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Effect of weak measurement on quantum correlations

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We investigate the local quantum uncertainty (LQU) in weak measurement. An expression of weak LQU is explicitly determined. Also, we consider some cases of three special X states, Werner state, circulant two-qubit states, and Heisenberg model via LQU in normal and weak measurements. We find that the LQU in weak measurement is weaker than the case of strong measurement.

Keywords: local quantum uncertainty, weak measurement, weak local quantum uncertainty

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1. Introduction

Since its inception, quantum mechanics (QM) shows several counterintuitive concepts and so has been purported not to be a complete theory.[1–3] Among these quantum concepts, that of measurement plays a central role in quantum theory.[4,5] Indeed, the way in which quantum measurement acts on quantum systems is fundamentally and radically different from that of classical measurement.

The weak measurements,[6,7] for which the coupling of the system and the measuring device is weak, are proved to be soft measurements that do not strongly influence the state of the quantum system. In fact, even if an arbitrary quantum state loses its decoherence, when a projective measurement is made. It will be weakly disturbed and will not lose its coherence completely if a weak measurement is made. In particular, weak measurements have been shown to be universal in the sense that any generalized measurement can be obtained in the form of a series of weak measurements resulting in minor changes in the quantum state for all the results.[7] The low value attributed to an observable can take values outside the spectrum of the observable. Recently, it has been shown that weak measurements have been used to understand quantum systems in a coherent way, such as the uncertainty principle, realism, and non-locality.[8–12] They are therefore important for exploring fundamental physics issues and have technological implications.[13–17]

Another particular result is the type of correlations existing between different parts of a composite system, commonly called non-classical correlations.[1–3] One of the most powerful non-classical correlations is the entanglement, which is clear from its importance in different information processing tasks.[2] To fully understand the behavior of composite quantum systems, several quantum correlation measurements have been invented recently and have attracted much interest.[18] In particular, a new quantifier of quantum correlations called local quantum uncertainty (LQU) has recently been introduced in the literature.[19] Due to the measurement of a local observable, this quantum measurement quantifies uncertainty in a quantum state. Although, the LQU form for higher-dimensional systems is not available, such a quantifier is relevant in quantum metrology, particularly in quantum information measurement. Thus, LQU has good reason to be considered as a faithful measure of quantumness of quantum states.

In this paper, after reviewing briefly the skew information and LQU, we introduce, in a simple way, the formalism of the weak measurements, and then we determine weak local quantum uncertainty (WLQU) of two-qubit X states in comparison with strong LQU. As an application, we consider some cases of special X states: (i) z-Werner state, (ii) (ϵ, g)-circulant two qubit states, and Heisenberg XXX model. A conclusion closes the paper.

2. Skew information and LQU

2.1. Skew information

One of the intrinsic features of quantum mechanics is the quantified non-classical correlations. Skew information is an uncertainty relation which provides the statistical nature of measurement errors. The measurement of a single observable can also help to detect the uncertainty of a quantum observable. For a bipartite system AB with the quantum state ρ, an observable is said quantum certain if the measurement error of the observable is only due to the ignorance of the classical mixture ρ.[21] The skew information is a good quantifier to the

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uncertainty of an observable, and is given by[20]

$$\mathcal{I}(\rho_{AB},K)=-\frac{1}{2}\mathrm{Tr}\left([\sqrt{\rho_{AB}},K]^2\right), \tag{1}$$

where Tr denotes the trace and the symbol $[.,.]$ denotes the commutator. Wigner and Yanase introduced this quantity to express the amount of information stored in an inaccessible state by measuring the observable K, because of the non-commutativity between the state ρ and the observable K. Therefore, it serves as a measure of uncertainty of the observable K in the state ρ. The measurement of skew information has interesting properties for pure states ($\rho^2=\rho$), the variance $\mathcal{V}(\rho,K)=\mathrm{Tr}\rho K^2-(\mathrm{Tr}\rho K)^2$ is purely quantum in nature. Nevertheless, for a mixed state ($\mathrm{Tr}\rho^2<1$) it includes classical and quantum contributions. The Wigner–Yanase skew Information can be rewritten as

$$\mathcal{I}(\rho_{AB},K)=\mathrm{Tr}(\rho_{AB}(K)^2)-\mathrm{Tr}(\sqrt{\rho_{AB}}K\sqrt{\rho_{AB}}K)^2. \tag{2}$$

It has many good properties,[20,22] as (i) it is non-negative, (ii) reduces to the variance of the pure state ($\rho^2=\rho$), and (iii) is convex in ρ in the meaning where $\mathcal{I}(\sum_j\lambda_j\rho_j,K)<\sum_j\lambda_j\mathcal{I}(\rho_j,K)$, $\lambda_j\in\mathbb{R}$, which means that skew information decreases when the multiple states are mixed. For example, it disappears if and only if ρ and K commute and it is positive in the other cases; it does not increase under classical mixing. In particular, if we select an observable $K=K_A\otimes I$ with K_A that can be observed on subsystem A, $\mathcal{I}(\rho_{AB},K)$ is contractive under completely positive and trace-preserving maps Φ on B, that is, $\mathcal{I}((I_A\otimes\Phi)\rho_{AB},K_A\otimes I)\leq\mathcal{I}(\rho_{AB},K_A\otimes I)$.

The skew information is delimited from below by the amount of discord-type correlations in the state and its minimum value can actually be used as a measure of discord-type correlations, defined in Ref. [19] as the LQU. The authors have chosen the local observable $K_A=K_A^\Lambda\otimes I_B$ such that K_A^Λ is a Hermitian operator on subsystem A with non degenerate spectrum $K_A^\Lambda=\sum_i\alpha_i|i\rangle_A\langle i|$. The LQU based on the skew information is given by

$$\begin{aligned}\mathcal{U}(\rho_{AB}) &\equiv \min_{K_\Lambda^A}\mathcal{I}(\rho_{AB},K_A^\Lambda\otimes I_B)\\ &= -\frac{1}{2}\min_{K_\Lambda^A}\mathrm{Tr}\left([\sqrt{\rho_{AB}},K_A^\Lambda\otimes I_B]^2\right).\end{aligned} \tag{3}$$

In Eq. (3) the minimization is performed over all local maximally informative observables $K^\Lambda=K_A^\Lambda\otimes I_B$ on non-degenerate spectrum $K_A^\Lambda=\sum_i\alpha_i|i\rangle_A\langle i|$, where $\{|i\rangle_A\}$ is an orthonormal basis of A and the α_i are all different. That is, only observables of the form $K_A^\Lambda=U_A\Lambda U_A^\dagger$ are taken into account, where U_A is any local unitary transformation on subsystem A.[19] This quantity is not only a quantzation of the minimum amount of uncertainty in quantum state, but also a well-defined quantum correlations measurement. In particular, for $2\otimes d$ dimensional state ρ, by dropping the superscript Λ for brevity and choosing the non-degeneate observales as $K^A=\boldsymbol{n}\cdot\boldsymbol{\sigma}^A$ with $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ being the vector of Pauli operators, the LQU can be derived as[19]

$$\mathcal{U}(\rho)=1-\lambda_{\max}\{W\}, \tag{4}$$

where $\lambda_{\max}$ denotes the maximal eigenvalue of the matrix $W=(\omega_{ij})_{3\times3}$ whose matrix elements are given by

$$\omega_{ij}\equiv\mathrm{Tr}\left\{\sqrt{\rho_{AB}}(\sigma_i\otimes\boldsymbol{I})\sqrt{\rho_{AB}}(\sigma_j\otimes\boldsymbol{I})\right\}. \tag{5}$$

Here σ_i's ($i=1,2,3$) are the standard Pauli matrices in this case (for $d=2$) of subsystem A. It has been demonstrated that the LQU satisfies the full physical requirements of a measure of quantum correlations.[19]

3. Weak mesurement

Although the information obtained was lower than the evaluation of the von Neumann projection, the disturbance was much smaller. More significantly, the quantum states were recoverable after a weak measurement.[6] Then, a test is implemented to minimize the impact on the initial quantum state which induces a relatively small modification of the quantum states. This is called the weak measurement. It is a way of manipulating and controlling the quantum state. The weak operators of the measurement are provided by[7]

$$\begin{aligned}P(x) &= \sqrt{\frac{1-\tanh x}{2}}\Pi_0+\sqrt{\frac{1+\tanh x}{2}}\Pi_1,\\ P(-x) &= \sqrt{\frac{1+x}{2}}\Pi_0+\sqrt{\frac{1-\tanh x}{2}}\Pi_1,\end{aligned} \tag{6}$$

where x is a parameter indicating the strength of the measurement process, Π_0 and Π_1 are the two orthogonal rank-1 projectors that satisfy $\Pi_0+\Pi_1=1$. The weak measurement operators satisfy $P^\dagger(x)P(x)+P(-x)P^\dagger(-x)=1$. The larger the x value, the stronger the weak measurement. One can easily find that if and only if $x\to\infty$, the weak measure is close to the projective measurement. These operators have the following properties:

i) $P(0)=\frac{1}{\sqrt{2}}$ resulting in no state change;

ii) in the strong measurement limit we have the projective measurement operators, i.e., $\lim\limits_{x\to\infty}P(-x)=\Pi_0$ and $\lim\limits_{x\to\infty}P(x)=\Pi_1$ (for $\lim\limits_{x\to\infty}\tanh x=1$);

iii) $[P(x),P(-x)]=0$.

To see that these operators indeed correspond to weak measurement, consider the action of the weak operators given by $\lim\limits_{x\to\infty}\tanh x=1$,

$$P(\pm x)=\frac{1}{2}(\zeta_1+\zeta_2)I\pm\frac{1}{2}(\zeta_1-\zeta_2)\sigma_z, \tag{7}$$

where $\zeta_1=\sqrt{\frac{1-\tanh x}{2}}$ and $\zeta_2=\sqrt{\frac{1+\tanh x}{2}}$. Indeed, it is possible to parameterize a particular local observable $P(\pm x)$ with non-degenerate spectrum $\{\zeta_1,\zeta_2\}$[23]

$$\begin{aligned}P(\pm x) &= U_A\left(\frac{1}{2}(\zeta_1+\zeta_2)I^A\pm\frac{1}{2}(\zeta_1-\zeta_2)\sigma_z^A\right)U_A^\dagger\\ &= \frac{1}{2}(\zeta_1+\zeta_2)I^A\pm\frac{1}{2}(\zeta_1-\zeta_2)\boldsymbol{n}\cdot\boldsymbol{\sigma}^A,\end{aligned} \tag{8}$$

where $\boldsymbol{n}$ is a unit vector, as it follows from the concept of skew information.

4. LQU in weak measurement

In this section, we can give a rigorous definition of the skew information and LQU based on a weak measurement of quantum correlations. For the case of bipartite states,[23] defined on a Hilbert space $2\otimes d$ (qubit–qudit states), the local quantum based on skew information has the closed form.[24] In order to find a closed form for the weak local quantum (WLQ), we would like to give the connection between the LQU and the WLQU.

Theorem 1 Any 2-dimensional observable (8) can be decomposed as $P(\pm x)=\alpha I_A \pm \beta \boldsymbol{n}\cdot\boldsymbol{\sigma}_A$ [19] with the parameters α and β being respectively determined by the strength of the weak measurement, while $\boldsymbol{n}$ varies on the unit sphere. Therefore, equation (1) can also be rewritten as

$$\mathcal{WI}(\rho,P(\pm x))=\beta^2\mathcal{I}(\rho,\boldsymbol{n}\cdot\boldsymbol{\sigma}_A).$$

Proof After a simple calculation, one can see that the LQ of skew information defined in Ref. [19] can be rewritten as

$$\begin{aligned}\mathcal{WI}(\rho,P(\pm x)) &= -\frac{1}{2}\mathrm{Tr}\{\alpha[\sqrt{\rho},I_A]+\beta[\sqrt{\rho},\boldsymbol{n}\cdot\boldsymbol{\sigma}_A]\}^2\\ &= -\frac{\beta^2}{2}\mathrm{Tr}\{[\sqrt{\rho},\boldsymbol{n}\cdot\boldsymbol{\sigma}_A]\}^2\\ &= \beta^2\mathcal{I}(\rho,\boldsymbol{n}\cdot\boldsymbol{\sigma}_A).\end{aligned}$$

Therefore, for qubit–qudit systems, the choice of the spectrum Λ does not affect the quantification of non-classical correlations, and without loss of generality, we suppose that the local observables are of the form $K_A=\boldsymbol{n}\cdot\boldsymbol{\sigma}_A$. After having simplified the form of the observables over which we need to optimise (the minimisation now works on $\boldsymbol{n}$), we can write the LQU in the following fashion:

$$\begin{aligned}\mathcal{WU}_A^{\Lambda}(\rho) &= \min_{K^{\Lambda}}\mathcal{WI}\left(\rho,K_{\pm}^{\Lambda}\right)=\beta^2\min_{n\cdot\sigma}\mathcal{I}(\rho,\boldsymbol{n}\cdot\boldsymbol{\sigma}_A),\\ \mathcal{WU}_A^{\Lambda}(\rho) &= \beta^2\min_{K^{\Lambda}}\left\{1-\mathrm{Tr}\left(\sqrt{\rho}\,(n_i.\boldsymbol{\sigma}_i\otimes I)\sqrt{\rho}\,(n_j\cdot\boldsymbol{\sigma}_j\otimes I)\right)\right\}\\ &= \beta^2(1-\lambda_{\max}),\end{aligned}$$

where $\lambda_{\max}$ is the maximum eigenvalue of the matrix W whose matrix elements are $(W)_{ij}=\mathrm{Tr}\left(\sqrt{\rho}\,(\boldsymbol{\sigma}_i\otimes I)\sqrt{\rho}\,(\boldsymbol{\sigma}_j\otimes I)\right)$. Theorem 1 provides a succinct and significant relation between the WLQU and LQU. We are more interested here in skew information and LQU. From this theorem, we can find that the WLQU has also a closed form like the LQU and that the WLQU vanishes if and only if the state is a classical quantum state and for $\beta=0$. Since $0<\beta<1$, Theorem 1 implies that the WLQU in the weak measurement process captures only the partial quantumness of operator density ρ.

Now, under normal and weak measurements, we have obtained the analytic formulas of LQU. It should be noted that in the case of a weak measurement or a projective measure, the analytic statements of LQU are very independent of the dimension d. The relationship between quantum entanglement and local quantum uncertainty, as well as their development in noisy channels, will be discussed below.

5. WLQU for two-qubit X states

Two-qubit states belonging to the Hilbert space $\mathcal{H}^2\otimes\mathcal{H}^2$, constitute the main constitutive elements for encoding correlations in quantum information theory. The basis of calculation of $\mathcal{H}^2\otimes\mathcal{H}^2$ is $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$. We consider the general state X with two-qubits written in matrix form with respect to the usual basis

$$\rho_{AB}=\begin{pmatrix}\rho_{11}&0&0&\rho_{14}\\0&\rho_{22}&\rho_{23}&0\\0&\rho_{32}&\rho_{33}&0\\\rho_{41}&0&0&\rho_{44}\end{pmatrix}. \tag{9}$$

As a density matrix, the coefficients ρ_{ij} are complex numbers and satisfy the following conditions: $\sum_{i=1}^{4}\rho_{ii}=1$, $\rho_{22}\rho_{33}\geq|\rho_{23}|^2$, $\rho_{11}\rho_{44}\geq|\rho_{14}|^2$, $\rho_{ii}\in\mathbb{R}$, $\rho_{23}=\rho_{32}^*$, $\rho_{14}=\rho_{41}^*$. Then the Fano–Bloch decomposition form of ρ_{AB} is

$$\rho_{AB}=\frac{1}{4}\sum_{ij}R_{ij}\sigma_i\otimes\sigma_j, \tag{10}$$

where the correlation matrix $R_{ij}=\mathrm{Tr}(\rho_{AB}\sigma_i\otimes\sigma_j)$ with σ_i $(i,j=0,1,2,3)$ being the Pauli matrices. It is easy to verify that the square root of ρ_{AB} [25] also has the X-structure expressed by

$$\sqrt{\rho_{AB}}=\frac{1}{4}\sum_{ij}\mathcal{R}_{ij}\sigma_i\otimes\sigma_j \tag{11}$$

with $\mathcal{R}_{ij}=\mathrm{Tr}\left(\sqrt{\rho_{AB}}\sigma_i\otimes\sigma_j\right)$. The non-vanishing matrix correlation elements $\mathcal{R}_{ij}$ are explicitly given by

$$\sqrt{\rho_{AB}}=\frac{1}{4}\sum_{ij}\mathcal{R}_{ij}\sigma_i\otimes\sigma_j \tag{12}$$

with $\mathcal{R}_{ij}=\mathrm{Tr}\left(\sqrt{\rho_{AB}}\sigma_i\otimes\sigma_j\right)$. The non-vanishing matrix correlation elements $\mathcal{R}_{ij}$ are explicitly given by

$$\begin{aligned}\mathcal{R}_{00} &= \left(\sqrt{t_1+2\sqrt{d_1}}+\sqrt{t_2+2\sqrt{d_2}}\right),\\ \mathcal{R}_{03} &= \frac{1}{2}\frac{R_{30}+R_{03}}{\sqrt{t_1+2\sqrt{d_1}}}-\frac{1}{2}\frac{R_{30}-R_{03}}{\sqrt{t_2+2\sqrt{d_2}}},\\ \mathcal{R}_{30} &= \frac{1}{2}\frac{R_{30}+R_{03}}{\sqrt{t_1+2\sqrt{d_1}}}+\frac{1}{2}\frac{R_{30}-R_{03}}{\sqrt{t_2+2\sqrt{d_2}}},\\ \mathcal{R}_{11} &= \frac{1}{2}\frac{R_{11}+R_{22}}{\sqrt{t_2+2\sqrt{d_2}}}+\frac{1}{2}\frac{R_{11}-R_{22}}{\sqrt{t_1+2\sqrt{d_1}}},\\ \mathcal{R}_{22} &= \frac{1}{2}\frac{R_{11}+R_{22}}{\sqrt{t_2+2\sqrt{d_2}}}-\frac{1}{2}\frac{R_{11}-R_{22}}{\sqrt{t_1+2\sqrt{d_1}}},\\ \mathcal{R}_{33} &= \left(\sqrt{t_1+2\sqrt{d_1}}-\sqrt{t_2+2\sqrt{d_2}}\right),\end{aligned} \tag{13}$$

with $t_1=\rho_{11}+\rho_{44}$, $t_2=\rho_{22}+\rho_{33}$, $d_1=\rho_{11}\rho_{44}-\rho_{14}\rho_{41}$, and $d_2=\rho_{22}\rho_{33}-\rho_{23}\rho_{32}$. At this stage, we have the tools to evaluate the matrix elements defined by[25]

$$(W)_{ij}=\mathrm{Tr}\left(\sqrt{\rho_{AB}}\,(\sigma_i\otimes I)\sqrt{\rho_{AB}}\,(\sigma_j\otimes I)\right). \tag{14}$$

The eigenvalues of matrix $(W)_{ij}$ are explicitly given by[25]

$$\omega_{11} = \frac{1}{4}\left[4\left(\sqrt{\lambda_1}+\sqrt{\lambda_4}\right)\left(\sqrt{\lambda_2}+\sqrt{\lambda_3}\right)+\frac{\left(R_{11}^2-R_{22}^2\right)+\left(R_{03}^2-R_{30}^2\right)}{\left(\sqrt{\lambda_1}+\sqrt{\lambda_4}\right)\left(\sqrt{\lambda_2}+\sqrt{\lambda_3}\right)}\right],$$
$$\omega_{22} = \frac{1}{4}\left[4\left(\sqrt{\lambda_1}+\sqrt{\lambda_4}\right)\left(\sqrt{\lambda_2}+\sqrt{\lambda_3}\right)+\frac{\left(R_{22}^2-R_{11}^2\right)+\left(R_{03}^2-R_{30}^2\right)}{\left(\sqrt{\lambda_1}+\sqrt{\lambda_4}\right)\left(\sqrt{\lambda_2}+\sqrt{\lambda_3}\right)}\right],$$
$$\omega_{33} = \frac{1}{2}\left[\left(\sqrt{\lambda_1}+\sqrt{\lambda_4}\right)^2+\left(\sqrt{\lambda_2}+\sqrt{\lambda_3}\right)^2+\frac{1}{4}\frac{\left(R_{03}+R_{30}\right)^2-\left(R_{11}-R_{22}\right)^2}{\left(\sqrt{\lambda_1}+\sqrt{\lambda_4}\right)^2}+\frac{1}{4}\frac{\left(R_{03}-R_{30}\right)^2-\left(R_{11}+R_{22}\right)^2}{\left(\sqrt{\lambda_2}+\sqrt{\lambda_3}\right)^2}\right],$$
$$\omega_{12} = \omega_{21} = \frac{1}{2}\frac{R_{11}R_{21}+R_{22}R_{12}}{\left(\sqrt{\lambda_1}+\sqrt{\lambda_4}\right)\left(\sqrt{\lambda_2}+\sqrt{\lambda_3}\right)}, \quad \omega_{13}=\omega_{31}=\omega_{23}=\omega_{32}=0, \tag{15}$$

where λ_i $(i=1,2,3,4)$ denote the eigenvalues of the density matrix ρ_{AB}, which are given by

$$\lambda_1 = \frac{1}{2}\left(t_1+\sqrt{t_1^2-4d_1}\right), \quad \lambda_3 = \frac{1}{2}\left(t_1-\sqrt{t_1^2-4d_1}\right),$$
$$\lambda_2 = \frac{1}{2}\left(t_2+\sqrt{t_2^2-4d_2}\right), \quad \lambda_4 = \frac{1}{2}\left(t_2-\sqrt{t_2^2-4d_2}\right), \tag{16}$$

in terms of the Fano–Bloch components R_{ij}. For the case of bipartite $2\otimes d$ systems and from Eq. (4) the LQU can be evaluated as

$$\mathcal{U}(\rho_{AB}) = 1-\max\left(\lambda_{\max}^{\omega}\right). \tag{17}$$

The last equation shows that it is difficult to determine the great value among the eigenvalues of matrix W. Then, in order to study the LQU in our system, we will examine the situation numerically. The next section will be focused on the application of all such background on some cases of two-qubit states, for example, z-Werner states, cicrulant two-qubit state, and Heisenberg model.

6. Application

To illustrate our method of computation of the WLQU, let us now consider the case of some special two-qubit X-states for which quantum correlations have recently been evaluated in the literature using this quantifier. We will focus on three special X states: (i) z-Werner state, (ii) cicrulant two-qubit state, and (iii) Heisenberg model.

6.1. The z-Werner state

The two-qubit of one-parameter states given by a $d\times d$ dimensional Werner state is a mixture of symmetric and antisymmetric projection operators represented by the density matrix

$$\rho_z = \frac{2z}{d^2+d}P_{\text{sys}}+\frac{2(1-z)}{d^2-d}P_{\text{ass}}, \tag{18}$$

where $0\le z\le 1$, $P_{\text{sys}}=\frac{1}{2}(1+\mathcal{F})$, and $P_{\text{ass}}=\frac{1}{2}(1-\mathcal{F})$ are the projectors as well as the flip operator defined by $\mathcal{F}=\sum_{ij}|i\rangle\langle j|\otimes|j\rangle\langle i|$. When $d=2$, the density matrix representing a Werner state is given by

$$\rho_z = \begin{pmatrix} \frac{z}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2}-\frac{z}{3} & \frac{2z}{3}-\frac{1}{2} & 0 \\ 0 & \frac{2z}{3}-\frac{1}{2} & \frac{1}{2}-\frac{z}{3} & 0 \\ 0 & 0 & 0 & \frac{z}{3} \end{pmatrix}. \tag{19}$$

In the Fano–Bloch representation, the z-Werner state is

$$\rho_z = \frac{1}{4}\left(\sigma_0\otimes\sigma_0+\sum_{i=1}^{3}R_{ii}\sigma_i\otimes\sigma_i\right),$$

where the non-vanishing correlation matrix elements are given by

$$R_{11}=R_{22}=R_{33}=\left(\frac{4z}{3}-1\right).$$

The square root of the state ρ_z can be easily calculated from Ref. [25], it is straight forward to check that the matrix W (14) is diagonal. The diagonal elements are given by

$$\omega_{11} = \omega_{22} = \frac{2\sqrt{z}}{3}\left(\sqrt{\left(\sqrt{3(1-z)}+z\right)^2-z(z+1)}\right),$$
$$\omega_{33} = \frac{2}{3}\left(z+\sqrt{3(1-z)z}\right).$$

Having the analytical expressions of ω_{11} and ω_{33}, the LQU and WLQU measurements of quantum correlations in the Werner states ρ_z are then given by

$$\mathcal{U}(\rho_z) = 1-\max(\omega_{11},\omega_{33}),$$
$$\mathcal{WU}(\rho_z) = \beta^2\left(1-\max(\omega_{11},\omega_{33})\right), \tag{20}$$

where $\beta=\dfrac{\zeta_1-\zeta_2}{2}$.

The highest value of LQU reaches $\mathcal{U}(\rho_z)=1$ at $z=0$ and also LQU reaches $\mathcal{U}(\rho_z)\simeq 0.35$ at $z=1$. The behavior of LQU as a function of parameter z is depicted in Fig. 1(a) and WLQU versus z for different values of x is reported in Fig. 1(b). From Fig. 1(a), we observe two different regimes of variation of LQU. At small values of parameter z, the LQU

decreases to zero when z takes the value $(z = 0.75)$. On other term, a sudden change is observed along to $(z = 0.75)$ in which the $\mathcal{U}(\rho_z)$ starts to grow up to attend its maximal value $(\mathcal{U}(\rho_z) \simeq 0.35)$. Similarly, the same behavior is obtained for WLQU, when we change the values of x. Then, we denote that the weak LQU takes an important value for the higher values of parameter x (Fig. 1(b)). A comparison between $\mathcal{U}(\rho_z)$ and WLQU $(\mathcal{WU}(\rho_z))$ for initial qubit–qudit systems shows that WLQU $(\mathcal{WU}(\rho_z))$ is smaller than $\mathcal{U}(\rho_z)$ for smaller values of x and they approach each other for larger values of x. Quantum correlations can be extracted by optimally projective measurement through continuously infinitesimal weak measurements. However, the total amount extracted by using a series of weak measurements is less than the quantity extracted from the projective measurements. This explains the fact that the continuous infinitesimal weak measurements cannot realize the projective measurement. In other words, weak measurements destroy (extract) a very weak part of the quantum correlation compared with the projection measurement which destroys the quantum correlation of a quantum system and makes it classical. Mathematically, using Fig. 2, we note that the quantity based on LQU extracted by the weak measurements is much lower than that extracted by the projection measurements. Moreover, we conclude that the quantum correlation extracted by the weak measurement is less than that of projective measurement. This means that the LQU by weak measurement destroys much less quantum correlation.

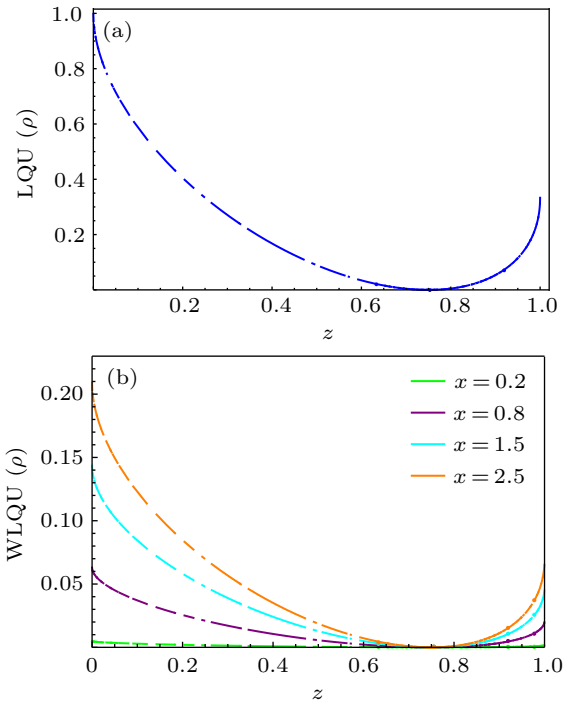


Fig. 1. Effect of measurement strength on LQU and WLQU for various z with $x = 0.2$, $x = 0.8$, $x = 1.5$.

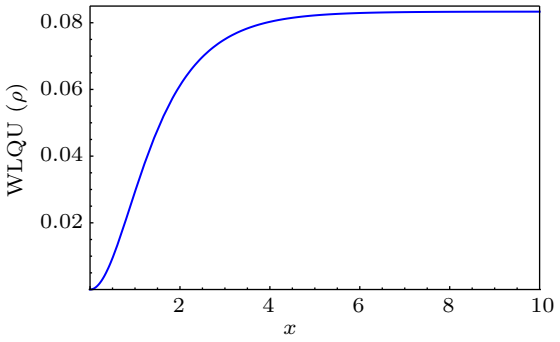


Fig. 2. Weak LQU for different values of x with $z = 1$.

6.2. Circulant two-qubit states

Now, let us consider the following family of circulant two-qubit states:

$$\rho_{\epsilon,g} = \frac{1}{2+\epsilon+\epsilon^{-1}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \epsilon & g & 0 \\ 0 & g & \epsilon^{-1} & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

This family does not belong to the Bell diagonal states considered in Ref. [26]. Based on the method of the evaluation of LQU and quantum discord X states (see Ref. [27]), we can rewrite the circulant two-qubit states $\rho_{\epsilon,g}$ on the form of the Fano–Bloch representation

$$\rho_{\epsilon,g} = \frac{1}{4}\left(\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} R_{ii}\sigma_i \otimes \sigma_i\right),$$

where the non vanishing correlation matrix elements are given by $R_{11} = 1$; $R_{11} = \frac{2\epsilon(1+g)}{(1+\epsilon)^2}$; $R_{22} = \frac{2\epsilon(g-1)}{(1+\epsilon)^2}$; $R_{03} = \frac{1-\epsilon}{1+\epsilon}$; $R_{30} = \frac{\epsilon-1}{1+\epsilon}$; $R_{33} = -\frac{(\epsilon-1)^2}{(1+\epsilon)^2}$. The square root of the state $\rho_{\epsilon,g}$ can be easily calculated from Ref. [25]. It is straighforward to check that the matrix W (14) is diagonal and the diagonal elements are given by

$$\omega_{11} = 2\sqrt{\frac{\epsilon}{1+\epsilon}}\sqrt{\left(\frac{1+\epsilon^2}{2}+\epsilon\sqrt{1-g^2}\right)} + \frac{2\epsilon g\sqrt{\frac{\epsilon}{1+\epsilon}}}{\sqrt{\left(\frac{1+\epsilon^2}{2}+\epsilon\sqrt{1-g^2}\right)}},$$

$$\omega_{22} = 2\sqrt{\frac{\epsilon}{1+\epsilon}}\sqrt{\left(\frac{1+\epsilon^2}{2}+\epsilon\sqrt{1-g^2}\right)} - \frac{2\epsilon g\sqrt{\frac{\epsilon}{1+\epsilon}}}{\sqrt{\left(\frac{1+\epsilon^2}{2}+\epsilon\sqrt{1-g^2}\right)}},$$

$$\omega_{33} = \frac{\left(1+\epsilon^2\right)\left(1-\epsilon^2\right)^2 + 8g^2\epsilon^3\sqrt{1-g^2}}{(1+\epsilon)^2\left(\left(1-\epsilon^2\right)^2+4g^2\epsilon^2\right)}.$$

The off diagonal matrix elements ω_{ij} are zero. It is simply verified that the states have $\omega_{11} \geq \omega_{33}$ or $\omega_{11} \leq \omega_{33}$ for any

value of ϵ and g.[27] Therefore, the LQU and WLQU have respectively the following forms:

$$\mathcal{U}(\rho_{\epsilon,g}) = 1-\omega_{\max},$$
$$\mathcal{WU}(\rho_{\epsilon,g}) = \left(\frac{\zeta_1-\zeta_2}{2}\right)^2(1-\omega_{\max}).$$

In Fig. 3, we report the behavior of the LQU and WLQU versus the parameter ϵ for different values of g with $x=1.2$.

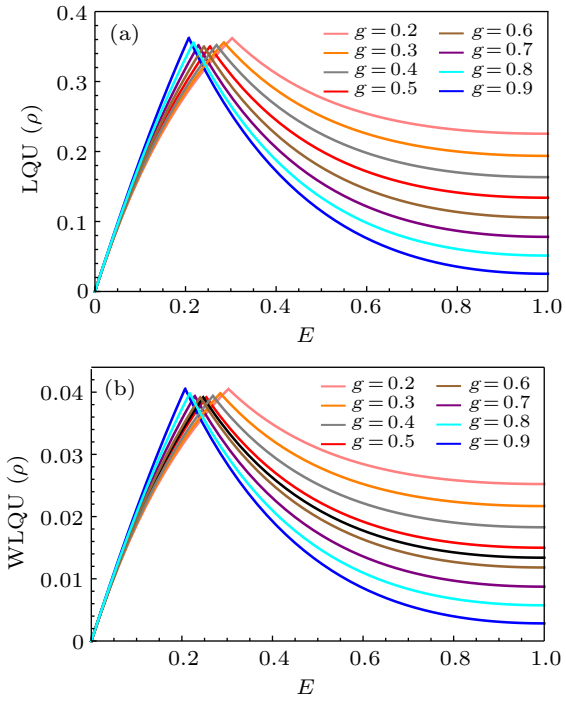


Fig. 3. LQU and WLQU as functions of ϵ for various values of g at $x=1.2$.

To start, we observe the phenomenon of sudden birth in Fig. 3(a). Afterwards, the system undergoes a transition when the LQU arrives at a maximal value. Also, this maximum is strongly dependent on the values taken by the parameter g. After sudden birth, LQU increases to a maximal value at certain parameter ϵ for different values of parameter g, we say that there is a transition at these points of ϵ. Then, it starts to decrease to some minimum value at $(\epsilon=1)$. A similar behavior is observed in the case of the WLQU (Fig. 3(b)). Note that the curves with the following pairs of g: $(0.2,0.9)$; $(0.3,0.8)$; $(0.4,0.7)$; $(0.5,0.6)$ have the same maximum values of the LQU and WLQU and they are suddenly decreasing for the first values of the pairs of g (for the smaller number) compared to the second values (for the greater value) of the pairs g which decrease gradually, with a growing value of ϵ. This shows that the variation of quantum correlations of a two-qubit measured by the LQU and WLQU is not monotonic with the growing g, and exhibits a resonant behavior that firstly increases and then decreases as g grows. This implies that the dynamics of the LQU and WLQU is synchronous, but the LQU is always greater than the WLQU for all g.

6.2.1. The Heisenberg *XXX* model

To be realist, we consider the Heisenberg XXX model composed of two qubits which are immersed in a constant external magnetic field B. Then, the Hamiltonian for the system can be written as

$$H = J\left(\sigma_1^x\sigma_2^x+\sigma_1^y\sigma_2^y+\sigma_1^z\sigma_2^z\right)+B\left(\sigma_1^z+\sigma_2^z\right), \tag{21}$$

where J is the coupling constant, and B is the z-component of the external magnetic field acting on qubits 1 and 2. In the two-qubit computational basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$, Hamiltonian (21) has the following form:

$$H = \begin{pmatrix} 2B+J & 0 & 0 & 0 \\ 0 & -J & 2J & 0 \\ 0 & 2J & -J & 0 \\ 0 & 0 & 0 & -2B+J \end{pmatrix}. \tag{22}$$

The eigenvalues and the corresponding eigenvectors of the Hamiltonien are

$$\begin{aligned} E_1 &= -3J, \quad |\psi_1\rangle = \frac{1}{\sqrt{2}}(|10\rangle-|01\rangle), \\ E_2 &= J, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|10\rangle+|01\rangle), \\ E_3 &= J-2B, \quad |\psi_3\rangle = |11\rangle, \\ E_4 &= 2B+J, \quad |\psi_4\rangle = |00\rangle. \end{aligned} \tag{23}$$

A system, in equilibrium state at temperature T, is described by operator density

$$\rho(T) = \frac{\exp(-\beta H)}{Z} = \frac{1}{Z}\sum_{i=1}^{4}\exp(-\beta E_i), \tag{24}$$

where $Z=\mathrm{Tr}[\exp(-H/kT)]$ is the partition function of the system and $\beta=1/kT$, and k is the Boltzman constant. For simplicity, we choose $k=1$. In the standard basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$, the thermal state $\rho(T)$ is written as

$$\rho(T) = \frac{1}{Z}\begin{pmatrix} \exp\left(-\frac{2B+J}{T}\right) & 0 & 0 & 0 \\ 0 & \frac{\exp\left(\frac{3J}{T}\right)+\exp\left(-\frac{J}{T}\right)}{2} & \frac{\exp\left(-\frac{J}{T}\right)-\exp\left(\frac{3J}{T}\right)}{2} & 0 \\ 0 & \frac{\exp\left(-\frac{J}{T}\right)-\exp\left(\frac{3J}{T}\right)}{2} & \frac{\exp\left(\frac{3J}{T}\right)+\exp\left(-\frac{J}{T}\right)}{2} & 0 \\ 0 & 0 & 0 & \exp\left(\frac{2B-J}{T}\right) \end{pmatrix}, \tag{25}$$

where

$$Z = \exp\left(\frac{3J}{T}\right)+\exp\left(-\frac{J}{T}\right)+\exp\left(\frac{2B-J}{T}\right)+\exp\left(-\frac{2B+J}{T}\right).$$

In the Fano–Bloch representation, the states $\rho(T)$ are written as

$$\rho(T)=\frac{1}{4}\sum_{\alpha\beta}^{3}R_{\alpha\beta}\sigma_\alpha\otimes\sigma_\beta, \tag{26}$$

where the non-vanishing correlation matrix elements of $R_{\alpha\beta}$ are given by

$$\begin{aligned}
R_{00} &= \frac{1}{Z}\exp\left(-\frac{J}{T}\right)\left(1+2\cosh\left(\frac{2B}{T}\right)\right),\\
R_{11} &= \frac{1}{Z}\exp\left(-\frac{J}{T}\right),\\
R_{03} &= \frac{1}{Z}\exp\left(-\frac{2B+J}{T}\right)\\
&\quad\times\left(1-\exp\left(\frac{4B}{T}\right)-\exp\left(\frac{2(B+2J)}{T}\right)\right),\\
R_{30} &= \frac{1}{Z}\exp\left(-\frac{2B+J}{T}\right)\\
&\quad\times\left(1-\exp\left(\frac{4B}{T}\right)+\exp\left(\frac{2(B+2J)}{T}\right)\right),\\
R_{22} &= \frac{1}{Z}\exp\left(-\frac{J}{T}\right),\\
R_{33} &= \frac{1}{Z}\exp\left(-\frac{J}{T}\right)\left(-1+2\cosh\left(\frac{2B}{T}\right)\right),\\
R_{12} &= \frac{\mathrm{i}}{Z}\exp\left(\frac{3J}{T}\right),\quad R_{21}=-\frac{\mathrm{i}}{Z}\exp\left(\frac{3J}{T}\right).
\end{aligned} \tag{27}$$

From Ref. [25], it is easy to show that the square root of the state $\rho(T)$ also has a structure in X state expressed by

$$\sqrt{\rho(T)}=\begin{pmatrix}\widetilde{\rho}_{11}&0&0&0\\0&\widetilde{\rho}_{22}&\widetilde{\rho}_{23}&0\\0&\widetilde{\rho}_{32}&\widetilde{\rho}_{33}&0\\0&0&0&\widetilde{\rho}_{44}\end{pmatrix}, \tag{28}$$

where the diagonal elements of $\sqrt{\rho(T)}$ are given by

$$\begin{aligned}
\widetilde{\rho}_{11} &= \frac{\sqrt{\exp\left(-\frac{2B+J}{T}\right)}\left(1+\exp\left(\frac{2B+J}{T}\right)\sqrt{\exp\left(-\frac{2J}{T}\right)}\right)}{\sqrt{Z}\left(\exp\left(\frac{2B}{T}\right)+\exp\left(\frac{J}{T}\right)\sqrt{\exp\left(-\frac{2J}{T}\right)}\right)},\\
\widetilde{\rho}_{22} &= \frac{1}{2}\sqrt{\frac{1}{Z}\exp\left(-\frac{J}{T}\right)}\left(1+\exp\left(\frac{4J}{T}\right)\right),\\
\widetilde{\rho}_{33} &= -\frac{1}{2}\sqrt{\frac{1}{Z}\exp\left(-\frac{J}{T}\right)}\left(-1+\exp\left(\frac{4J}{T}\right)\right),\\
\widetilde{\rho}_{44} &= \frac{\exp\left(\frac{-J}{T}\right)}{\sqrt{Z\exp\left(-\frac{2B+J}{T}\right)}},
\end{aligned}$$

and the off-diagonal matrix elements are

$$\begin{aligned}
\widetilde{\rho}_{23} &= -\frac{1}{2}\sqrt{\frac{1}{Z}\exp\left(-\frac{J}{T}\right)}\left(-1+\exp\left(\frac{4J}{T}\right)\right),\\
\widetilde{\rho}_{32} &= \frac{1}{2}\sqrt{\frac{1}{Z}\exp\left(-\frac{J}{T}\right)}\left(1+\exp\left(\frac{4J}{T}\right)\right).
\end{aligned}$$

Then, the eigenvalues of matrix elements (14) become

$$\begin{aligned}
\omega_{11} = \omega_{22} &= \frac{\sqrt{\exp\left(-\frac{2(B+J)}{T}\right)}\left(\left(\exp\left(\frac{2B}{T}\right)+\exp\left(\frac{J}{T}\right)\sqrt{\exp\left(-\frac{2J}{T}\right)}\right)^2+\exp\left(\frac{4J}{T}\right)\left(-1+\exp\left(\frac{4B}{T}\right)\right)\right)}{Z\left(\exp\left(\frac{2B}{T}\right)+\exp\left(\frac{J}{T}\right)\sqrt{\exp\left(-\frac{2J}{T}\right)}\right)},\\
\omega_{33} &= \frac{1}{Z}\exp\left(-\frac{2B+J}{T}\right)\left(1+\exp\left(\frac{4B}{T}\right)+\exp\left(\frac{2(B+4J)}{T}\right)\right).
\end{aligned} \tag{29}$$

We note that the eigenvalue $\omega_{11}=\omega_{22}$ so that $\omega_{\max}=\max(\omega_{11},\omega_{33})$. It is easy to verify that the states for which $\omega_{11}\geq\omega_{33}$ or $\omega_{11}\leq\omega_{33}$ are parametrized by J. Then, for different value of J the LQU and WLQU in the state (25) can be expressed by

$$\begin{aligned}
\mathcal{U}(\rho_{\alpha,\beta}) &= 1-\omega_{\max},\\
\mathcal{WU}(\rho_{\alpha,\beta}) &= \left(\frac{\zeta_1-\zeta_2}{2}\right)^2(1-\omega_{\max}).
\end{aligned} \tag{30}$$

Therefore, we must consider the situations where $\omega_{\max}$ is ω_{11} or ω_{33}. This question is examined numerically in what follows for various values of the coupling constant J, in presence of external magnetic fields as well as the effect on the quantum correlations in the state $\rho(T)$.

Let us now analyze the LQU and WLQU in Heisenberg XXX model. From the thermal behavior of the LQU for different values of parameter J with $T=1$ and $T=5$ in Figs. 4(a) and 4(b), the LQU decreases with increasing J and when we modify the parameter $T=5$ to $T=1$, it increases for the smaller values of J despite the thermal effects tending to reduce the amount of quantum correlations. Moreover, LQU via weak measurement is weaker than that in the projective mea-

surement for different values of J with $T = 1$ and $T = 5$.

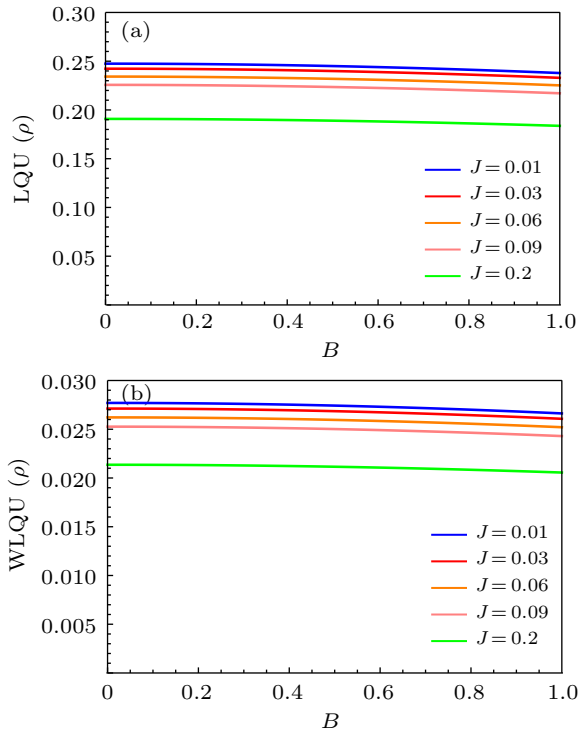


Fig. 4. Evolution of (a) LQU and (b) WLQU for suitable parameter of B, where $T = 1$ for different values of J.

7. Concluding remarks

The LQU is potentially a powerful way to describe the quantum correlations in quantum systems. This is mainly due to its accuracy and ease of use from a computational point of view. In this vision, we studied the LQU in weak measurement. After presenting the distinctive feature of the weak measurement with respect to the projective one, we have determined the WLQU whose form is considered closed by the LQU. We focused on the WLQU for the case of two-qubit X states, and then, for a comparison purpose, we have determined the corresponding WLQU with respect to the LQU for the cases of three special X states: (i) z-Werner state, (ii) (ϵ, g) circulant two-qubit state, and (iii) Heisenberg XXX model.

Quantum correlation can be extracted by optimally projective measurement through continuously by the infinitesimal weak measurements. However, the total amount extracted by using a series of weak measurements is less than the quantity extracted from the projective measurements. This explains the fact that the continuous infinitesimal weak measurements cannot realize the projective measurement. In other words, weak measurements destroy i.e., extract a very weak part of the quantum correlation compared with the projection measurement which destroys the quantum correlation of a quantum system and makes it classical.

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