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Citation: Chin. Phys. B . 2020, 29(7): 070304 . **doi:** 10.1088/1674-1056/ab8e2e
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Tighter constraints of multiqubit entanglement in terms of Rényi-α entropy*

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(Received 14 March 2020; revised manuscript received 20 April 2020; accepted manuscript online 29 April 2020)

Quantum entanglement plays essential roles in quantum information processing. The monogamy and polygamy relations characterize the entanglement distributions in the multipartite systems. We present a class of monogamy inequalities related to the μ-th power of the entanglement measure based on Rényi-α entropy, as well as polygamy relations in terms of the μ-th power of Rényi-α entanglement of assistance. These monogamy and polygamy relations are shown to be tighter than the existing ones.

Keywords: monogamy relations, polygamy relations, Rényi-α entropy, Hamming weight

PACS: 03.67.Mn, 03.65.Ud, 03.65.Yz **DOI:** 10.1088/1674-1056/ab8e2e

1. Introduction

Quantum entanglement is one of the most quintessential features of quantum mechanics, which distinguishes the quantum world from the classical one and plays essential roles in quantum information processing,[1–5] revealing the basic understanding of the nature of quantum correlations. One distinct property of quantum entanglement is that a quantum system entangled with another system limits its sharing with other systems, known as the monogamy of entanglement.[6,7] The monogamy of entanglement can be used as a resource to distribute a secret key which is secure against unauthorized parties.[8,9] It also plays a significant role in many fields of physics such as foundations of quantum mechanics,[10,34] condensed matter physics,[11] statistical mechanics,[34] and even black-hole physics.[12,13]

The monogamy inequality was first introduced by Coffman–Kundu–Wootters (CKW) by using tangle as a bipartite entanglement measure in three-qubit systems,[14] and then generalized to multiqubit systems based on various entanglement measures.[15] The assisted entanglement is a dual concept to bipartite entanglement measure, which shows polygamy relations in multiparty quantum systems. For a three-qubit state ρ_{ABC}, a polygamy inequality was introduced as[16] $\tau^{\mathrm{a}}(\rho_{A|BC}) \leq \tau^{\mathrm{a}}(\rho_{A|B}) + \tau^{\mathrm{a}}(\rho_{A|C})$, where $\tau^{\mathrm{a}}(\rho_{A|BC}) = \max \sum_i p_i \tau(|\psi_i\rangle_{A|B})$ is the tangle of assistance,[16,17] with the maximum taking over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle\psi_i|$. This tangle-based polygamy inequality was extended to multiqubit systems and also high-dimensional quantum systems in terms of various entropy entanglement measures.[18,19] General polygamy inequalities of entanglement are also established in arbitrary dimensional multipartite quantum systems.[20–25]

In this paper, we investigate the monogamy and polygamy constraints based on the μ-th power of entanglement measures in terms of the Rényi-α entropy for multiqubit systems. By using the Hamming weight of binary vectors, we present a class of monogamy inequalities for multiqubit entanglement based on the μ-th power of Rényi-α entanglement (RαE)[26] for $\mu \geq 1$. For $0 \leq \mu \leq 1$, we introduce a class of tight polygamy inequalities based on the μ-th power of the Rényi-α entanglement of assistance (RαEoA). Then, we show that both the monogamy inequalities with $\mu \geq 1$ and the polygamy inequalities with $0 \leq \mu \leq 1$ can be further improved to be tighter under certain conditions. These monogamy and polygamy relations are shown to be tighter than the existing ones. Moreover, our monogamy inequality is shown to be more effective for the counterexamples of the CKW monogamy inequality in higher-dimensional systems.

2. Preliminaries

We first recall the conceptions of Rényi-α entropy, Rényi-α entanglement, and multiqubit monogamy and polygamy inequalities. For any $\alpha > 0$, $\alpha \neq 1$, the Rényi-α entropy of a quantum state ρ is defined as[27]

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log(\mathrm{tr}\rho^\alpha).$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11765016 and 11675113), the Natural Science Foundation of Beijing, China (Grant No. KZ201810028042), and Beijing Natural Science Foundation, China (Grant No. Z190005).

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$S_\alpha(\rho)$ reduces to the von Neumann entropy when α approaches to 1.

The Rényi-α entanglement (RαE) $E_\alpha(|\psi\rangle_{AB})$ of a bipartite pure state $|\psi\rangle_{AB}$ is defined as

$$E_\alpha(|\psi\rangle_{AB}) = S_\alpha(\rho_A),$$

where $\rho_A = \mathrm{Tr}_B|\psi\rangle_{AB}\langle\psi|$ is the reduced state of system A. For a mixed state ρ_{AB}, the Rényi-α entanglement is given by

$$E_\alpha(\rho_{A|B}) = \min \sum_i p_i E_\alpha(|\psi_i\rangle_{A|B}),$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i|\psi_i\rangle_{AB}\langle\psi_i|$.

As a dual concept to RαE, the Rényi-α entanglement of assistance (RαEoA) is introduced as

$$E_\alpha^{\mathrm{a}}(\rho_{A|B}) = \max \sum_i p_i E_\alpha(|\psi_i\rangle_{A|B}), \tag{1}$$

where the maximum is taken over all possible pure state decompositions of ρ_{AB}.[28]

For any multiqubit state $\rho_{AB_0\cdots B_{N-1}}$, a monogamous inequality has been presented in Ref. [28] for $\alpha \geq 2$,

$$E_\alpha(\rho_{A|B_0\cdots B_{N-1}}) \geq \sum_{i=0}^{N-1} E_\alpha(\rho_{A|B_i}), \tag{2}$$

where $E_\alpha(\rho_{A|B_0\cdots B_{N-1}})$ is the RαE of $\rho_{AB_0\cdots B_{N-1}}$ with respect to the bipartition between A and $B_0\cdots B_{N-1}$, and $E_\alpha(\rho_{A|B_i})$ is the RαE of the reduced density matrix ρ_{AB_i}, $i = 0,\ldots,N-1$.

In addition, a class of polygamy inequalities has been obtained for multiqubit systems,

$$E_\alpha^{\mathrm{a}}(\rho_{A|B_0\cdots B_{N-1}}) \leq \sum_{i=0}^{N-1} E_\alpha^{\mathrm{a}}(\rho_{A|B_i}), \tag{3}$$

for $0 \leq \alpha \leq 2$, $\alpha \neq 1$, where $E_\alpha(\rho_{AB_0\cdots B_{N-1}})$ is the RαEoA of $\rho_{AB_0\cdots B_{N-1}}$ with respect to the bipartition between A and $B_0\cdots B_{N-1}$, and $E_\alpha^{\mathrm{a}}(\rho_{A|B_i})$ is the RαEoA of the reduced density matrix ρ_{AB_i}, $i = 0,\ldots,N-1$.

In Ref. [29], Kim established a class of tight monogamy inequalities of multiqubit entanglement in terms of Hamming weight. For any nonnegative integer j with binary expansion $j = \sum_{i=0}^{n-1} j_i 2^i$, where $\log_2 j \leq n$ and $j_i \in \{0,1\}$ for $i = 0,\ldots,n-1$, one can always define a unique binary vector associated with j, $\boldsymbol{j} = (j_0, j_1, \ldots, j_{n-1})$. The Hamming weight $\omega_{\mathrm{H}}(\boldsymbol{j})$ of the binary vector $\boldsymbol{j}$ is defined to be the number of 1's in its coordinates.[35] Moreover, the Hamming weight $\omega_{\mathrm{H}}(\boldsymbol{j})$ is bounded above by $\log_2 j$,

$$\omega_{\mathrm{H}}(\boldsymbol{j}) \leq \log_2 j \leq j. \tag{4}$$

Kim proposed the tight constraints of multiqubit entanglement based on Hamming weights[29]

$$\left[E_\alpha(\rho_{A|B_0B_1\ldots B_{N-1}})\right]^\mu \geq \sum_{j=0}^{N-1} \mu^{\omega_{\mathrm{H}}(j)} \left[E_\alpha(\rho_{A|B_j})\right]^\mu \tag{5}$$

for $\mu \geq 1$, and

$$\left[E_\alpha^{\mathrm{a}}(\rho_{A|B_0B_1\ldots B_{N-1}})\right]^\mu \leq \sum_{j=0}^{N-1} \mu^{\omega_{\mathrm{H}}(j)} \left[E_\alpha^{\mathrm{a}}(\rho_{A|B_j})\right]^\mu \tag{6}$$

for $0 \leq \mu \leq 1$. Inequalities (5) and (6) are then further written as

$$\left[E_\alpha(\rho_{A|B_0B_1\ldots B_{N-1}})\right]^\mu \geq \sum_{j=0}^{N-1} \mu^j \left[E_\alpha(\rho_{A|B_j})\right]^\mu$$

for $\mu \geq 1$, and

$$\left[E_\alpha^{\mathrm{a}}(\rho_{A|B_0B_1\ldots B_{N-1}})\right]^\mu \leq \sum_{j=0}^{N-1} \mu^j \left[E_\alpha^{\mathrm{a}}(\rho_{A|B_j})\right]^\mu$$

for $0 \leq \mu \leq 1$.

In the following we show that these inequalities above can be further improved to be much tighter under certain conditions, which provide tighter constraints on the multiqubit entanglement distribution.

3. Tighter constraints of multiqubit entanglement in terms of RαE

We first present a class of tighter monogamy and polygamy inequalities of multiqubit entanglement in terms of the μ-th power of RαE. We need the following results.[32] Suppose k is a real number, $0 < k \leq 1$. Then for any $0 \leq x \leq k$, we have

$$(1+x)^\mu \geq 1 + \frac{(1+k)^\mu - 1}{k^\mu} x^\mu \tag{7}$$

for $\mu \geq 1$, and

$$(1+x)^\mu \leq 1 + \frac{(1+k)^\mu - 1}{k^\mu} x^\mu \tag{8}$$

for $0 \leq \mu \leq 1$. Based on inequality (7), we have the following theorem for RαE.

Theorem 1 For any multiqubit state $\rho_{AB_0\ldots B_{N-1}}$ and $\alpha \geq 2$, we have

$$\left[E_\alpha(\rho_{A|B_0B_1\ldots B_{N-1}})\right]^\mu \geq \sum_{j=0}^{N-1} \left(\frac{(1+k)^\mu - 1}{k^\mu}\right)^{\omega_{\mathrm{H}}(j)} \left[E_\alpha(\rho_{A|B_j})\right]^\mu, \tag{9}$$

where $\mu \geq 1$, $\boldsymbol{j} = (j_0,\ldots,j_{n-1})$ is the vector from the binary representation of j, and $\omega_{\mathrm{H}}(\boldsymbol{j})$ is the Hamming weight of $\boldsymbol{j}$.

Proof We first prove that

$$\left[\sum_{j=0}^{N-1} E_\alpha(\rho_{A|B_j})\right]^\mu \geq \sum_{j=0}^{N-1} \left(\frac{(1+k)^\mu - 1}{k^\mu}\right)^{\omega_{\mathrm{H}}(j)} \left[E_\alpha(\rho_{A|B_j})\right]^\mu. \tag{10}$$

Without loss of generality, we assume that the qubit subsystems $B_0,\ldots,B_{N-1}$ are so labeled such that

$$kE_\alpha(\rho_{A|B_j}) \geq E_\alpha(\rho_{A|B_{j+1}}) \geq 0 \tag{11}$$

for $j = 0,1,\ldots,N-2$ and some $0 < k \leq 1$.

We first show that the inequality (10) holds for the case of $N = 2^n$. For $n = 1$, let ρ_{AB_0} and ρ_{AB_1} be the two-qubit reduced

density matrices of a three-qubit pure state $\rho_{AB_0B_1}$. We obtain

$$\left[E_\alpha(\rho_{A|B_0})+E_\alpha(\rho_{A|B_1})\right]^\mu=\left[E_\alpha(\rho_{A|B_0})\right]^\mu\left(1+\frac{E_\alpha(\rho_{A|B_1})}{E_\alpha(\rho_{A|B_0})}\right)^\mu. \tag{12}$$

Combining inequalities (7) and (11), we have

$$\left(1+\frac{E_\alpha(\rho_{A|B_1})}{E_\alpha(\rho_{A|B_0})}\right)^\mu\geq 1+\frac{(1+k)^\mu-1}{k^\mu}\left(\frac{E_\alpha(\rho_{A|B_1})}{E_\alpha(\rho_{A|B_0})}\right)^\mu. \tag{13}$$

From formulas (12) and (13), we obtain

$$\left[E_\alpha(\rho_{A|B_0})+E_\alpha(\rho_{A|B_1})\right]^\mu\geq\left[E_\alpha(\rho_{A|B_0})\right]^\mu+\frac{(1+k)^\mu-1}{k^\mu}\left[E_\alpha(\rho_{A|B_1})\right]^\mu.$$

Therefore, the inequality (10) holds for $n=1$.

We assume that the inequality (10) holds for $N=2^{n-1}$ with $n\geq 2$, and prove the case of $N=2^n$. For an $(N+1)$-qubit pure state $\rho_{AB_0B_1\ldots B_{N-1}}$, we have $E_\alpha(\rho_{A|B_{j+2^{n-1}}})\leq k^{2^{n-1}}E_\alpha(\rho_{A|B_j})$ from inequality (11). Therefore,

$$0\leq\frac{\sum_{j=2^{n-1}}^{2^n-1}E_\alpha(\rho_{A|B_j})}{\sum_{j=0}^{2^{n-1}-1}E_\alpha(\rho_{A|B_j})}\leq k^{2^{n-1}}\leq k,$$

$$\left(\sum_{j=0}^{N-1}E_\alpha(\rho_{A|B_j})\right)^\mu=\left(\sum_{j=0}^{2^{n-1}-1}E_\alpha(\rho_{A|B_j})\right)^\mu\times\left(1+\frac{\sum_{j=2^{n-1}}^{2^n-1}E_\alpha(\rho_{A|B_j})}{\sum_{j=0}^{2^{n-1}-1}E_\alpha(\rho_{A|B_j})}\right)^\mu.$$

Thus, we have

$$\left(\sum_{j=0}^{N-1}E_\alpha(\rho_{A|B_j})\right)^\mu\geq\left(\sum_{j=0}^{2^{n-1}-1}E_\alpha(\rho_{A|B_j})\right)^\mu+\frac{(1+k)^\mu-1}{k^\mu}\left(\sum_{j=2^{n-1}}^{2^n-1}E_\alpha(\rho_{A|B_j})\right)^\mu.$$

According to the induction hypothesis, we obtain

$$\left(\sum_{j=0}^{2^{n-1}-1}E_\alpha(\rho_{A|B_j})\right)^\mu\geq\sum_{j=0}^{2^{n-1}-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)}\times\left[E_\alpha(\rho_{A|B_j})\right]^\mu.$$

By relabeling the subsystems, the induction hypothesis leads to

$$\left(\sum_{j=2^{n-1}}^{2^n-1}E_\alpha(\rho_{A|B_j})\right)^\mu\geq\sum_{j=2^{n-1}}^{2^n-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)-1}\times\left[E_\alpha(\rho_{A|B_j})\right]^\mu.$$

Thus, we have

$$\left(\sum_{j=0}^{2^n-1}E_\alpha(\rho_{A|B_j})\right)^\mu\geq\sum_{j=0}^{2^n-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)}\times\left[E_\alpha(\rho_{A|B_j})\right]^\mu.$$

Now consider a (2^n+1)-qubit state

$$\Gamma_{AB_0B_1\ldots B_{2^n-1}}=\rho_{AB_0B_1\ldots B_{N-1}}\otimes\sigma_{B_N\ldots B_{2^n-1}}, \tag{14}$$

which is the tensor product of $\rho_{AB_0B_1\ldots B_{N-1}}$ and an arbitrary (2^n-N)-qubit state $\sigma_{B_N\ldots B_{2^n-1}}$. We have

$$\left[E_\alpha(\Gamma_{A|B_0B_1\ldots B_{2^n-1}})\right]^\mu\geq\sum_{j=0}^{2^n-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)}\left[E_\alpha(\Gamma_{A|B_j})\right]^\mu,$$

where $\Gamma_{A|B_j}$ is the two-qubit reduced density matrix of $\Gamma_{AB_0B_1\ldots B_{2^n-1}}$, $j=0,1,\ldots,2^n-1$. Therefore,

$$\begin{aligned}\left[E_\alpha(\rho_{A|B_0B_1\ldots B_{N-1}})\right]^\mu&=\left[E_\alpha(\Gamma_{A|B_0B_1\ldots B_{2^n-1}})\right]^\mu\\&\geq\sum_{j=0}^{2^n-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)}\left[E_\alpha(\Gamma_{A|B_j})\right]^\mu\\&=\sum_{j=0}^{N-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)}\left[E_\alpha(\rho_{A|B_j})\right]^\mu,\end{aligned}$$

where $\Gamma_{A|B_0B_1\ldots B_{2^n-1}}$ is separated to the bipartition $AB_0\ldots B_{N-1}$ and $B_N\ldots B_{2^n-1}$, $E_\alpha\left(\Gamma_{A|B_0B_1\cdots B_{2^n-1}}\right)=E_\alpha\left(\rho_{A|B_0B_1\cdots B_{N-1}}\right)$, $E_\alpha\left(\Gamma_{A|B_j}\right)=0$ for $j=N,\ldots,2^n-1$, and $\Gamma_{AB_j}=\rho_{AB_j}$ for each $j=0,\ldots,N-1$.

Since $\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)}\geq\mu^{\omega_{\rm H}(j)}$ for $\mu\geq 1$, for any multiqubit state $\rho_{AB_0B_1\cdots B_{N-1}}$ we have the following relation:

$$\begin{aligned}\left[E_\alpha(\rho_{A|B_0B_1\ldots B_{N-1}})\right]^\mu&\geq\sum_{j=0}^{N-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)}\left[E_\alpha(\rho_{A|B_j})\right]^\mu\\&\geq\sum_{j=0}^{N-1}\mu^{\omega_{\rm H}(j)}\left[E_\alpha(\rho_{A|B_j})\right]^\mu.\end{aligned}$$

Therefore, our inequality (9) in Theorem 1 is always tighter than the inequality (5).

In fact, the tighter monogamy inequality (9) holds not only for multiqubit systems, but also for some multipartite higher-dimensional quantum systems, which can be proved in a similar way as in Ref. [29]. Here, we show that inequality (9) is also more efficient than inequality (5) for such higher-dimensional quantum systems. Let us consider the counterexample of the CKW inequality in tripartite quantum systems[30]

$$|\psi\rangle_{ABC}=\frac{1}{\sqrt{6}}(|123\rangle-|132\rangle+|231\rangle-|213\rangle+|312\rangle-|321\rangle). \tag{15}$$

One has $E_\alpha(|\psi\rangle_{A|BC})=S_\alpha(\rho)$. Taking $\alpha=3$, we have $E_\alpha(|\psi\rangle_{A|BC})=\log 3$ and the RαE of the two-qubit reduced density matrices are

$$E_\alpha(\rho_{A|B})=E_\alpha(\rho_{A|C})=-\frac{1}{2}\log{\rm tr}\sigma_A^3=1.$$

In the case $k=1$, for $\mu\geq 1$, we have

$$\begin{aligned}y_1&\equiv\left[E_\alpha(\rho_{A|B})\right]^\mu+\frac{(1+k)^\mu-1}{k^\mu}\left[E_\alpha(\rho_{A|C})\right]^\mu\\&=1+\frac{(1+k)^\mu-1}{k^\mu}=2^\mu,\\y_2&\equiv\left[E_\alpha(\rho_{A|B})\right]^\mu+\mu\left[E_\alpha(\rho_{A|C})\right]^\mu=1+\mu.\end{aligned}$$

Therefore, one gets

$$\left[E_\alpha(\rho_{A|B})\right]^\mu+\frac{(1+k)^\mu-1}{k^\mu}\left[E_\alpha(\rho_{A|C})\right]^\mu\geq\left[E_\alpha(\rho_{A|B})\right]^\mu+\mu\left[E_\alpha(\rho_{A|C})\right]^\mu,$$

where $\mu \geq 1$, see Fig. 1. In other words, our new monogamy inequality is indeed tighter than the previous one given in Ref. [29].

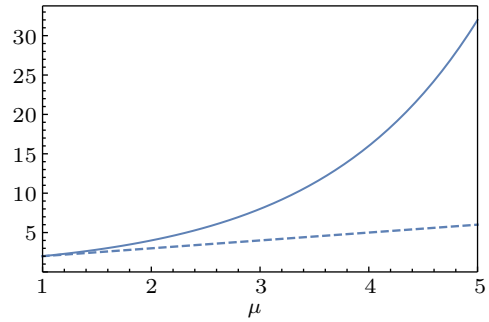


Fig. 1. Rényi-α entanglement with respect to μ: the solid line is for y_1 and the dashed line for y_2 from the result in Ref. [29].

Under certain conditions, the inequality (9) can even be improved further to become a much tighter inequality.

Theorem 2 For $\mu \geq 1$, $\alpha \geq 2$, and real number $0 < k \leq 1$, any multiqubit state $\rho_{AB_0\ldots B_{N-1}}$ satisfies

$$\left[E_\alpha(\rho_{A|B_0\ldots B_{N-1}})\right]^\mu \geq \sum_{j=0}^{N-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^j\left[E_\alpha(\rho_{A|B_j})\right]^\mu, \tag{16}$$

if

$$kE_\alpha(\rho_{A|B_j}) \geq \sum_{j=i+1}^{N-1} E_\alpha(\rho_{A|B_j}) \tag{17}$$

for $i = 0,1,\ldots,N-2$.

Proof We need to show

$$\left(\sum_{j=0}^{N-1} E_\alpha\left(\rho_{A|B_j}\right)\right)^\mu \geq \sum_{j=0}^{N-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^j\left(E_\alpha\left(\rho_{A|B_j}\right)\right)^\mu. \tag{18}$$

For any multiqubit state $\rho_{AB_0\ldots B_{N-1}}$, it is easy to show that

$$\left(\sum_{j=0}^{N-1} E_\alpha\left(\rho_{A|B_j}\right)\right)^\mu = \left(E_\alpha\left(\rho_{A|B_0}\right)\right)^\mu\left(1+\frac{\sum_{j=1}^{N-1}E_\alpha\left(\rho_{A|B_j}\right)}{E_\alpha\left(\rho_{A|B_0}\right)}\right)^\mu,$$

$$\left(1+\frac{\sum_{j=1}^{N-1}E_\alpha\left(\rho_{A|B_j}\right)}{E_\alpha\left(\rho_{A|B_0}\right)}\right)^\mu \geq 1+\left(\frac{(1+k)^\mu-1}{k^\mu}\right)\left(\frac{\sum_{j=1}^{N-1}E_\alpha\left(\rho_{A|B_j}\right)}{E_\alpha\left(\rho_{A|B_0}\right)}\right)^\mu.$$

Thus,

$$\begin{aligned}&\left(\sum_{j=0}^{N-1} E_\alpha\left(\rho_{A|B_j}\right)\right)^\mu \\ &\geq \left(E_\alpha\left(\rho_{A|B_0}\right)\right)^\mu+\left(\frac{(1+k)^\mu-1}{k^\mu}\right)\left(\sum_{j=1}^{N-1}E_\alpha\left(\rho_{A|B_j}\right)\right)^\mu \\ &\geq \sum_{j=0}^{N-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^j\left(E_\alpha\left(\rho_{A|B_j}\right)\right)^\mu,\end{aligned}$$

where the second inequality is due to the induction hypothesis.

In fact, according to Eq. (4), for any $\mu \geq 1$, one has

$$\begin{aligned}\left[E_\alpha(\rho_{A|B_0\ldots B_{N-1}})\right]^\mu &\geq \sum_{j=0}^{N-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^j\left(E_\alpha\left(\rho_{A|B_j}\right)\right)^\mu \\ &\geq \sum_{j=0}^{N-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\mathrm{H}}(j)}\left[E_\alpha(\rho_{A|B_j})\right]^\mu.\end{aligned}$$

For the case of $\mu < 0$, we can also derive a tighter upper bound of $E_\alpha^\mu(\rho_{A|B_0B_1\ldots B_{N-1}})$.

Theorem 3 For any multiqubit state $\rho_{AB_0\ldots B_{N-1}}$ with $E_\alpha(\rho_{AB_i}) \neq 0$, $i = 0,1,\ldots,N-1$, we have

$$\left[E_\alpha(\rho_{A|B_0B_1\ldots B_{N-1}})\right]^\mu \leq \frac{1}{N-1}\sum_{j=0}^{N-1}\left[E_\alpha(\rho_{A|B_j})\right]^\mu \tag{19}$$

for all $\mu < 0$ and $\alpha \geq 2$.

Proof Similar to the proof in Ref. [31] for arbitrary three-qubit states, we have

$$\begin{aligned}\left[E_\alpha(\rho_{A|B_0B_1})\right]^\mu &\leq \left[E_\alpha^2(\rho_{A|B_0})+E_\alpha^2(\rho_{A|B_1})\right]^{\mu/2} \\ &= \left[E_\alpha(\rho_{A|B_0})\right]^\mu\left(1+\frac{E_\alpha^2(\rho_{A|B_1})}{E_\alpha^2(\rho_{A|B_0})}\right)^{\mu/2} \\ &< \left[E_\alpha(\rho_{A|B_0})\right]^\mu,\end{aligned} \tag{20}$$

where the first inequality is from $\mu < 0$, and the second inequality is due to $\left(1+\frac{E_\alpha^2(\rho_{A|B_1})}{E_\alpha^2(\rho_{A|B_0})}\right)^\mu < 1$. Moreover, we have

$$\left[E_\alpha(\rho_{A|B_0B_1})\right]^\mu < \left[E_\alpha(\rho_{A|B_1})\right]^\mu. \tag{21}$$

Combining inequalities (20) and (21), we obtain

$$\left[E_\alpha(\rho_{A|B_0B_1})\right]^\mu < \frac{1}{2}\left\{\left[E_\alpha(\rho_{A|B_0})\right]^\mu+\left[E_\alpha(\rho_{A|B_1})\right]^\mu\right\}.$$

Thus, we have

$$\begin{aligned}&\left[E_\alpha(\rho_{A|B_0B_1\ldots B_{N-1}})\right]^\mu \\ &< \frac{1}{2}\left\{\left[E_\alpha(\rho_{A|B_0})\right]^\mu+\left[E_\alpha(\rho_{A|B_1\ldots B_{N-1}})\right]^\mu\right\} \\ &< \frac{1}{2}\left[E_\alpha(\rho_{A|B_0})\right]^\mu+\left(\frac{1}{2}\right)^2\left[E_\alpha(\rho_{A|B_1})\right]^\mu \\ &\quad+\left(\frac{1}{2}\right)^2\left[E_\alpha(\rho_{A|B_2\ldots B_{N-1}})\right]^\mu \\ &< \ldots \\ &< \frac{1}{2}\left[E_\alpha(\rho_{A|B_0})\right]^\mu+\left(\frac{1}{2}\right)^2\left[E_\alpha(\rho_{A|B_1})\right]^\mu+\ldots \\ &\quad+\left(\frac{1}{2}\right)^{N-2}\left[E_\alpha(\rho_{A|B_{N-2}})\right]^\mu+\left(\frac{1}{2}\right)^{N-2}\left[E_\alpha(\rho_{A|B_{N-1}})\right]^\mu.\end{aligned} \tag{22}$$

One can get a set of inequalities through the cyclic permutation of the pair indices $B_0, B_1, \ldots, B_{N-1}$ in inequality (22). Summing up these inequalities, we obtain inequality (19).

4. Tighter constraints of multiqubit entanglement in terms of RαEoA

We consider now the RαEoA defined in Eq. (1), and provide a class of polygamy inequalities satisfied by the multiqubit entanglement in terms of RαEoA.

Theorem 4 For any multiqubit state $\rho_{AB_0\ldots B_{N-1}}$ and $0 \le \mu \le 1$, $0 < \alpha < 2$, $\alpha \ne 1$, we have

$$\left[E_\alpha^{\rm a}(\rho_{A|B_0B_1\ldots B_{N-1}})\right]^\mu \le \sum_{j=0}^{N-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)}\left[E_\alpha^{\rm a}(\rho_{A|B_j})\right]^\mu. \tag{23}$$

Proof Similar to proof in Ref. [29], we just need to prove

$$\left[\sum_{j=0}^{N-1}E_\alpha^{\rm a}(\rho_{A|B_j})\right]^\mu \le \sum_{j=0}^{N-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)}\left[E_\alpha^{\rm a}(\rho_{A|B_j})\right]^\mu. \tag{24}$$

Firstly, assume that the qubit subsystems $B_0,\ldots,B_{N-1}$ satisfy

$$kE_\alpha^{\rm a}(\rho_{A|B_j}) \ge E_\alpha^{\rm a}(\rho_{A|B_{j+1}}) \ge 0, \tag{25}$$

where $j = 0,1,\ldots,N-2$ and $0 < k \le 1$. Similar to the proof of Theorem 1, we first show that the inequality (24) holds for a three-qubit pure state $\rho_{AB_0B_1}$. We have

$$\begin{aligned}
&\left[E_\alpha^{\rm a}(\rho_{A|B_0})+E_\alpha^{\rm a}(\rho_{A|B_1})\right]^\mu\\
=\;&\left[E_\alpha^{\rm a}(\rho_{A|B_0})\right]^\mu\left(1+\frac{E_\alpha^{\rm a}(\rho_{A|B_1})}{E_\alpha^{\rm a}(\rho_{A|B_0})}\right)^\mu\\
\le\;&\left[E_\alpha^{\rm a}(\rho_{A|B_0})\right]^\mu\left[1+\frac{(1+k)^\mu-1}{k^\mu}\left(\frac{E_\alpha^{\rm a}(\rho_{A|B_1})}{E_\alpha^{\rm a}(\rho_{A|B_0})}\right)^\mu\right]\\
=\;&\left[E_\alpha^{\rm a}(\rho_{A|B_0})\right]^\mu+\frac{(1+k)^\mu-1}{k^\mu}\left[E_\alpha^{\rm a}(\rho_{A|B_1})\right]^\mu,
\end{aligned}$$

where the inequality is due to inequality (8).

Then we assume that the inequality (24) holds for $N = 2^{n-1}$ with $n \ge 2$. Consider the case of $N = 2^n$. For an $(N+1)$-qubit pure state $\rho_{AB_0B_1\ldots B_{N-1}}$ with its two-qubit reduced density matrices ρ_{AB_j}, $j = 0,1,\ldots,N-1$, we have $E_\alpha^{\rm a}(\rho_{A|B_{j+2^{n-1}}}) \le k^{2^{n-1}}E_\alpha^{\rm a}(\rho_{A|B_j})$ due to the ordering of subsystems in the inequality (25). Then, we obtain

$$\begin{aligned}
&0 \le \frac{\sum_{j=2^{n-1}}^{2^n-1}E_\alpha^{\rm a}(\rho_{A|B_j})}{\sum_{j=0}^{2^{n-1}-1}E_\alpha^{\rm a}(\rho_{A|B_j})} \le k^{2^{n-1}} \le k \le 1,\\
&\left(\sum_{j=0}^{N-1}E_\alpha^{\rm a}(\rho_{A|B_j})\right)^\mu\\
=\;&\left(\sum_{j=0}^{2^{n-1}-1}E_\alpha^{\rm a}(\rho_{A|B_j})\right)^\mu\times\left(1+\frac{\sum_{j=2^{n-1}}^{2^n-1}E_\alpha^{\rm a}(\rho_{A|B_j})}{\sum_{j=0}^{2^{n-1}-1}E_\alpha^{\rm a}(\rho_{A|B_j})}\right)^\mu.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left(\sum_{j=0}^{N-1}E_\alpha^{\rm a}(\rho_{A|B_j})\right)^\mu\\
\le\;&\left(\sum_{j=0}^{2^{n-1}-1}E_\alpha(\rho_{A|B_j})\right)^\mu+\frac{(1+k)^\mu-1}{k^\mu}\left(\sum_{j=2^{n-1}}^{2^n-1}E_\alpha^{\rm a}(\rho_{A|B_j})\right)^\mu.
\end{aligned}$$

According to the induction hypothesis, we obtain

$$\begin{aligned}
&\left(\sum_{j=0}^{2^{n-1}-1}E_\alpha^{\rm a}(\rho_{A|B_j})\right)^\mu\\
\le\;&\sum_{j=0}^{2^{n-1}-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)}\left[E_\alpha^{\rm a}(\rho_{A|B_j})\right]^\mu.
\end{aligned}$$

By relabeling the subsystems, the induction hypothesis leads to

$$\begin{aligned}
&\left(\sum_{j=2^{n-1}}^{2^n-1}E_\alpha^{\rm a}(\rho_{A|B_j})\right)^\mu\\
\le\;&\sum_{j=2^{n-1}}^{2^n-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)-1}\left[E_\alpha^{\rm a}(\rho_{A|B_j})\right]^\mu.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\sum_{j=0}^{2^n-1}E_\alpha^{\rm a}(\rho_{A|B_j})\right)^\mu\\
\le\;&\sum_{j=0}^{2^n-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)}\left[E_\alpha^{\rm a}(\rho_{A|B_j})\right]^\mu.
\end{aligned}$$

Consider the (2^n+1)-qubit state (17). We have

$$\begin{aligned}
&\left[E_\alpha^{\rm a}(\rho_{A|B_0B_1\ldots B_{N-1}})\right]^\mu\\
=\;&\left[E_\alpha^{\rm a}(\Gamma_{A|B_0B_1\ldots B_{2^n-1}})\right]^\mu\\
\le\;&\sum_{j=0}^{2^n-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)}\left[E_\alpha^{\rm a}(\Gamma_{A|B_j})\right]^\mu\\
=\;&\sum_{j=0}^{N-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^{\omega_{\rm H}(j)}\left[E_\alpha^{\rm a}(\rho_{A|B_j})\right]^\mu.
\end{aligned}$$

Since $\frac{(1+k)^\mu-1}{k^\mu} \le \mu$ for $0 \le \mu \le 1$, it is easy to see that inequality (23) is tighter than inequality (6).

As an example, let us consider the three-qubit W-state[33]

$$|W\rangle_{ABC} = \frac{1}{\sqrt{3}}(|211\rangle+|121\rangle+|112\rangle). \tag{26}$$

We have $E_\alpha^{\rm a}(|\psi\rangle_{A|BC}) = S_\alpha(\rho) = \log 3 - \frac{2}{3}$ and

$$E_\alpha^{\rm a}(\rho_{A|B}) = E_\alpha^{\rm a}(\rho_{A|C}) = -\frac{1}{2}\log{\rm tr}\sigma_A^3 = \frac{2}{3}.$$

In the case of $k = 1$ and $0 \le \mu \le 1$, we have

$$\begin{aligned}
y_3 \equiv\;&\left[E_\alpha^{\rm a}(\rho_{A|B})\right]^\mu+\frac{(1+k)^\mu-1}{k^\mu}\left[E_\alpha^{\rm a}(\rho_{A|C})\right]^\mu\\
=\;&\left(\frac{2}{3}\right)^\mu+\frac{(1+k)^\mu-1}{k^\mu}\left(\frac{2}{3}\right)^\mu\\
=\;&\left(\frac{2}{3}\right)^\mu 2^\mu,
\end{aligned}$$

$$y_4 \equiv \left[E_\alpha^{\rm a}(\rho_{A|B})\right]^\mu+\mu\left[E_\alpha^{\rm a}(\rho_{A|C})\right]^\mu = \left(\frac{2}{3}\right)^\mu(1+\mu).$$

Therefore, we obtain

$$\begin{aligned}
&\left[E_\alpha^{\rm a}(\rho_{A|B})\right]^\mu+\frac{(1+k)^\mu-1}{k^\mu}\left[E_\alpha^{\rm a}(\rho_{A|C})\right]^\mu\\
\le\;&\left[E_\alpha(\rho_{A|B})\right]^\mu+\mu\left[E_\alpha(\rho_{A|C})\right]^\mu,
\end{aligned}$$

where $0 \le \mu \le 1$, see Fig. 2.

Similar to the improvement from the inequality (9) to the inequality (16), we can also improve the polygamy inequality in Theorem 4. The proof is similar to that of Theorem 2.

Theorem 5 For $0 \le \mu \le 1$, $0 < \alpha < 2$, $\alpha \ne 1$, and $0 < k \le 1$, we have for any multiqubit state $\rho_{AB_0\ldots B_{N-1}}$,

$$\left[E_\alpha^{\rm a}(\rho_{A|B_0\ldots B_{N-1}})\right]^\mu \le \sum_{j=0}^{N-1}\left(\frac{(1+k)^\mu-1}{k^\mu}\right)^j\left[E_\alpha^{\rm a}(\rho_{A|B_j})\right]^\mu, \tag{27}$$

if

$$kE_\alpha^{\mathrm{a}}(\rho_{A|B_i}) \geq \sum_{j=i+1}^{N-1} E_\alpha^{\mathrm{a}}(\rho_{A|B_j}) \tag{28}$$

for $i = 0, 1, \ldots, N-2$.

Since $\omega_{\mathrm{H}}(j) \leq j$, for $0 \leq \mu \leq 1$ we obtain

$$\begin{aligned}\left[E_\alpha^{\mathrm{a}}(\rho_{A|B_0\ldots B_{N-1}})\right]^\mu &\leq \sum_{j=0}^{N-1}\left(\frac{(1+k)^\mu - 1}{k^\mu}\right)^j \left(E_\alpha^{\mathrm{a}}\left(\rho_{A|B_j}\right)\right)^\mu \\ &\leq \sum_{j=0}^{N-1}\left(\frac{(1+k)^\mu - 1}{k^\mu}\right)^{\omega_{\mathrm{H}}(j)} \left[E_\alpha^{\mathrm{a}}(\rho_{A|B_j})\right]^\mu.\end{aligned}$$

Therefore, for any multiqubit state $\rho_{AB_0\ldots B_{N-1}}$ satisfying the condition (28), the inequality (27) of Theorem 5 is tighter than the inequality (23) of Theorem 4.

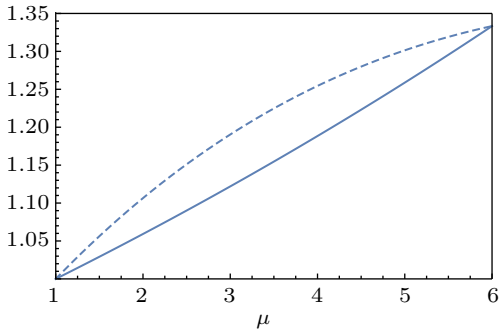


Fig. 2. Rényi-α entanglement y with respect to μ: the solid line is for y_3 and the dashed line for y_4 from the result in Ref. [29].

5. Conclusion

Quantum entanglement is the essential resource in quantum information. The monogamy and polygamy relations characterize the entanglement distributions in the multipartite systems. Tighter monogamy and polygamy inequalities give finer characterization of the entanglement distribution. In this article, by using the Hamming weights of binary vectors we have proposed a class of monogamy inequalities related to the μ-th power of the entanglement measure based on Rényi-α entropy, polygamy relations in terms of the μ-th powered of RαEoA for $0 \leq \mu \leq 1$. These new monogamy and polygamy relations are shown to be tighter than the existing ones. Moreover, it has been shown that our monogamy inequality is effective for the counterexamples of the CKW monogamy inequality in higher-dimensional systems. Our results may motivate further investigations on the entanglement distribution in multipartite systems.

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