 + \eta(P_0 + P_3)]|ad|\}. \quad (27)$$

The concurrence $C_d^{P\psi}$ decreases with the increase of decoherent strength r and increases with memory parameter η increasing in Fig. 5(a). In the case of $\eta = 1$, the concurrence $C_d^{P\psi}$ is unaffected by the PD decoherence.

After the WMR scheme, the concurrence $C_r^{P\psi}$ of the density matrix $\rho_r^{P\psi}$ is

$$C_r^{P\psi} = \max\left\{0, \Delta_r^{P\psi} = \frac{2\bar{q}\bar{p}[\bar{\eta}(P_0 - P_3)^2 + \eta(P_0 + P_3)]|ad|}{|a|^2\bar{q}^2 + |d|^2\bar{p}^2}\right\}. \quad (28)$$

According to Eq. (17), we also can obtain the maximal concurrence $C_r^{P\psi}$ in Eq. (28). The maximal concurrence $C_r^{P\psi}$ of the state $\rho_r^{P\psi}$ appears at the strength of measurement reversal $\bar{q} = |d|\bar{p}/|a|$. So, with the optimal value q, $\chi^{P\psi} = 1/2|ad| > 1$ with $|a| \neq \sqrt{2}/2$, implying $C_r^{P\psi}$ is always larger than $C_d^{P\psi}$. We have demonstrated that WMR can indeed be used to protect entanglement against the correlated PD channel when the initial state is the Bell-like state $|\psi\rangle$. Here, the probability of success is $P^{P\psi} = |a|^2\bar{q}^2 + |d|^2\bar{p}^2$, which is independent of r and η.

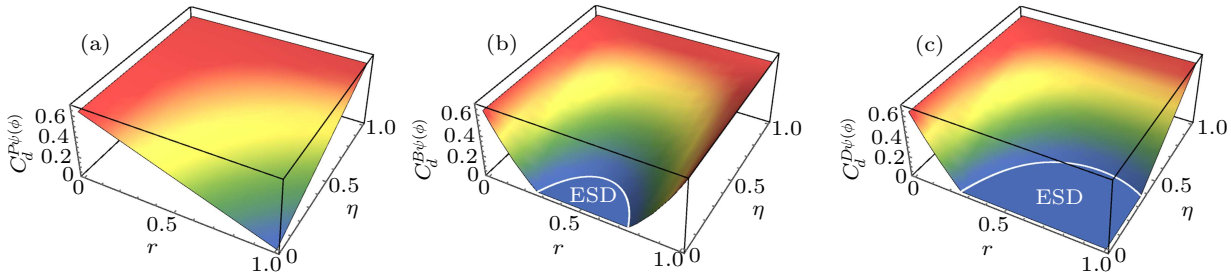


Fig. 5. In the quantum channels with memory, the concurrence $C_d^{P(B,D)\psi(\phi)}$ is a function of decoherent strength r and the memory parameter η for the two initial entangled states $|\psi\rangle = a|00\rangle + d|11\rangle$ and $|\phi\rangle = b|01\rangle + c|10\rangle$. We assume $|a| = 1/3$ and $|b| = 1/3$. (a) PD channel; (b) BF channel; (c) depolarizing channel.

3.2.2. Entanglement for the initial state $|\phi\rangle$

For the Bell-like state $|\phi\rangle$, the concurrence $C_d^{P\phi}$ of the corresponding time-dependent density matrix $\rho^{P\phi}$ is

$$C_d^{P\phi} = 2\max\{0, \Delta_d^{P\phi} = [\bar{\eta}(P_0 - P_3)^2 + \eta(P_0 + P_3)]\,|bc|\}, \quad (29)$$

which means the evolution of the concurrence $C_d^{P\phi}$ is the same as that of $C_d^{P\psi}$as shown in Fig. 5(a).

After the WMR scheme, the concurrence of the density matrix $\rho_r^{P\phi}$ is

$$C_r^{P\phi} = 2\max\{0, \Delta_r^{P\phi} = [\bar{\eta}(P_0 - P_3)^2 + \eta(P_0 + P_3)]\,|bc|\}. \quad (30)$$

From Eqs. (29) and (30), we obtain that the concurrence $C_r^{P\phi}$ is equal to $C_d^{P\phi}$, which means that the WMR cannot be used to protect entanglement from the correlated PD channel with the initial state $|\phi\rangle$. Moreover, $C_r^{P\phi}$ is independent of the WM strength p and the QMR strength q. Here, the probability of success is calculated to be $P^{P\phi} = \bar{q}\bar{p}$, which is independent of r and η.

3.3. Bit flip channel with memory

The Kraus operators for a single qubit over the BF channel with memory are defined in terms of the Pauli operators $\sigma_0 = I$ and σ_1,

$$A_i = \sqrt{P_i}\sigma_i. \quad (31)$$

Similarly, if we consider two consecutive uses of the BF channel, there are no correlations between the two uses of the channel without memory, and the Kraus operators are given by[40]

$$E_{i,j}^{\mathrm{u}} = \sqrt{P_i P_j}\sigma_i \otimes \sigma_j. \quad (32)$$

In the presence of memory, the two uses of the channel exhibit some correlation, and the Kraus operators of this channel with memory are given by

$$E_{k,k}^{\mathrm{c}} = \sqrt{P_k}\sigma_k \otimes \sigma_k, \quad (33)$$

where $i, j, k = (0, 1)$ and $P_0 = 1 - r$, $P_1 = r$, $r = 1 - \mathrm{e}^{-\gamma t}$.

3.3.1. Entanglement for the initial state $|\psi\rangle$

According to Eq. (2), the concurrence $C_d^{B\psi}$ of the corresponding time-dependent density matrix $\rho^{B\psi}$ is

$$C_d^{B\psi} = \max\left\{0, \Delta_d^{B\psi} = 2(|\rho_{14}^{B\psi}| - \rho_{22}^{B\psi}), 2(|\rho_{23}^{B\psi}| - \sqrt{\rho_{11}^{B\psi}\rho_{44}^{B\psi}})\right\}, \quad (34)$$

which means the concurrence $C_d^{B\psi}$ decreases firstly and then increases as the decoherent strength r increases and increases with memory parameter η increasing. The minimum entanglement occurs at $r = 0.5$, even would experience ESD in Fig. 5(b). This means the BF channel with memory destroys entanglement firstly and then creates entanglement. Besides, the entanglement can be protected effectively when the memory parameter η of BF channel increases. In the case of $\eta = 1$, the concurrence $C_d^{B\psi}$ is unaffected by the BF decoherence.

After the WMR scheme, the concurrence $C_r^{B\psi}$ of the density matrix $\rho_r^{B\psi}$ is

$$C_r^{B\psi} = \max\left\{0, \Delta_r^{B\psi} = \frac{2\bar{q}(|X^{B\psi}| - V^{B\psi})}{U^{B\psi}\bar{q}^2 + 2V^{B\psi}\bar{q} + W^{B\psi}}, \frac{2\bar{q}(|Y^{B\psi}| - \sqrt{U^{B\psi}W^{B\psi}})}{U^{B\psi}\bar{q}^2 + 2V^{B\psi}\bar{q} + W^{B\psi}}\right\}, \quad (35)$$

where

$$X^{B\psi} = \bar{\eta}(P_0^2 ad^* + P_1^2 a^* d) + \eta(P_0 ad^* + P_1 a^* d)\bar{p},$$
$$Y^{B\psi} = \bar{\eta}P_0 P_1 \bar{p}(ad^* + a^* d),$$
$$U^{B\psi} = \bar{\eta}(P_0^2 |a|^2 + P_1^2 \bar{p}^2 |d|^2) + \eta(P_0 |a|^2 + P_1 \bar{p}^2 |d|^2),$$
$$V^{B\psi} = \bar{\eta}P_0 P[|a|^2 + \bar{p}^2 |d|^2],$$
$$W^{B\psi} = \bar{\eta}(P_0^2 \bar{p}^2 |d|^2 + P_1^2 |a|^2) + \eta(P_0 \bar{p}^2 |d|^2 + P_1 |a|^2).$$

The probability of success is $P^{B\psi} = U^{B\psi}\bar{q}^2 + 2V^{B\psi}\bar{q} + W^{B\psi}$.

According to Eq. (17), we can also obtain the maximal concurrence $C_r^{B\psi}$ in Eq. (35). The maximal amount of entanglement of $\rho_r^{B\psi}$ appears at the QMR strength $\bar{q} =$

$\sqrt{W^{B\psi}/U^{B\psi}}$. $\chi^{B\psi} = C_r^{B\psi}/C_d^{B\psi}$, where $\chi^{B\psi} > 1$ in the appropriate case in Fig. 6, means the WMR indeed helps to protect entanglement from correlated BF decoherence. For BF channel, the WMR protection is similar to AD channel. However, with the increase of the coefficient a, the range of WM strength p becomes narrower, even the protective effect weakens with the increase of p when $a > \sqrt{2}/2$.

In Fig. 7(a), we also investigate how the concurrence $C_r^{B\psi}$ changes with the WM strength p. One can obviously see that $C_r^{B\psi}$ increases linearly with increasing p but with a turning point where the concurrence reaches its maximal value. Moreover, it is clear that the stronger η is, the larger the concurrence $C_r^{B\psi}$ is. Therefore, if one wants a larger concurrence, η should be bigger and p cannot approach 1.

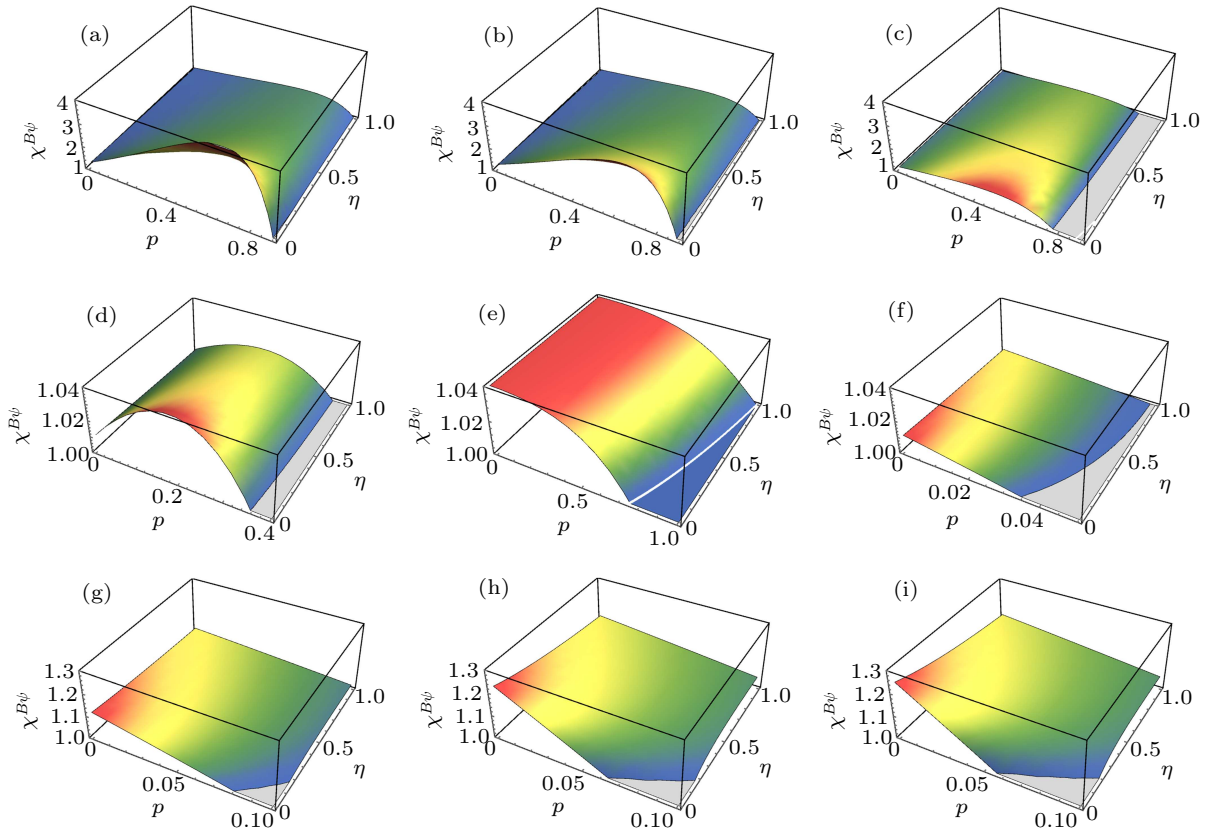


Fig. 6. In the BF channel with memory, $\chi^{B\psi}$ is a function of the weak measurement strength p and the memory parameter η for the initially entangled state $|\psi\rangle = a|00\rangle + d|11\rangle$. We assume $r = 0.2$, (a) $a = \sqrt{0.1}$; (b) $a = 1/3$; (c) $a = \sqrt{0.2}$; (d) $a = \sqrt{0.4}$; (e) $a = \sqrt{2}/2$; (f) $a = \sqrt{0.6}$; (g) $a = \sqrt{0.8}$; (h) $a = \sqrt{8}/3$; (i) $a = \sqrt{0.9}$.

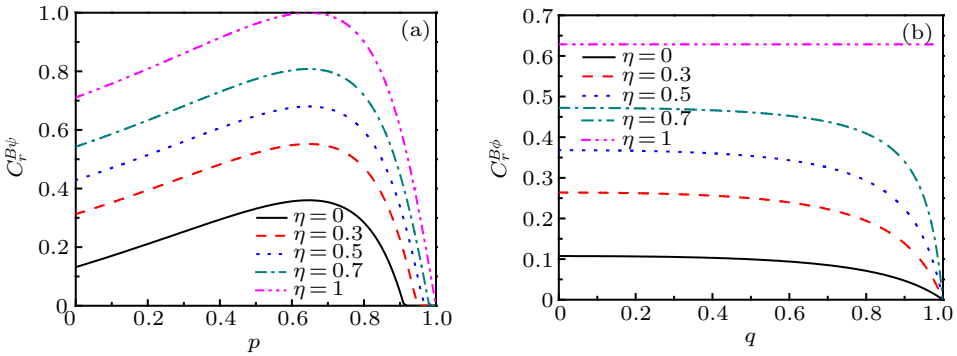


Fig. 7. In the BF channel with memory, the concurrence $C_r^{B\psi(\phi)}$ is a function of weak measurement strength p (q) for two initially entangled states $|\psi\rangle = a|00\rangle + d|11\rangle$ and $|\phi\rangle = b|01\rangle + c|10\rangle$. We assume $r = 0.2$, (a) $|a| = 1/3$, (b) $|b| = 1/3$.

3.3.2. Entanglement for the initial state $|\phi\rangle$

The concurrence $C_d^{B\phi}$ of $\rho^{B\phi}$ is

$$C_d^{B\phi} = \max\left\{0, \Delta_d^{B\phi} = 2(|\rho_{14}^{B\phi}| - \sqrt{\rho_{22}^{B\phi}\rho_{33}^{B\phi}}), 2(|\rho_{23}^{B\phi}| - \rho_{11}^{B\phi})\right\}, \tag{36}$$

which means the evolution of the concurrence $C_d^{B\phi}$ is the same as that of $C_d^{B\psi}$ as shown in Fig. 5(b).

The concurrence of the density matrix $\rho_r^{B\psi}$ with the WMR scheme is

$$C_r^{B\phi} = \max\left\{0, \Delta_r^{B\phi} = \frac{2\bar{q}(|\rho_{14}^{B\phi}| - \sqrt{\rho_{22}^{B\phi}\rho_{33}^{B\phi}})}{\rho_{11}^{B\phi}\bar{q}^2 + (\rho_{22}^{B\phi} + \rho_{33}^{B\phi})\bar{q} + \rho_{44}^{B\phi}}, \frac{2\bar{q}(|\rho_{23}^{B\phi}| - \rho_{11}^{B\phi})}{\rho_{11}^{B\phi}\bar{q}^2 + (\rho_{22}^{B\phi} + \rho_{33}^{B\phi})\bar{q} + \rho_{44}^{B\phi}}\right\}. \quad (37)$$

The probability of success is $P^{B\phi} = \rho_{11}^{B\phi}\bar{q}^2 + (\rho_{22}^{B\phi} + \rho_{33}^{B\phi})\bar{q} + \rho_{44}^{B\phi}$.

We find an interesting result that the $\chi^{B\phi}$ is independent of the WM strength p. Therefore, in Fig. 4(b), we plot that $\chi^{B\phi}$ behaves with η and q. As shown in Fig. 4(b), we can see that $\chi^{B\phi}$ is always smaller than 1. In Fig. 7(b), we show how $C_r^{B\phi}$ behaves with q. It is clear that $C_r^{B\phi}$ decreases with q increasing, implying the WMR indeed cannot protect concurrence against the correlated BF channel with the initial state $|\phi\rangle$. In the case of $\eta = 1$, the concurrence $C_r^{B\phi}$ is unaffected by the BF decoherence.

3.4. Depolarizing channel with memory

Finally, we will study the quantum entanglement of two qubits over the depolarizing channel. The depolarizing channel is a Pauli channel with particularly nice symmetric properties, and it describes the process in which the density matrix is dynamically replaced by the state $I/2$. I denotes the identity matrix of the qubit. The Kraus operators for a single qubit are given by[41]

$$A_i = \sqrt{P_i}\sigma_i, \quad (38)$$

where $P_0 = 1 - r$, $P_1 = P_2 = P_3 = r/3$, $r = 1 - \mathrm{e}^{-\gamma t}$.

By definition, the Kraus operators of the depolarizing channel without memory corresponding to no correlations between the two uses of the channel are described as

$$E_{i,j}^{\mathrm{u}} = \sqrt{P_i P_j}\sigma_i \otimes \sigma_j. \quad (39)$$

The case of two consecutive uses of a channel with partial memory exhibits some correlation, the task of constructing Kraus operators $E_{k,k}^{\mathrm{c}}$ is given as

$$E_{k,k}^{\mathrm{c}} = \sqrt{P_k}\sigma_k \otimes \sigma_k, \quad (40)$$

where $i, j, k = (0, 1, 2, 3)$.

3.4.1. Entanglement for the initial state $|\psi\rangle$

The concurrence $C_d^{D\psi}$ of $|\psi\rangle$ is calculated to be

$$C_d^{D\psi} = \max\{0, \Delta_d^{D\psi} = 2(|\rho_{14}^{D\psi}| - \rho_{22}^{D\psi})\}, \quad (41)$$

which means the concurrence $C_d^{D\psi}$ decreases with the increasing of decoherent strength r when memory parameter η is smaller, even would experience ESD; for bigger η, $C_d^{D\psi}$ decreases firstly and then increases with the increasing of decoherent strength r, and $C_d^{D\psi}$ increases with memory parameter η increasing in Fig. 5(c). This also means the depolarizing channel with memory destroys entanglement firstly and then creates entanglement. In the case of $\eta = 1$, the concurrence $C_d^{D\psi}$ is unaffected by the depolarizing decoherence.

The concurrence $C_r^{D\psi}$ of the density matrix $\rho_r^{D\psi}$ with the WMR scheme is

$$C_r^{D\psi} = \max\left\{0, \Delta_r^{D\psi} = \frac{2\bar{q}(|X^{D\psi}| - V^{D\psi})}{U^{D\psi}\bar{q}^2 + 2V^{D\psi}\bar{q} + W^{D\psi}}\right\}, \quad (42)$$

where

$$\begin{aligned}
X^{D\psi} &= \{\bar{\eta}[(P_0 - P_3)^2 ad^* + (P_1 - P_2)^2 a^* d] \\
&\quad + \eta[(P_0 + P_3)ad^* + (P_1 + P_2)a^* d]\}\bar{p}, \\
U^{D\psi} &= \bar{\eta}[(P_0 + P_3)^2 |a|^2 + (P_1 + P_2)^2 \bar{p}^2 |d|^2] \\
&\quad + \eta[(P_0 + P_3)|a|^2 + (P_1 + P_2)\bar{p}^2 |d|^2], \\
V^{D\psi} &= \bar{\eta}(P_0P_1 + P_0P_2 + P_1P_3 + P_2P_3)[|a|^2 + \bar{p}^2 |d|^2], \\
W^{D\psi} &= \bar{\eta}[(P_0 + P_3)^2 \bar{p}^2 |d|^2 + (P_1 + P_2)^2 |a|^2] \\
&\quad + \eta[(P_0 + P_3)\bar{p}^2 |d|^2 + (P_1 + P_2)|a|^2].
\end{aligned}$$

The probability of success is $P^{D\psi} = U^{D\psi}\bar{q}^2 + 2V^{D\psi}\bar{q} + W^{D\psi}$. According to Eq. (17), we also can obtain the maximal concurrence $C_r^{D\psi}$ in Eq. (42). The maximal amount of entanglement of the state $\rho_r^{D\psi}$ appears at the QMR strength $\bar{q} = \sqrt{W^{D\psi}/U^{D\psi}}$. $\chi^{D\psi}$ and $C_r^{D\psi}$ have same trend as $\chi^{B\psi}$ and $C_r^{B\psi}$, but there is a numercial difference in Fig. 6.

3.4.2. Entanglement for the initial state $|\phi\rangle$

The concurrence $C_d^{D\phi}$ of the density matrix $\rho^{D\phi}$ is calculated to be

$$C_d^{D\phi} = \max\{0, \Delta_d^{D\phi} = 2(\left|\rho_{23}^{D\phi}\right| - \rho_{11}^{D\phi})\}. \quad (43)$$

which means the evolution of the concurrence $C_d^{D\phi}$ is the same as that of $C_d^{D\psi}$ as shown in Fig. 5(c).

The concurrence of the density matrix $\rho_r^{D\phi}$ is

$$C_r^{D\phi} = \max\left\{0, \Delta_r^{D\phi} = \frac{2\bar{q}(\left|\rho_{23}^{D\phi}\right| - \rho_{11}^{D\phi})}{\rho_{11}^{D\phi}\bar{q}^2 + (\rho_{22}^{D\phi} + \rho_{33}^{D\phi})\bar{q} + \rho_{44}^{D\phi}}\right\}. \quad (44)$$

The probability of success is $P^{D\phi} = \rho_{11}^{D\phi}\bar{q}^2 + (\rho_{22}^{D\phi} + \rho_{33}^{D\phi})\bar{q} + \rho_{44}^{D\phi}$. The evolution of the concurrence $C_r^{D\phi}$ has the same trend as $C_r^{B\phi}$, but there is a numercial difference in Fig. 4(c).

4. Conclusion

In this paper, we have investigated the practical application of the WMR in protecting entanglement when two qubits go through four typical kinds of quantum noise channels with memory, *i.e.*, the amplitude damping channel, the phase damping channel, the bit flip channel, and the depolarizing channel.

Some analytical or numerical results are presented. For the initial state $|\psi\rangle$, one of the results shows that the protection effect of the WMR scheme decreases with the increase of coefficient a until the coefficient is $a = \sqrt{2}/2$, and then the protection effect increases with the increase of a, but the effect is still not as good as that of a with smaller value. Another result shows that our WMR scheme can indeed be useful for battling against decoherence under all these quantum channels with memory. For the other initial state $|\varphi\rangle$, the effect of the protection scheme is the same regardless of the coefficient b, and our WMR scheme can protect the entanglement for the amplitude damping channel with memory. We have also studied the relationship between the concurrence, the memory parameter, the weak measurement strength and the weak measurement reversal strength. We found that the WMR and the memory effects can effectively reduce the ESD region against quantum channels and even amplify the entanglement. This WMR protection scheme is conducive to the system which maintains maximum entanglement in the nosie channel. Although the method is risky (*i.e.*, a stronger procedure is required for longer preservation, which decreases the probability of success), this procedure for entanglement preservation is useful in entanglement distillation protocols and some quantum communication tasks.

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