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Exact analytical results for a two-level quantum system under a Lorentzian-shaped pulse field*

Qiong-Tao Xie(谢琼涛)[1,†] and Xiao-Liang Liu(刘小良)[2]

[1] College of Physics and Electronic Engineering, Hainan Normal University, Haikou 571158, China
[2] School of Physics and Electronics, Central South University, Changsha 410083, China

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We investigate a two-level quantum system driven by a Lorentzian-shaped pulse field. An analytical solution is presented in terms of the confluent Heun functions. It is shown that for specially chosen parameter conditions, there are a number of the exact analytical solutions in an explicit form. The dependence of the final transition probabilities in the two levels on the system parameters is derived analytically and confirmed numerically.

Keywords: exact solution, Lorentzian-shaped pulse, two-level system

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1. Introduction

During the past few decades, two-level quantum systems driven by various laser pulses have attracted extensive interest of research.[1–6] It has been shown that these systems not only exhibit many interesting physical phenomena, but also have important applications in quantum information and quantum simulation. Naturally, an exact analytical solution is of fundamental importance for understanding these phenomena and applications. In most of the previous works, three typical pulse shapes have been studied theoretically and experimentally, i.e., the hyperbolic secant, the Lorentzian, and the Gaussian shapes.[1–6] For the hyperbolic secant pulse, the exact analytical solution has been found.[7] The exact analytical solutions for the latter two typical pulses are still lacking. It is thus an important problem how to find the exact analytical solutions for the two typical pulses.

In recent years, Heun equation and its confluent forms have been used to construct the analytical solutions for a number of specific pulse shapes.[8–13] The analytical solutions to these pulses are given by the Heun-related special functions defined by infinite series. These equations contain the well-known hypergeometric and confluent hypergeometric equations as particular cases which have widely been applied to construct the exact analytical solutions for various types of pulse shapes.[14–21] Different from the hypergeometric and confluent hypergeometric functions, the coefficients in these Heun-related special functions obey a three-term recurrence relation, and their asymptotic behavior in different parameter regions is not well-known.[22,23] The final transition probabilities in the two levels cannot be obtained in an explicit form.[8–13] For a general parameter condition, these analytical solutions cannot provide a good understanding of the properties of the systems.

In the present work, our main aim is to give certain exact analytical results for the well-known model where a two-level quantum system is driven by a Lorentzian-shaped pulse field. Two main findings are presented. Firstly, it is shown that this model is related to the confluent Heun equation (CHE), and its analytical solution is given in terms of the confluent Heun functions (CHFs). Secondly, it is found that for some specially chosen system parameters, the CHFs become finite series. This allows us to find an infinite number of exact analytical solutions. The dependence of the final transition probability on the system parameters is derived analytically. In addition, we show that the exact analytical results can be valid in some non-Hermitian situations.

2. Model and exact analytical solutions

We begin with a physical model which describes the interaction between an atom and a pulse electric field. In the dipole approximation and the radiation gauge, this model is described by the following Hamiltonian:[24]

$$H = H_{\rm atom} - \boldsymbol{d}\cdot\boldsymbol{E}(t), \tag{1}$$

where $H_{\rm atom}$ is the free-atom Hamiltonian, $\boldsymbol{d}$ is the atomic dipole moment operator, and the electric field $\boldsymbol{E}(t)$ takes the form

$$\boldsymbol{E}(t) = E(t)\hat{\epsilon} = E_0 F_{\rm e}(t)\cos(\omega t)\hat{\epsilon}, \tag{2}$$

with peak magnitude E_0, polarization $\hat{\epsilon}$, and envelope function $F_{\rm e}(t)$. The atomic Hamiltonian has a set of eigenstates $|n\rangle$ associated with the eigenvalues $\hbar\omega_n$, $H_{\rm am}|n\rangle = \hbar\omega_n$ $(n = 1,2,3,\ldots)$. The pulse electric field is chosen suitably to drive the transition between an initial state $|i\rangle$ and a final state $|f\rangle$. The single-photon transition between the two states is not

*Project supported by the Natural Science Foundation of Hainan Province, China (Grant No. 2019RC179).
†Corresponding author. E-mail: qiongtaoxie@yahoo.com
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dipole allowed. They are coupled to the intermediate states $|k\rangle$ which are far detuned from the single-photon resonance. In experiments, the two states are chosen, so that they have a longer lifetime.[25] Here for brevity, we take $i = 1$ and $j = 2$. The energy separation between the two energy levels is $\Delta_0 = \hbar(\omega_2 - \omega_1)$. In the following, we shall show that under certain conditions, this model can be simplified into a two-level model. After expanding the state vector as

$$|\Psi\rangle = \sum_n c_n \mathrm{e}^{-\mathrm{i}\omega_n t}|n\rangle, \tag{3}$$

we obtain

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}c_n = \sum_k c_n \mathrm{e}^{\mathrm{i}\omega_{kn}t}\langle k|\boldsymbol{d}\cdot\hat{\epsilon}|n\rangle E(t), \tag{4}$$

where $\omega_{kn} = \omega_k - \omega_n$ is the frequency separation between the two states $|k\rangle$ and $|n\rangle$. We consider the case where there is no diploe coupling between the intermediate sates, and at once obtain

$$\mathrm{i}\frac{\mathrm{d}c_1}{\mathrm{d}t} = -F_\mathrm{e}(t)\sum_k c_k\Omega_{1k}\mathrm{e}^{\mathrm{i}\omega_{1k}t}, \tag{5}$$

$$\mathrm{i}\frac{\mathrm{d}c_2}{\mathrm{d}t} = -F_\mathrm{e}(t)\sum_k c_k\Omega_{2k}\mathrm{e}^{\mathrm{i}\omega_{2k}t}, \tag{6}$$

$$\mathrm{i}\frac{\mathrm{d}c_k}{\mathrm{d}t} = -F_\mathrm{e}(t)(c_1\Omega_{k1}\mathrm{e}^{\mathrm{i}\omega_{k1}t} + c_2\Omega_{k2}\mathrm{e}^{\mathrm{i}\omega_{k2}t}), \tag{7}$$

where $\Omega_{kn} = -d_{kn}E_0/\hbar$ is the Rabi frequencies, and d_{kn} is the electric diploe matric elements, $d_{kn} = \langle k|\boldsymbol{d}\cdot\hat{\epsilon}|n\rangle$. Since all the intermediate states are far detuned from the single-photon resonance, they can be eliminated adiabatically, and thus the system can be approximated as a two-level system.[25] In addition, under the condition of $\Delta_0 \ll \omega$, we apply the rotating-wave approximation by dropping all the oscillating terms with the frequencies $2\omega \pm \Delta_0$. This results in an effective two-level quantum system

$$\mathrm{i}\frac{\mathrm{d}c_1}{\mathrm{d}t} = \Delta_1 G(t)c_1 - \Omega^*\mathrm{e}^{-\mathrm{i}\Delta_0 t}G(t)c_2/2, \tag{8}$$

$$\mathrm{i}\frac{\mathrm{d}c_2}{\mathrm{d}t} = \Delta_2 G(t)c_2 - \Omega\,\mathrm{e}^{\mathrm{i}\Delta_0 t}G(t)c_1/2. \tag{9}$$

For a detailed derivation, the reader is referred to Appendix A in Ref. [25]. Here $G(t) = F_\mathrm{e}^2(t)$ is the profile of the pulse, Δ_p is the ac Stark shift of state p,

$$\Delta_p = -1/2\sum_k \Omega_{pk}\Omega_{kp}\frac{\omega_{kp}}{\omega_{kp}^2 - \omega^2}, \tag{10}$$

and Ω is the two-photon Rabi frequency

$$\Omega = \sum_k \Omega_{2k}\Omega_{k1}\frac{\omega_{k1}}{\omega_{k1}^2 - \omega^2}. \tag{11}$$

If one further takes the following transforms:

$$c_{1,2} = \mathrm{e}^{-\mathrm{i}\theta_{1,2}(t)}a_{1,2}, \tag{12}$$

with

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \frac{1}{2}(\Delta_0 + (\Delta_1 + \Delta_2)G(t)), \tag{13}$$

$$\frac{\mathrm{d}\theta_2}{\mathrm{d}t} = -\frac{1}{2}(\Delta_0 - (\Delta_1 + \Delta_2)G(t)), \tag{14}$$

one has

$$\mathrm{i}\frac{\mathrm{d}a_1}{\mathrm{d}t} = \frac{\nu(t)}{2}a_2 + \frac{f(t)}{2}a_1, \tag{15}$$

$$\mathrm{i}\frac{\mathrm{d}a_2}{\mathrm{d}t} = \frac{\nu(t)}{2}a_1 - \frac{f(t)}{2}a_2, \tag{16}$$

where $f(t)$ and $\nu(t)$ have the form

$$f(t) = f_0 + f_1 G(t), \quad \nu(t) = \nu_1 G(t), \tag{17}$$

with

$$f_0 = -\Delta_0, \quad f_1 = \Delta_1 - \Delta_2, \quad \nu_1 = -\Omega. \tag{18}$$

Here for simplicity, we have assumed that the Rabi frequency Ω is real. For the Lorentzian-shaped pulse field, the intensity profile $G(t)$ is given as[25,26]

$$G(t) = \frac{1}{1 + g^2t^2}, \tag{19}$$

where the parameter g is related to the width scale τ, $g = \pi/\tau$. By eliminating $a_2(t)$ and $a_1(t)$, we obtain two second-order differential equations for $a_1(t)$ and $a_2(t)$

$$\frac{\mathrm{d}^2a_1}{\mathrm{d}t^2} - \frac{\dot{\nu}}{\nu}\frac{\mathrm{d}a_1}{\mathrm{d}t} + \Big(\frac{f^2}{4} + \frac{\nu^2}{4} + \mathrm{i}\frac{\dot{f}}{2} - \mathrm{i}\frac{f}{2}\frac{\dot{\nu}}{\nu}\Big)a_1 = 0, \tag{20}$$

$$\frac{\mathrm{d}^2a_2}{\mathrm{d}t^2} - \frac{\dot{\nu}}{\nu}\frac{\mathrm{d}a_2}{\mathrm{d}t} + \Big(\frac{f^2}{4} + \frac{\nu^2}{4} - \mathrm{i}\frac{\dot{f}}{2} + \mathrm{i}\frac{f}{2}\frac{\dot{\nu}}{\nu}\Big)a_2 = 0, \tag{21}$$

where the dot denotes the derivative with respect to t, $\dot{f} = \mathrm{d}f/\mathrm{d}t$ and $\dot{\nu} = \mathrm{d}\nu/\mathrm{d}t$. It follows from the above two equations that if equation (20) has a solution of $\phi(t)$, equation (21) has a solution of $\phi^*(t)$. This property will simplify our later discussions. Once the analytical solution to $a_1(t)$ is obtained, $a_2(t)$ is obtained from Eq. (15). With an appropriate variable transform, equation (15) is reduced to the CHE, and the analytical solution to $a_1(t)$ is expressed in terms of the CHFs (see details in Appendix A). Although the asymptotic behavior of the CHFs in different parameter ranges is not clear, the CHFs can become finite series under some special conditions.[22,23] This gives an infinite number of the exact analytical solutions to $a_{1,2}$ in terms of elementary functions. In the present work, we mainly focus on these exact analytical solutions.

It is found that if the parameters $f_{0,1}$, ν_1, and g satisfy certain specific parameter relations (see details in Appendix A)

$$f_1^2 + \nu_1^2 = 4(N+2)^2g^2, \quad F_N(f_0, f_1, g) = 0, \tag{22}$$

the exact analytical solutions of $a_{1,2}(t)$ can be given in an explicit form. Here $N \geq 0$ is an integer, and $F_N(f_0, f_1, g)$ is a function of the parameters $f_{0,1}$ and g. Since the expression of

$F_N(f_0,f_1,g)$ is very cumbersome for $N>2$, here we only list the three cases of $N=0,1,2$ as examples

$$F_0(f_0,f_1,g)=f_0f_1+4g^2, \tag{23}$$

$$F_1(f_0,f_1,g)=(f_0f_1+4g^2)(f_0f_1+12g^2)-4f_0^2g^2, \tag{24}$$

$$F_2(f_0,f_1,g)=(f_0f_1+4g^2)(f_0f_1+12g^2)(f_0f_1+24g^2) -16f_0^2g^2(f_0f_1+16g^2). \tag{25}$$

With the exact analytical solutions in hand, we can obtain the final transition probability at once, $P_1^{\rm f}=|a_1(+\infty)|^2$ ($P_2^{\rm f}=1-P_1^{\rm f}$). For simplicity, it is assumed that the system is initially in the first level, $a_1(t_0)=1$ and $a_2(t_0)=0$. In the case of $N=0$, the final transition probability is given as

$$P_1^{\rm f}=\left(1-\frac{f_1^2}{8(g^2+g^4t_0^2)}\right)^2+\frac{f_1^2t_0^2}{4(1+g^2t_0^2)^2}. \tag{26}$$

In this situation, for given parameters of g and f_1/g, we have $f_0/g=-4g/f_1$, and $\nu_1/g=\sqrt{16-f_1^2/g^2}$ from the parameter relations (22) and (23). Here without loss of generality, we may take $f_1>0$ and $\nu_1>0$. For a real coupling strength $\nu_1>0$, we require $f_1/g<4$. Now we discuss some properties of the final transition probability $P_1^{\rm f}$. Firstly, it is found that for arbitrarily chosen values of f_1 and g, we have $P_1^{\rm f}=1$ for $t_0=-\infty$. This indicates that if the system is initially in the first state, it remains in this state after the duration of the pulse. Secondly, from the condition $\mathrm{d}P_1^{\rm f}/\mathrm{d}f_1=0$, it is found that at $f_1=2\sqrt{2}g$, $P_1^{\rm f}$ reaches its minimal value

$$P_{1,\min}^{\rm f}=1-\frac{1}{(1+g^2t_0^2)^2}, \tag{27}$$

which depends on the choice of the initial dimensionless time gt_0. It is evident that we have $P_{1,\min}^{\rm f}=0$ for $t_0=0$. This result indicates that a complete population inversion occurs at $t_0=0$, and the choice of $t_0\neq 0$ leads to a smaller population inversion. The above analytical results are demonstrated in Fig. 1 where we plot $P_1(t)$ and $P_1^{\rm f}$ for four different initial time t_0. Since the parameters f_0, f_1, ν_1, and g have the same units of Hz, the ratios between these parameters are thus essential. Here we fix $g=1$ and $f_1/g=2\sqrt{2}$. So we have $f_0/g=-4g/f_1=-\sqrt{2}$ and $\nu_1/g=\sqrt{16-f_1^2/g^2}=2\sqrt{2}$. In the case of $N=1$, we focus on the situation with $f_1=0$, and obtain the final transition probability

$$P_1^{\rm f}=\frac{(3gt_0+2g^3t_0^3)^2}{4(1+g^2t_0^2)^3}+\frac{3}{4(1+g^2t_0^2)^3}. \tag{28}$$

In this case, for a given parameter of g, we have $f_0/g=2\sqrt{3}$ and $\nu_1/g=6$ from the parameter relations (22) and (24). The final transition probability $P_1^{\rm f}$ only depends on the dimensionless time gt_0. Similarly, it is found that we have $P_1^{\rm f}=1$ for $t_0=-\infty$. For $t_0=0$, the minimal transition probability is achieved, $P_{1,\min}^{\rm f}=3/4$. These analytical results are shown in Fig. 2.

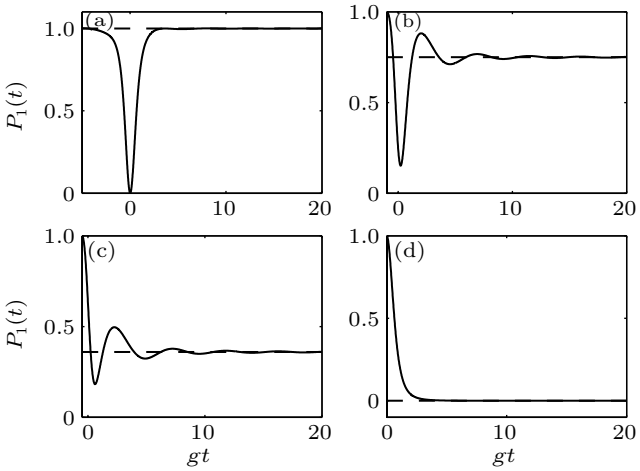


Fig. 1. Transition probability $P_1(t)=|a_1(t)|^2$ for (a) $gt_0=-5$, (b) $gt_0=-1$, (c) $gt_0=-0.5$, and (d) $gt_0=0$. The parameters are given as $g=1$, $f_1/g=2\sqrt{2}$, $f_0/g=-4g/f_1=-\sqrt{2}$, and $\nu_1/g=\sqrt{16-f_1^2/g^2}=2\sqrt{2}$. The initial conditions are $a_1(t_0)=1$ and $a_2(t_0)=0$. The dashed lines are the exact results of the final transition probability $P_1^{\rm f}$.

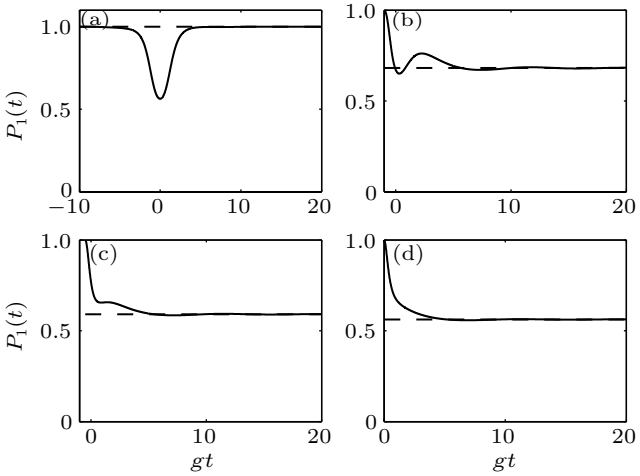


Fig. 2. Transition probability $P_1(t)=|a_1(t)|^2$ for (a) $gt_0=-5$, (b) $gt_0=-1$, (c) $gt_0=-0.5$, and (d) $gt_0=0$ with $f_1/g=0$. The parameters are given as $g=1$, $f_0/g=2\sqrt{3}$, and $\nu_1/g=6$. The initial conditions are $a_1(t_0)=1$ and $a_2(t_0)=0$. The dashed lines are the exact results of the final transition probability $P_1^{\rm f}$.

Finally, we show that our exact analytical results are also applicable for the non-Hermitian situation. For example, the final transition probability $P_1^{\rm f}$ in Eq. (26) is valid in the case of $f_1/g>4$ where the coupling strength ν_1 becomes imaginary. Now the imaginary coupling strength has been realized in different systems.[27–33] In the resulting non-Hermitian system, for the initial conditions of $a_1(t_0)=1$ and $a_2(t_0)=0$, the conserved quantity becomes

$$|a_1(t)|^2-|a_2(t)|^2=1, \tag{29}$$

instead of $|a_1(t)|^2+|a_2(t)|^2=1$ in the Hermitian situation. In this situation, we introduce the normalized transition probability defined by $P_{1,n}(t)=P_1(t)/(P_1(t)+P_2(t))$, so that the normalized final transition probability is given by $P_{1,n}^{\rm f}=P_1^{\rm f}/(2P_1^{\rm f}-1)$. It is found that $P_{1,n}^{\rm f}=1$ for $t_0=-\infty$, and it

takes a minimal value at $t_0 = 0$. In Fig. 3, we plot the normalized transition probability in the case where we take $g = 1$ and $f_1/g = 5$. This leads to $f_0/g = -4g/f_1 = -4/5$, and $\nu_1/g = \sqrt{16 - f_1^2/g^2} = \mathrm{i}\sqrt{3}$. It is observed that our exact analytical results are also applicable for the non-Hermitian situation.

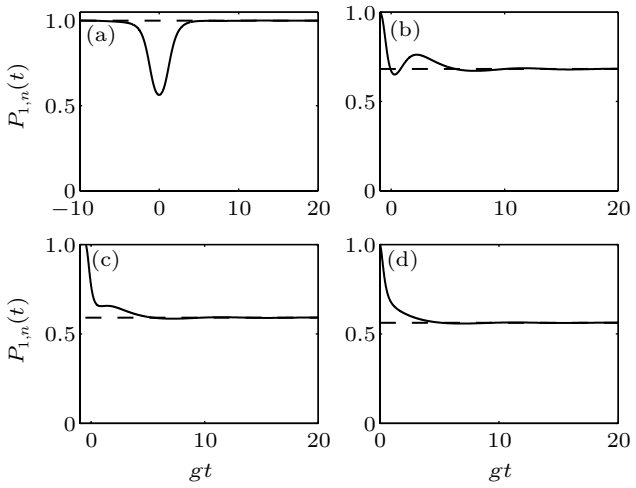


Fig. 3. Normalized transition probability $P_{1,n}(t) = |a_1(t)|^2/(|a_1(t)|^2 + |a_2(t)|^2)$ for (a) $gt_0 = -10$, (b) $gt_0 = -1$, (c) $gt_0 = -0.5$, and (d) $gt_0 = 0$. The parameters are given as $g = 1$, $f_1/g = 5$, $f_0/g = -4g/f_1 = -4/5$, and $\nu_1/g = \sqrt{16 - f_1^2/g^2} = \mathrm{i}\sqrt{3}$. The initial conditions are $a_1(t_0) = 1$ and $a_2(t_0) = 0$. The dashed lines are the exact results of the normalized final transition probability $P_{1,n}^{\mathrm{f}}$.

3. Conclusion

We have investigated the two-level quantum system interacting with the Lorentzian-shaped pulse. The analytical solution is presented in terms of the CHFs. It is shown that under certain special parameter conditions, the CHFs become finite series. This allows us to obtain certain exact analytical solutions in an explicit form. We also derived the parametric dependence of the final transition probability analytically. It is shown that the choice of the initial time affects the final transition probability strongly. In addition, it is found that our exact analytical results are also applicable to the non-Hermitian situation where the coupling strength becomes imaginary.

Appendix A: Analytical solutions in terms of CHFs

We present the analytical solution in terms of CHFs. We first make the transformations

$$z = \frac{1}{2}(1 - \mathrm{i}gt),$$
$$a_1(z) = \mathrm{e}^{\mathrm{i}\lambda_1 t}(1 - \mathrm{i}gt)^{\lambda_2}(1 + \mathrm{i}gt)^{\lambda_3}\phi(z), \tag{A1}$$

with

$$\lambda_1 = \frac{f_0}{2}, \quad \lambda_2 = \lambda_3 = -\frac{\sqrt{f_1^2 + \nu_1^2}}{4g}, \tag{A2}$$

and then obtain the confluent Heun equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} + \left(\alpha + \frac{\beta+1}{z} + \frac{\gamma+1}{z-1}\right)\frac{\mathrm{d}\phi}{\mathrm{d}z} + \frac{pz+q}{z(z-1)}\phi = 0, \tag{A3}$$

where $p = \delta + \alpha(\beta+\gamma+2)/2$ and $q = \eta + \beta/2 + (\gamma-\alpha)(\beta+1)/2$. The parameters α, β, γ, δ, and η read $\alpha = -2f_0/g$, $\beta = \gamma = -\sqrt{f_1^2 + \nu_1^2}/2g$, $\delta = -2f_0/g$, and $\eta = (f_0 g + f_0 f_1/2 + f_1^2/8 + \nu_1^2/8)/g^2$. The CHE has a local Frobenius solution around $z = 0$ of the form

$$\phi(z) = \mathrm{HC}(\alpha,\beta,\gamma,\delta,\eta,z) = \sum_{n=0}^{\infty} h_n z^n, \tag{A4}$$

where $\mathrm{HC}(\alpha,\beta,\gamma,\delta,\eta,z)$ is the CHF. The coefficients h_n in the CHF are defined by the three-term recurrence relation, $R_n h_n = P_{n-1}h_{n-1} + Q_{n-1}h_{n-2}$ $(n \geq 1)$ with the initial conditions $h_0 = 1$ and $h_{-1} = 0$. Here $R_n = 1 + \beta/n$, $P_n = 1 + (\beta+\gamma-\alpha-1)/n + (\eta - \beta/2 + (\gamma-\alpha)(\beta+1)/2)/n^2$, and $Q_n = (\delta + \alpha(\beta+\gamma)/2 + \alpha(n-1))/n^2$. Mathematically, the radius of convergence of the CHF is $|z| < 1$. It follows from $|z| = \frac{1}{2}(\sqrt{1+g^2t^2})$ that the solution $\phi(t)$ is not valid for $|t| > \sqrt{3}/g$. However, there exist necessary conditions

$$\delta = -\left(N + \frac{\gamma+\beta+2}{2}\right)\alpha, \tag{A5}$$
$$h_{N+1} = 0, \tag{A6}$$

under which the CHF $\mathrm{HC}(\alpha,\beta,\gamma,\delta,\eta,z)$ can be terminated as a polynomial of degree N for z. By applying the conditions (A5) in $\phi(t)$, we obtain the first parameter relation

$$f_1^2 + \nu_1^2 = 4(N+2)^2 g^2. \tag{A7}$$

From the condition (A6), we can obtain the second parameter relation $F_N(f_0, f_1, g) = 0$. Here we list the three simple cases of $N = 0, 1, 2$ as examples

$$F_0(f_0,f_1,g) = f_0 f_1 + 4g^2, \tag{A8}$$
$$F_1(f_0,f_1,g) = (f_0 f_1 + 4g^2)(f_0 f_1 + 12g^2) - 4f_0^2 g^2, \tag{A9}$$
$$F_2(f_0,f_1,g) = (f_0 f_1 + 4g^2)(f_0 f_1 + 12g^2)(f_0 f_1 + 24g^2) - 16 f_0^2 g^2 (f_0 f_1 + 16g^2). \tag{A10}$$

In such a way, we obtain an analytical solution in terms of elementary functions

$$\psi_N^1(t) = \frac{\mathrm{e}^{\mathrm{i}\lambda_1 t}}{(1+g^2t^2)^{N/2+1}} \sum_{n=0}^{N} \frac{h_n}{2^n}(1 - \mathrm{i}gt)^n. \tag{A11}$$

Another linearly independent analytical solution can be constructed in the following way:

$$\psi_N^2(t) = \psi_N^1(t) \int \frac{\mathrm{e}^{\kappa(t)}}{(\psi_N^1(t))^2}\,\mathrm{d}t, \tag{A12}$$

where $\kappa(t) = \int (\dot{\nu}(t)/\nu(t))\mathrm{d}t = -\ln(1+g^2t^2)$. The general solution to $a_1(t)$ is given as

$$a_1(t) = c_1\psi_N^1(t) + c_2\psi_N^2(t), \tag{A13}$$

where $c_{1,2}$ are two constants determined by the initial states. For simplicity, we assume that the initial states are $a_1(t_0) = 1$

and $a_2(t_0)=0$, and then obtain

$$c_1=\frac{\dot{\psi}_N^2(t_0)+\mathrm{i}f(t_0)\psi_N^2(t_0)/2}{\psi_N^1(t_0)\dot{\psi}_N^2(t_0)-\psi_N^2(t_0)\dot{\psi}_N^1(t_0)}, \tag{A14}$$

$$c_2=-\frac{\dot{\psi}_N^1(t_0)+\mathrm{i}f(t_0)\psi_N^1(t_0)/2}{\psi_N^1(t_0)\dot{\psi}_N^2(t_0)-\psi_N^2(t_0)\dot{\psi}_N^1(t_0)}. \tag{A15}$$

Here $\psi_N^1(t)\dot{\psi}_N^2(t)-\psi_N^2(t)\dot{\psi}_N^1(t)=1/(1+g^2t^2)$ is the Wronskian of the two analytical solutions. Therefore, one can obtain an infinite number of the exact analytical solutions of $a_{1,2}$ in this way in principle.

Now we apply the exact analytical solutions to give the final transition probability $P_1^{\mathrm{f}}=|a_1(+\infty)|^2$. It is found that as $t\to+\infty$, we have $\psi_N^1(t)\to 0$. So the final transition probability P_1^{f} is given as

$$P_1^{\mathrm{f}}=|c_2|^2|\lim_{t\to+\infty}\psi_N^2(t)|^2=\left|\frac{2^N c_2}{\lambda_1 h_N}\right|^2. \tag{A16}$$

In deriving the last expression in the final transition probability, we have applied the L'Hôpital rule

$$\lim_{t\to+\infty}\psi_N^2(t)=-\lim_{t\to+\infty}\frac{1}{\dot{\psi}_N^1(t)(1+g^2t^2)}=\frac{\mathrm{i}2^N}{(-\mathrm{i})^N\lambda_1 h_N}. \tag{A17}$$

This means that the final transition probability is determined by the exact analytical solution $\psi_N^1(t)$ completely. In the following, we present some analytical results for the final transition probability in the two simple cases of $N=0,1$ as examples.

In the case of $N=0$, we have the two linearly independent solutions

$$\psi_0^1(t)=\frac{\mathrm{e}^{-\mathrm{i}\frac{2g^2t}{f_1}}}{1+g^2t^2}, \tag{A18}$$

$$\psi_0^2(t)=\frac{\mathrm{i}f_1\,\mathrm{e}^{\mathrm{i}\frac{2g^2t}{f_1}}}{32(1+g^2t^2)g^4}(f_1^2-8g^2-4\mathrm{i}f_1g^2t-8g^4t^2). \tag{A19}$$

The final transition probability takes the simple form

$$P_1^{\mathrm{f}}=|a_1(+\infty)|^2=\left(1-\frac{f_1^2}{8(g^2+g^4t_0^2)}\right)^2+\frac{f_1^2t_0^2}{4(1+g^2t_0^2)^2}. \tag{A20}$$

Here without loss of generality, we take $f_1>0$ and $\nu_1>0$. For a real coupling strength $\nu_1=\sqrt{16g^2-f_1^2}$, we require $f_1<4g$. But the final transition probability is also valid for an imaginary coupling strength under the condition of $f_1>4g$.

In the case of $N=1$, the explicit expressions for the final transition probability are more complex. In the situation of $f_1=0$, the final transition probability takes the simple form

$$P_1^{\mathrm{f}}=\frac{(3gt_0+2g^3t_0^3)^2}{4(1+g^2t_0^2)^3}+\frac{3}{4(1+g^2t_0^2)^3}, \tag{A21}$$

with $f_0=2\sqrt{3}g$ and $\nu_1=6g$.

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