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Quasi-canonicalization for linear homogeneous nonholonomic systems*

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For conservative linear homogeneous nonholonomic systems, there exists a cotangent bundle with the symplectic structure $\mathrm{d}\pi^\mu \wedge \mathrm{d}\xi_\mu$, in which the motion equations of the system can be written into the form of the canonical equations by the set of quasi-coordinates π^μ and quasi-momenta ξ_μ. The key to construct this cotangent bundle is to define a set of suitable quasi-coordinates π^μ by a first-order linear mapping, so that the reduced configuration space of the system is a Riemann space with no torsion. The Hamilton–Jacobi method for linear homogeneous nonholonomic systems is studied as an application of the quasi-canonicalization. The Hamilton–Jacobi method can be applied not only to Chaplygin nonholonomic systems, but also to non-Chaplygin nonholonomic systems. Two examples are given to illustrate the effectiveness of the quasi-canonicalization and the Hamilton–Jacobi method.

Keywords: quasi-canonicalization, nonholonomic systems, first-order linear mapping, Hamilton–Jacobi method

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1. Introduction

The linear differential constrained system[1–3] is an important research object of analytical mechanics.[4–11] And linear differential constraints exist widely in many engineering problems, such as robots, bicycles, skateboards, electromechanical coupling systems, and so on. Many of these linear differential constraints are homogeneous.[12–24]

Linear differential constrained systems include holonomic systems and linear nonholonomic systems. When all differential constraints are integrable, the linear differential constrained system is called a holonomic system, and can be reduced into a geometric constrained system. When some differential constraints are not integrable, the linear differential constrained system is a nonholonomic system. In classical analytical mechanics, the correct motion equations of linear differential constrained systems can be written by the multiplier method. But because the motion equations with multipliers is a differential-algebraic equations, it is usually difficult to solve directly.

However the motion problem of conservative holonomic systems has been perfectly solved. In this process, the canonicalization of holonomic systems is a key. By introducing generalized coordinates q^μ, the motion equations of conservative holonomic systems can be firstly reduced to the Lagrange equation without multiplier

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^\nu}\right) - \frac{\partial L}{\partial q^\nu} = 0. \tag{1}$$

Then, the canonicalization of holonomic systems is realized by the Legendre transformation and introducing generalized momentum $p_\mu = \partial L/\partial \dot{q}^\mu$. That is, the motion equations can be written into the canonical Hamilton's equations

$$\dot{q}^\mu = \frac{\partial H}{\partial p_\mu}, \quad \dot{p}_\mu = -\frac{\partial H}{\partial q^\mu}. \tag{2}$$

On the basis of the canonicalization, some important methods to solve motion differential equations (such as the Hamilton–Jacobi method[25–32] and the symplectic algorithm[33]) have been proposed.

From the geometric point of view, the canonicalization shows that a holonomic system has a natural symplectic structure.[27–34] If the symplectic form on the cotangent bundle of a holonomic system is recorded as $\Omega = \mathrm{d}q^\mu \wedge \mathrm{d}p_\mu$, the Hamilton's equations (2) can be written as

$$\mathrm{i}_X \Omega = \mathrm{d}H. \tag{3}$$

Many important studies of conservative holonomic systems depend on this symplectic structure. For example, the Hamilton–Jacobi method for conservative holonomic systems is closely related to the Lagrange Submanifold in the cotangent bundle; The key of the symplectic algorithm is that the

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algorithm can keep the symplectic structure of Hamiltonian systems.

It is generally believed that nonholonomic systems have not the natural symplectic structures unless the Helmholtz condition is satisfied. For example, if the nonholonomic constraint of a dynamical system is

$$f(q^\mu, \dot{q}^\mu) = 0, \tag{4}$$

the motion equations of the system can be written as the following Hamilton form by the multiplier method

$$\dot{q}^\mu = \frac{\partial H}{\partial p_\mu}, \quad \dot{p}_\mu = -\frac{\partial H}{\partial q^\mu} + \lambda \frac{\partial f}{\partial \dot{q}^\mu}. \tag{5}$$

In general, because the nonholonomic constraining force is not a conservative force, equations (5) cannot be transformed into the form of Eqs. (2). In other words, the canonicalization of the nonholonomic systems can not be generally achieved by generalized coordinates q^μ and generalized momenta p_μ. It is the fundamental reason why the Hamilton–Jacobi method and symplectic algorithms cannot be directly extended from holonomic systems to nonholonomic systems by generalized coordinates q^μ and generalized momenta p_μ.

Because quasi-velocities is more consistent with geometric characteristics of the nonholonomic constraint than generalized velocities, applying nonholonomic frame, which consists of independent quasi-velocities

$$\frac{\partial}{\partial \pi^\mu} = \psi_\mu^\nu \frac{\partial}{\partial q^\nu},$$

instead of generalized coordinate bases $\partial/\partial q^\mu$ is a more suitable choice for the study of nonholonomic systems. By introducing quasi-velocities, the motion equations of a nonholonomic system can be written as the Hamel equations[35–39]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \zeta^\mu}\right) - \psi_\mu^\nu \frac{\partial L}{\partial q^\nu} = c_{\nu\mu}^\sigma \frac{\partial L}{\partial \zeta^\sigma} \zeta^\nu, \tag{6}$$

where the quantities ζ^μ and the quantities $c_{\nu\mu}^\sigma$ are respectively defined by

$$\dot{q}^\mu \frac{\partial}{\partial q^\mu} = \zeta^\nu \frac{\partial}{\partial \pi^\nu}, \tag{7}$$

$$\left[\frac{\partial}{\partial \pi^\nu}, \frac{\partial}{\partial \pi^\mu}\right] = c_{\nu\mu}^\sigma \frac{\partial}{\partial \pi^\sigma}. \tag{8}$$

Although the nonholonomic constraining force disappears from the Hamel equations, the right side of Eqs. (6) is still not zero. So in general, the Hamel equations cannot be written as the form of Eqs. (2) directly through a Legendre transformation.

In this paper, by the first-order linear nonholonomic mapping theory,[40–43] we will give a method to realize the quasi-canonicalization of linear homogeneous nonholonomic systems. That is, by introducing a set of suitable quasi-coordinates π^μ and quasi-momenta ξ_μ, the motion equations of conservative linear homogeneous nonholonomic systems can be written as

$$\dot{\pi}^\mu = \frac{\partial H}{\partial \xi_\mu}, \quad \dot{\xi}_\mu = -\frac{\partial H}{\partial \pi^\mu}. \tag{9}$$

From the geometrical point of view, it means that a special symplectic structure $\mathrm{d}\pi^\mu \wedge \mathrm{d}\xi_\mu$ can be defined for linear homogeneous nonholonomic systems. The symplectic structure $\mathrm{d}\pi^\mu \wedge \mathrm{d}\xi_\mu$ is given in the third section of this paper.

In this paper, we call the canonicalization of nonholonomic systems quasi-canonicalization. This is to distinguish it from the canonicalization of holonomic systems, and to emphasize that the canonicalization of nonholonomic systems depends on the introduction of suitable quasi-coordinates π^μ.

In the third section of this paper, it can be seen that the application scope of the quasi-canonicalization method is linear homogeneous differential constrained systems. Therefore, the quasi-canonicalization is actually an extension of the canonicalization of holonomic system. That is, we extend the canonicalization from holonimic systems to linear homogeneous differential constrained systems which include holonimic systems and linear homogeneous nonholonomic system. It is easy to prove that for holonomic systems, quasi-coordinates π^μ and the quasi-canonicalization can be reduced respectively to generalized coordinates q^μ and the canonicalization mentioned in the third paragraph above.

Obviously, the quasi-canonicalization provides a new way to extend the Hamilton–Jacobi method and symplectic algorithms from holonomic systems to nonholonomic systems. Due to the symmetry of Chaplygin nonholonomic systems, the Hamilton–Jacobi method of Chaplygin nonholonomic systems has been proposed based on the symmetry reduction.[44–46] But the quasi-canonicalization method does not depend on the symmetry of nonholonomic systems. So the Hamilton–Jacobi method based on the quasi-canonicalization can be applied not only to Chaplygin systems, but also to non-Chaplygin systems. An example of studying non-Chaplygin systems by the Hamilton–Jacobi method is provided in the fifth section of this paper.

The structure of this paper is organized as follows: the first-order linear nonholonomic mapping theory is briefly introduced in the second section of this paper. The quasi-canonicalization method for linear homogeneous differential constrained systems is proposed in the third section. In this section, we firstly study the motion equations of linear homogeneous differential constrained systems on Riemann–Cartan configuration spaces. On this basis, we point out that there are nonholonomic mappings, by which we can write the motion equations of linear homogeneous differential constrained sys-

tems as Eqs. (9) (we call Eqs. (9) the quasi-Hamilton equations in this paper). And we explain the special symplectic structure corresponding to the quasi-Hamilton equations at the end of the third section. In the fourth section of this paper, we study the Hamilton–Jacobi method for linear homogeneous nonholonomic systems as an application of the quasi-canonicalization. Finally, two examples (one is a Chaplygin system, and the other is a non-Chaplygin system) are given to illustrate the effectiveness of the quasi-canonicalization and the Hamilton–Jacobi method.

The Einstein's summation convention is used throughout in this paper, such as $i,j,k,l = 1,2,\ldots,n$; $\mu,\nu,\rho,\sigma = 1,2,\ldots,m$; $\alpha,\beta = 1,2,\ldots,n-m$. The same kind of geometric quantities are represented by the same letter symbol, and the same kind of geometric quantities in different spaces are distinguished by different subscripts and superscripts. For example, g_{ij} and $g_{\mu\nu}$ represent respectively the metric of the space M and the space Π in this paper.

2. The theory of first-order linear nonholonomic mappings

The idea of the first-order linear mapping method comes from the work of Kleinert *et al.*[47–51] In order to construct a Riemann–Cartan space with torsion, they proposed an embedding method by which a Riemann–Cartan space can be embeded into a known Euclidean space. Later, we extended the embedding method to the theory of first-order linear mappings.

Here we only briefly introduce the first-order linear mapping from flat spaces to Riemann–Cartan spaces. More about the first-order linear mapping theory can be seen in Refs. [52–55].

Take an n-dimensional flat space M with coordinates x^i. The metrics g_{ij} and the connections Γ^k_{ij} of M satisfy

$$g_{ij} = g_{ji}, \quad \Gamma^k_{ij} = 0. \tag{10}$$

If a full rank first-order linear mapping

$$\dot{x}^i = b^i_\mu \dot{\pi}^\mu \tag{11}$$

on M is not integrable, it is called a nonholonomic mapping. In this case, $\dot{\pi}^\mu$ is usually regarded as a symbol rather then the derivative of the variable π^μ respect to time. So π^μ and $\dot{\pi}^\mu$ are respectively called quasi-coordinates and quasi-velocities. If the mapping (11) is integrable, it is called a holonomic mapping. In this case, π^μ and $\dot{\pi}^\mu$ are reduced into generalized coordinates q^μ and generalized velocities $\dot{q}^\mu$ respectively.

The space with a set of quasi-coordinates π^μ is called the image space of the mapping (11), and it is recorded as Π in this paper. The metrics and the connections of the low dimensional space Π can be calculated by the mapping (11).

Proposition 1 The metrics and the connections of the space Π defined by the mapping (11) given respectively by

$$g_{\mu\nu} = g_{ij} b^i_\mu b^j_\nu, \tag{12}$$

$$\Gamma^\mu_{\nu\sigma} = a^\mu_i b^i_{\nu,\sigma}, \tag{13}$$

where

$$b^i_{\nu,\sigma} \equiv \frac{\partial b^i_\nu}{\partial \pi^\sigma} \equiv \frac{\partial b^i_\nu}{\partial x^j} b^j_\sigma,$$

$\left(a^\mu_i\right)$ is the right inverse of matrix $\left(b^i_\mu\right)$, and compatible with metric $g_{\mu\nu}$. The so-called "compatible with metric $g_{\mu\nu}$" means that a^μ_i satisfies

$$g^{\mu\nu} = g^{ij} a^\mu_i a^\nu_j, \tag{14}$$

where $\left(g^{\mu\nu}\right) = \left(g_{\mu\nu}\right)^{-1}$, and $\left(g^{ij}\right) = \left(g_{ij}\right)^{-1}$.

Proof The length of a vector $\left(\dot{\pi}^\mu\right)$ satisfies

$$g_{\mu\nu}\dot{\pi}^\mu \dot{\pi}^\nu = g_{ij}\dot{x}^i\dot{x}^j = g_{ij} b^i_\mu b^j_\nu \dot{\pi}^\mu \dot{\pi}^\nu. \tag{15}$$

So we can obtain

$$g_{\mu\nu} = g_{ij} b^i_\mu b^j_\nu. \tag{16}$$

This is the first conclusion of Proposition 1.

Utilizing Eqs. (11), (14), and (16), we can verify

$$a^\mu_i = g^{\mu\nu} g_{ij} b^j_\nu, \tag{17}$$

$$\dot{\pi}^\mu = a^\mu_i \dot{x}^i. \tag{18}$$

Substituting the mapping (11) into the compatibility condition

$$\frac{\mathrm{D}\dot{\pi}^\mu}{\mathrm{d}t} = \frac{\mathrm{d}\dot{\pi}^\mu}{\mathrm{d}t} + \Gamma^\mu_{\nu\sigma}\dot{\pi}^\nu\dot{\pi}^\sigma = a^\mu_i \frac{\mathrm{d}\dot{x}^i}{\mathrm{d}t}, \tag{19}$$

we can obtain

$$\Gamma^\mu_{\nu\sigma} = a^\mu_i b^i_{\nu,\sigma}. \tag{20}$$

This is the second conclusion of Proposition 1. □

It can be seen from Eq. (13) that the torsion $S^\mu_{\nu\sigma} = \frac{1}{2}\left(\Gamma^\mu_{\nu\sigma} - \Gamma^\mu_{\sigma\nu}\right)$ of Π is generally not equal to zero. It indicates that Π is generally a Riemann–Cartan space with torsion, except for the following cases.

Corollary 1 The torsion of Π is equal to zero on the condition

$$a^\mu_i \left(b^i_{\mu,\nu} - b^i_{\nu,\mu}\right) = 0. \tag{21}$$

Proof According to the definition of torsion $S^\mu_{\nu\sigma}$ and Eq. (13), the condition (21) can be obviously hold. □

Corollary 2 When the mapping (11) is a holonomic mapping, the torsion of Π is equal to zero.

Proof When the mapping (11) is integrable, $b^i_{\mu,\nu} = b^i_{\nu,\mu}$. According to the definition of torsion $S^\mu_{\nu\sigma}$, we can easily get $S^\mu_{\nu\sigma} = 0$. □

When the torsion of Π is zero, Π will reduce to a Riemann space. In this case, Π is a submanifold of M.

It can be seen that Corollary 2 is the reduction of Corollary 1. In this paper, when the mapping (11) is holonomic, we record the image space as Q. Since Q has a topological structure induced from M, it is an embedded submanifold with generalized coordinates q^μ in M. When the mapping (11) is a nonholonomic mapping satisfying condition (21), although the image space is also a Riemann space, it is not an embedded submanifold in M. So we still record the image space of a nonholonomic mapping satisfying condition (21) as Π in this paper.

3. Quasi-canonicalization for linear homogeneous nonholonomic systems

In this section, we study the motion equations of dynamical systems on Riemann–Cartan configuration spaces. On this basis, we can realize the quasi-canonicalization of linear homogeneous nonholonomic systems by the first-order linear nonholonomic mapping satisfying the condition (21).

First, let us briefly review some basic knowledge. When we study a dynamical system composed of N particles, the configuration space of the system is an $(n=3N)$-dimensional flat space M with coordinates x^i. the metrics of M are

$$g_{ij} = m_{\mathrm{int}((i-1)/3+1)}\delta_{ij}, \tag{22}$$

where m_i $(i=1,2,\ldots,N)$ is the mass of the i-th particle, int( ) is the integral function, and the same index on the right side does not mean summation. the connections of M are

$$\Gamma^i_{jk} = 0. \tag{23}$$

According to Newton's law, the motion equations of the system is

$$\frac{\mathrm{d}\dot{x}^i}{\mathrm{d}t} = g^{ij}f_j, \tag{24}$$

where f_i is the components of the force on the system.

If the system is subjected to $n-m$ first-order linear homogeneous differential constraints

$$e^\alpha_i \dot{x}^i = 0 \quad (\alpha = 1,2,\ldots,n-m), \tag{25}$$

f_j in Eq. (24) can be decomposed into the active force F_j and the constraining force R_j, *i.e.*,

$$f_j = F_j + R_j. \tag{26}$$

In addition,

$$R_j = C_\alpha e^\alpha_j, \tag{27}$$

where C_α are coefficients.

If an independent first-order linear differential equations are defined as

$$\dot{\pi}^\mu = \bar{a}^\mu_i \dot{x}^i,$$

$$e^\alpha_i \dot{x}^i = 0, \tag{28}$$

we can solve the first-order linear mapping

$$\dot{x}^i = b^i_\mu \dot{\pi}^\mu \tag{29}$$

that implies the constraints (25). Thus, according to Proposition 1, the mapping (29) maps a low-dimensional configuration space Π with quasi-coordinates $\dot{\pi}^\mu$ from the configuration space M. In general, Π is a Riemann–Cartan space with torsion.

Next, we study the various motion equations of constrained systems in the reduced Riemann–Cartan configuration space Π. Especially, we will provide the quasi-canonical equations for conservative linear homogeneous nonholonomic systems in Proposition 5 to realize the quasi-canonicalization of these nonholonomic systems.

Proposition 2 (Quasi-Newton's law) For a linear homogeneous differential constrained system, the motion equations on the reduced Riemann–Cartan configuration space is

$$\frac{\mathrm{d}\dot{\pi}^\mu}{\mathrm{d}t} = g^{\mu\nu}F_\nu, \tag{30}$$

where $F_\nu = b^j_\nu F_j$, and F_j is the components of the active force on the system.

Proof Substituting Eq. (29) into Eq. (24) yields

$$g_{ij}b^j_\nu\left(b^i_\mu\ddot{\pi}^\mu + b^i_{\mu,\sigma}\dot{\pi}^\mu\dot{\pi}^\sigma\right) = b^j_\nu f_j. \tag{31}$$

Utilizing Eqs. (12), (13), and (17), we can verify

$$b^j_\nu R_j = 0. \tag{32}$$

So

$$b^j_\nu f_j = b^j_\nu F_j = F_\nu. \tag{33}$$

Substituting Eq. (33) into Eq. (31) yields

$$g_{ij}b^j_\nu b^i_\mu\ddot{\pi}^\mu + g_{ij}b^j_\nu b^i_{\mu,\sigma}\dot{\pi}^\mu\dot{\pi}^\sigma = F_\nu. \tag{34}$$

Utilizing Eqs. (12), (13), and (17), we can simplify Eq. (34) to

$$\frac{\mathrm{D}\dot{\pi}^\mu}{\mathrm{d}t} = \ddot{\pi}^\mu + \Gamma'^\mu_{\nu\sigma}\dot{\pi}^\nu\dot{\pi}^\sigma = g^{\mu\nu}F_\nu. \tag{35}$$

□

Proposition 3 (Quasi-Lagrange equation) For a conservative linear homogeneous differential constrained system, the motion equations on the reduced Riemann-Cartan configuration space can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\pi}^\mu}\right) - \frac{\partial L}{\partial\pi^\mu} = 2S^\sigma_{\mu\nu}\frac{\partial L}{\partial\dot{\pi}^\sigma}\dot{\pi}^\nu, \tag{36}$$

where L is the Lagrangian function of the system,

$$\frac{\partial L}{\partial\pi^\mu} = b^i_\mu\frac{\partial L}{\partial x^i}.$$

Proof The kinetic energy of a linear homogeneous differential constrained system can be written as

$$T=\frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j=\frac{1}{2}g_{ij}b^i_\mu b^j_\nu\dot{\pi}^\mu\dot{\pi}^\nu=\frac{1}{2}g_{\mu\nu}\dot{\pi}^\mu\dot{\pi}^\nu. \tag{37}$$

We can verify

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\pi}^\mu}\right)&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\pi}^\mu}\right)-\Gamma^\sigma_{\mu\nu}\left(\frac{\partial T}{\partial\dot{\pi}^\sigma}\right)\dot{\pi}^\nu\\&=g_{\mu\nu}\left(\dot{\pi}^\nu+\Gamma^\nu_{\sigma\theta}\dot{\pi}^\sigma\dot{\pi}^\theta\right)\\&=g_{\mu\nu}\frac{\mathrm{D}\dot{\pi}^\nu}{\mathrm{d}t}.\end{aligned} \tag{38}$$

Substituting Eq. (30) into Eq. (38), we can obtain the following second quasi-Lagrange equation on the Riemann–Cartan configuration space

$$\frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\pi}^\mu}\right)=F_\mu. \tag{39}$$

The active force on the conservative constrained system can be written as the gradient of the potential energy $U(x^i)$, *i.e.*,

$$F_i=-\frac{\partial U}{\partial x^i}, \tag{40}$$

we can define naturally

$$\frac{\partial U}{\partial\pi^\mu}=b^i_\mu\frac{\partial U}{\partial x^i}, \tag{41}$$

and verify

$$F_\mu=-\frac{\partial U}{\partial\pi^\mu},\qquad\frac{\partial U}{\partial\dot{\pi}^\mu}=0. \tag{42}$$

Utilizing the Lagrangian function $L=T-U$, we can verify

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\pi}^\mu}\right)&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\pi}^\mu}\right)\\&=\frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\pi}^\mu}\right)+\Gamma^\sigma_{\mu\nu}\frac{\partial T}{\partial\dot{\pi}^\sigma}\dot{\pi}^\nu,\end{aligned} \tag{43}$$

$$\frac{\partial L}{\partial\pi^\mu}=\frac{\partial T}{\partial\pi^\mu}+F_\mu=\Gamma^\sigma_{\nu\mu}\frac{\partial T}{\partial\dot{\pi}^\sigma}\dot{\pi}^\nu+F_\mu. \tag{44}$$

Utilizing Eqs. (39), (43), and (44), we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\pi}^\mu}\right)-\frac{\partial L}{\partial\pi^\mu}=2S^\sigma_{\mu\nu}\frac{\partial L}{\partial\dot{\pi}^\sigma}\dot{\pi}^\nu. \tag{45}$$

□

The right side of Eq. (45) is called the torsion force. Obviously, due to the torsion force, equation (45) generally cannot be written as the canonical form. To achieve this, we must introduce the condition (21). So we can naturally get the following proposition.

Proposition 4 (Quasi-Hamilton's equations) For a conservative linear homogeneous differential constrained system, if a set of suitable quasi-coordinates is defined so that the configuration space of the system reduce to a Riemann space, then the motion equations of the system can be written as

$$\dot{\pi}^\mu=\frac{\partial H}{\partial\xi_\mu},\quad\dot{\xi}_\mu=-\frac{\partial H}{\partial\pi^\mu}, \tag{46}$$

where $\xi_\mu=\partial L/\partial\dot{\pi}^\mu$ is the quasi-momenta corresponding to the quasi-coordinates π^μ.

Proof According to Corollary 1, a set of suitable quasi-coordinates can be defined by a mapping satisfying condition (21), so that the reduced configuration space of the conservative linear homogeneous differential constrained system is a Riemann space with no torsion. So according to Proposition 4, the quasi-Lagrange equation of the system can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\pi}^\mu}\right)-\frac{\partial L}{\partial\pi^\mu}=0. \tag{47}$$

By means of the Legendre transformation $H=\dot{\pi}^\mu\xi_\mu-L$, equation (47) can be transformed into

$$\dot{\pi}^\mu=\frac{\partial H}{\partial\xi_\mu},\quad\dot{\xi}_\mu=-\frac{\partial H}{\partial\pi^\mu}. \tag{48}$$

□

According to Corollary 2 and Proposition 4, for a holonomic system, if the mapping (29) is a holonomic mapping, π^μ and ξ_μ will reduce into the generalized coordinates q^μ and the generalized momenta p_μ respectively. Analogously, the quasi-Hamilton's equations (46) will reduce into the canonical Hamilton's equations (2).

From the geometrical point of view, the reduced configuration space Q with generalized coordinates q^μ always is a Riemann space. So there always is a natural cotangent bundle T^*Q with canonical coordinates (q^μ,p_μ) for a holonomic system. This is the geometric basis of canonicalizations of holonomic systems. Obviously, the canonicalization process is independent of the selection of generalized coordinates q^μ.

From Corollary 1 and Proposition 4, for a linear homogeneous nonholonomic system, only when a set of suitable quasi-coordinates and quasi-momenta can be defined by the nonholonomic mapping satisfying the condition (21), the quasi-Hamilton's equations of the nonholonomic system can be written as Eq. (48). In other words, the quasi-canonicalization process is dependent of the selection of quasi-coordinates π^μ.

From the geometrical point of view, for a linear homogeneous nonholonomic system, the reduced configuration space Π is generally a Riemann–Cartan space with torsion. Because of the torsion, there is not a cotangent bundle on Π. Therefore, in order to achieve the quasi-canonicalization of the nonholonomic systems, a set of suitable quasi-coordinates must be defined, so that the reduced configuration space Π has no torsion. In this way, we can define the cotangent bundle $T^*\Pi$ with canonical coordinates (π^μ,ξ_μ). This is the geometric basis of quasi-canonicalizations for linear homogeneous nonholonomic systems.

$T^*\Pi$ and T^*Q are both submanifolds of the cotangent bundle T^*M with canonical coordinates (x^i,y_i). The difference is that T^*Q is embedded, while $T^*\Pi$ is only immersed.

The symplectic form on $T^*\Pi$ (or T^*Q) is the restriction of $\mathrm{d}x^i \wedge \mathrm{d}y_i$ on $T^*\Pi$ (or T^*Q). From

$$\begin{aligned}(\mathrm{d}x^i \wedge \mathrm{d}y_i)|_{T^*\Pi} &= b^i_\mu \mathrm{d}\pi^\mu \wedge \mathrm{d}(a^\nu_i \xi_\nu)\\ &= \mathrm{d}\pi^\mu \wedge \mathrm{d}\xi_\mu + b^i_\mu a^\nu_{i,\sigma}\xi_\nu \mathrm{d}\pi^\mu \wedge \mathrm{d}\pi^\sigma\\ &= \mathrm{d}\pi^\mu \wedge \mathrm{d}\xi_\mu - b^i_{\mu,\sigma}a^\nu_i \xi_\nu \mathrm{d}\pi^\mu \wedge \mathrm{d}\pi^\sigma\\ &= \mathrm{d}\pi^\mu \wedge \mathrm{d}\xi_\mu - \Gamma^\nu_{\mu\sigma}\xi_\nu \mathrm{d}\pi^\mu \wedge \mathrm{d}\pi^\sigma, \quad (49)\end{aligned}$$

it can be seen that if only the connection of Π is symmetric with respect to the lower index, $(\mathrm{d}x^i \wedge \mathrm{d}y_i)|_{T^*\Pi}$ gives the symplectic form $\mathrm{d}\pi^\mu \wedge \mathrm{d}\xi_\mu$ on the cotangent bundle $T^*\Pi$. Obviously, if the mapping (29) is holonomic, equation (49) reduces into $(\mathrm{d}x^i \wedge \mathrm{d}y_i)|_{T^*Q} = \mathrm{d}q^\mu \wedge \mathrm{d}p_\mu$, which is simply the symplectic form of holonomic systems.

4. Hamilton–Jacobi method for linear homogeneous nonholonomic systems

As an application of the above quasi-canonicalization, we study the Hamilton–Jacobi method for linear homogeneous nonholonomic systems.

Proposition 5 (Hamilton–Jacobi method) For a conservative linear homogeneous differential constrained system, if (π^μ, ξ_μ) is a set of canonical coordinates in which the motion equations of the system can be written as

$$\dot{\pi}^\mu = \frac{\partial H}{\partial \xi_\mu}, \quad \dot{\xi}_\mu = -\frac{\partial H}{\partial \pi^\mu}, \tag{50}$$

the corresponding Hamilton–Jacobi equation is

$$\frac{\partial S}{\partial t} + H\left(t, \pi^\mu, \xi_\mu\right) = 0, \tag{51}$$

where $\xi_\mu = \partial S/\partial \pi^\mu$. If a complete solution of the Hamilton–Jacobi equation can be found, then

$$\frac{\partial S}{\partial a^\mu} = b_\mu, \quad \xi_\mu = \frac{\partial S}{\partial \pi^\mu} \tag{52}$$

are $2m$ first integrals of the system, where b_μ are m arbitrary constants, and a^μ are m nontrivial constants in the complete solution of the Hamilton–Jacobi equation.

Proof Similar to the Hamilton–Jacobi method for holonomic systems, it can be directly verified that equations (52) are the first integrals satisfying the quasi-Hamilton's equations (50) by the theory of partial differential equations. □

Obviously, if the mapping (29) is a holonomic mapping, Proposition 5 will reduce into the classical Hamilton–Jacobi method for holonomic systems.

For a linear homogeneous nonholonomic system, since the mapping (29) is nonholonomic, the first integrals expressed by canonical coordinates (x^i, y_i) cannot be obtained directly from Eqs. (52).

In order to obtain the first integrals expressed by x^i and $\dot{x}^i$ (or y_i), we calculate the derivative of Eqs. (52) with respect to time, and then solve $\dot{\pi}^\mu$, which are functions of time. Then utilizing the mapping (29), we can get m first integrals expressed by x^i and $\dot{x}^i$.

Of course, from the derivative of Eqs. (52) with respect to time, we can also solve ξ_μ as functions of time. Then utilizing the mapping

$$y_i = a^\mu_i \xi_\mu, \tag{53}$$

we can get m first integrals expressed by x^i and y_i. But it is easy to prove that the two sets of m first integrals are equivalent. So according to Proposition 5 we can only obtain m independent first integrals expressed by x^i and $\dot{x}^i$ (or y_i).

5. Illustrative examples

Example 1 A particle with unit mass is subjected to the Chaplygin nonholonomic constraint

$$\dot{x}^1 + x^2\dot{x}^3 = 0. \tag{54}$$

We study the motion of the particle in the following two case:

(i) The potential energy is $U = x^2$.

(ii) The potential energy is $U = (1/3)(x^2)^3$.

(I) When $U = x^2$, the motion of the constrained particle can be solved by the traditional method. We first solve the problem by the multiplier method, then apply the quasi-canonicalization and the Hamilton–Jacobi method to solve it. It can be seen that the results of the two methods are equivalent.

The Lagrange equations with a multiplier λ of the particle are

$$\begin{cases}\ddot{x}^1 = \lambda,\\ \ddot{x}^2 = -1,\\ \ddot{x}^3 = \lambda x^2.\end{cases} \tag{55}$$

Combining the constraint equation (54) and Eqs. (55), we can obtain the motion equations without the multiplier λ as follows:

$$\begin{cases}\ddot{x}^1 = -\dfrac{x^2\dot{x}^3}{(x^2)^2+1},\\ \ddot{x}^2 = -1,\\ \ddot{x}^3 = -\dfrac{x^2\dot{x}^2\dot{x}^3}{(x^2)^2+1}.\end{cases} \tag{56}$$

Equations (56) can be solved directly. Its solution is

$$\begin{cases}x^1 = \dfrac{1}{2}\displaystyle\int \frac{c_3(t^2 - 2c_1t - 2c_2)}{\sqrt{(t^2-2c_1t-2c_2)^2+4}}\mathrm{d}t + c_5t + c_6,\\ x^2 = -\dfrac{1}{2}t^2 + c_1t + c_2,\\ x^3 = \displaystyle\int \frac{c_3}{\sqrt{(t^2-2c_1t-2c_2)^2+4}}\mathrm{d}t + c_4.\end{cases} \tag{57}$$

Since the constants in Eqs. (57) must satisfy the constraint equation, we substitute Eqs. (57) into Eq. (54), and then obtain

$$c_5 = 0. \tag{58}$$

Therefore, the motion of the particle can be expressed as

$$\begin{cases} x^1 = \dfrac{1}{2}\displaystyle\int \frac{c_3(t^2-2c_1t-2c_2)}{\sqrt{(t^2-2c_1t-2c_2)^2+4}}\,\mathrm{d}t + c_6, \\ x^2 = -\dfrac{1}{2}t^2 + c_1t + c_2, \\ x^3 = \displaystyle\int \frac{c_3}{\sqrt{(t^2-2c_1t-2c_2)^2+4}}\,\mathrm{d}t + c_4. \end{cases} \tag{59}$$

Next, we apply the quasi-canonicalization method to solve this problem.

We introduce the following nonholonomic mapping:

$$\begin{aligned} \dot{x}^1 &= \frac{x^2}{\sqrt{(x^2)^2+1}}\dot{\pi}^1, \\ \dot{x}^2 &= \dot{\pi}^2, \\ \dot{x}^3 &= -\frac{1}{\sqrt{(x^2)^2+1}}\dot{\pi}^1. \end{aligned} \tag{60}$$

It can be verified that the mapping (60) is nonholonomic, and imply the constraint (54). Then, from Eqs. (12) and (13), the metrics and the connections of the image space Π can be obtained as follows:

$$g_{11} = g_{22} = 1, \qquad g_{12} = g_{21} = 0, \tag{61}$$

$$\Gamma^{\sigma}_{\mu\nu} = 0. \tag{62}$$

Obviously, the torsion of Π is zero. It means that the reduced configuration space Π of the particle is a Riemann space with the quasi-coordinates π^1 and π^2. So the system can be quasi-canonized.

The quasi-Hamilton's equations of the system is

$$\begin{aligned} \frac{\mathrm{d}\pi^1}{\mathrm{d}t} &= \frac{\partial H}{\partial \xi_1}, \qquad \frac{\mathrm{d}\pi^2}{\mathrm{d}t} = \frac{\partial H}{\partial \xi_2}, \\ \frac{\mathrm{d}\xi_1}{\mathrm{d}t} &= -\frac{\partial H}{\partial \pi^1}, \qquad \frac{\mathrm{d}\xi_2}{\mathrm{d}t} = -\frac{\partial H}{\partial \pi^2}, \end{aligned} \tag{63}$$

where the Hamilton function is

$$H = \frac{1}{2}\left[(\xi_1)^2 + (\xi_2)^2\right] + \pi^2. \tag{64}$$

So the quasi-Hamilton's equations can be written as

$$\begin{aligned} \frac{\mathrm{d}\pi^1}{\mathrm{d}t} &= \xi_1, \qquad \frac{\mathrm{d}\pi^2}{\mathrm{d}t} = \xi_2, \\ \frac{\mathrm{d}\xi_1}{\mathrm{d}t} &= 0, \qquad \frac{\mathrm{d}\xi_2}{\mathrm{d}t} = -1, \end{aligned} \tag{65}$$

equations (65) can be solved directly. Its solution is

$$\begin{aligned} \pi_1 &= a^3t + a^2, \\ \pi^2 &= -\frac{1}{2}t^2 + a^4t + a^1, \\ \xi_1 &= a^3, \\ \xi_2 &= -t + a^4. \end{aligned} \tag{66}$$

We calculate the derivative of Eqs. (66) with respect to time, and then obtain

$$\dot{\pi}^1 = a^3, \quad \dot{\pi}^2 = -t + a^4. \tag{67}$$

Substituting Eqs. (67) into the mapping (60) yields

$$\begin{aligned} \dot{x}^1 &= \frac{a^3x^2}{\sqrt{(x^2)^2+1}}, \\ \dot{x}^2 &= -t + a^4, \\ \dot{x}^3 &= \frac{-a^3}{\sqrt{(x^2)^2+1}}. \end{aligned} \tag{68}$$

Equations (68) can be further integrated into

$$\begin{aligned} x^2 &= -\frac{1}{2}t^2 + a^4t + c_1, \\ x^1 &= \int \frac{a^3x^2}{\sqrt{(x^2)^2+1}}\,\mathrm{d}t + c_2, \\ x^3 &= \int \frac{-a^3}{\sqrt{(x^2)^2+1}}\,\mathrm{d}t + c_3, \end{aligned} \tag{69}$$

where x^2 in the latter two equations should be considered as a function of time, and the function relationship between x^2 and time is given by the first equation.

It can be seen that the solution (69) is equivalent to the solution (59). The correction of the solution (69) can also be verified by substituting Eq. (69) directly into Eq. (56) and the constraint equation (54).

(II) When $U = \frac{1}{3}(x^2)^3$, the motion of the constrained particle cannot be solved by the traditional method. But it can be solved by the quasi-canonicalization and the Hamilton–Jacobi method.

The motion equations of the particle with a multiplier is

$$\begin{cases} \ddot{x}^1 = \lambda, \\ \ddot{x}^2 = -(x^2)^2, \\ \ddot{x}^3 = \lambda x^2. \end{cases} \tag{70}$$

Combining the constraint equation (54) and Eqs. (70), we can obtain the motion equations without the multiplier λ as follows:

$$\begin{cases} \ddot{x}^1 = -\dfrac{\dot{x}^2\dot{x}^3}{(x^2)^2+1}, \\ \ddot{x}^2 = -(x^2)^2, \\ \ddot{x}^3 = -\dfrac{x^2\dot{x}^2\dot{x}^3}{(x^2)^2+1}. \end{cases} \tag{71}$$

Equation (71) is too complex to be solved directly.

Next, we try to solve this problem by the quasi-canonicalization and the Hamilton–Jacobi method. We also introduce the nonholonomic mapping (60). So the reduced configuration space Π of the particle is also a Riemann space with quasi-coordinates π^1 and π^2. And the system can also be quasi-canonized.

The quasi-Hamilton's equations is

$$\frac{\mathrm{d}\pi^1}{\mathrm{d}t}=\frac{\partial H}{\partial \xi_1},\qquad \frac{\mathrm{d}\pi^2}{\mathrm{d}t}=\frac{\partial H}{\partial \xi_2},$$
$$\frac{\mathrm{d}\xi_1}{\mathrm{d}t}=-\frac{\partial H}{\partial \pi^1},\qquad \frac{\mathrm{d}\xi_2}{\mathrm{d}t}=-\frac{\partial H}{\partial \pi^2}, \tag{72}$$

where the Hamilton function is

$$H=\frac{1}{2}\left[(\xi_1)^2+(\xi_2)^2\right]+\frac{1}{3}(\pi^2)^3. \tag{73}$$

So the Hamilton–Jacobi equation can be written as

$$\frac{\partial S}{\partial t}+\frac{1}{2}\left(\frac{\partial S}{\partial \pi^1}\right)^2+\frac{1}{2}\left(\frac{\partial S}{\partial \pi^2}\right)^2+\frac{1}{3}(\pi^2)^3=0. \tag{74}$$

Equation (74) can be solved by the separation of variables. The complete solution is

$$S=a^1t+a^2\pi^1+\int\left(-\sqrt{-2a^1-(a^2)^2-\frac{2}{3}(\pi^2)^3}\right)\mathrm{d}\pi^2+a^3, \tag{75}$$

where a^3 is a trivial arbitrary constant. According to Proposition 5, we can obtain the following four first integrals of the particle.

$$\begin{aligned}
b_1&=\int\left(-2a^1-(a^2)^2-\frac{2}{3}(\pi^2)^3\right)^{-1/2}\mathrm{d}\pi^2+t,\\
b_2&=\int a^2\left(-2a^1-(a^2)^2-\frac{2}{3}(\pi^2)^3\right)^{-1/2}\mathrm{d}\pi^2+\pi^1,\\
\xi_1&=a^2,\\
\xi_2&=-\sqrt{-2a^1-(a^2)^2-\frac{2}{3}(\pi^2)^3}.
\end{aligned} \tag{76}$$

In order to obtain the first integrals expressed by x^i and $\dot{x}^i$, we calculate the derivative of Eqs. (76) with respect to time, and then obtain

$$\begin{aligned}
\dot{\pi}^1&=a^2,\\
\dot{\pi}^2&=-\sqrt{-2a^1-(a^2)^2-\frac{2}{3}(\pi^2)^3}.
\end{aligned} \tag{77}$$

Substituting Eqs. (77) into the mapping (60) yields

$$\begin{aligned}
\dot{x}^1&=\frac{a^2x^2}{\sqrt{(x^2)^2+1}},\\
\dot{x}^2&=-\sqrt{-2a^1-(a^2)^2-\frac{2}{3}(x^2)^3},\\
\dot{x}^3&=-\frac{a^2}{\sqrt{(x^2)^2+1}}.
\end{aligned} \tag{78}$$

Equations (78 are three first integrals expressed by x^i and $\dot{x}^i$.

Equations (78) can be further integrated into

$$t+c_2=\int-\left(-2a^1-(a^2)^2-\frac{2}{3}(x^2)^3\right)^{-1/2}\mathrm{d}x^2,$$

$$\begin{aligned}
x^1&=\int\frac{a^2x^2}{\sqrt{(x^2)^2+1}}\mathrm{d}t+c_1,\\
x^3&=\int-\frac{a^2}{\sqrt{(x^2)^2+1}}\mathrm{d}t+c_3,
\end{aligned} \tag{79}$$

where x^2 in the latter two equations should be considered as a function of time, and the function relationship between x^2 and time is determined by the first equation.

The problem is completely solved by the Hamilton–Jacobi method.The correction of the solution (79) can be verified by substituting Eq. (79) directly into Eq. (71) and the constraint equation (54).

Example 2 We study the motion of a particle with unit mass. The potential energy $U=\frac{1}{3}(x^2)^3$. And the constraint equation is

$$2x^3\dot{x}^1+(x^1+x^2)\dot{x}^3=0. \tag{80}$$

Obviously, the constraint (80) is non-Chaplygin. And the motion of the constrained particle cannot be solved by the traditional method. But we obtain three independent first integrals of the system by the quasi-canonicalization and the Hamilton–Jacobi method.

We introduce the following nonholonomic mapping:

$$\begin{aligned}
\dot{x}^1&=\frac{x^1+x^2}{\sqrt{(x^1+x^2)^2+4(x^3)^2}}\dot{\pi}^1,\\
\dot{x}^2&=\dot{\pi}^2,\\
\dot{x}^3&=-\frac{2x^3}{\sqrt{(x^1+x^2)^2+4(x^3)^2}}\dot{\pi}^1.
\end{aligned} \tag{81}$$

It can be verified that the mapping (81) is nonholonomic, and imply the constraint (80). Then, from Eqs. (12) and (13), the metrics and the connections of the image space Π can be obtained as follows:

$$g_{11}=g_{22}=1,\qquad g_{12}=g_{21}=0, \tag{82}$$
$$\Gamma^{\sigma}_{\mu\nu}=0. \tag{83}$$

The torsion of Π is zero. The reduced configuration space Π of the particle is a Riemann space with the quasi-coordinates π^1 and π^2. So the system can be quasi-canonized.

The quasi-Hamilton's equations is

$$\frac{\mathrm{d}\pi^1}{\mathrm{d}t}=\frac{\partial H}{\partial \xi_1},\qquad \frac{\mathrm{d}\pi^2}{\mathrm{d}t}=\frac{\partial H}{\partial \xi_2},$$
$$\frac{\mathrm{d}\xi_1}{\mathrm{d}t}=-\frac{\partial H}{\partial \pi^1},\qquad \frac{\mathrm{d}\xi_2}{\mathrm{d}t}=-\frac{\partial H}{\partial \pi^2}, \tag{84}$$

where the Hamilton function is

$$H=\frac{1}{2}\left[(\xi_1)^2+(\xi_2)^2\right]+\frac{1}{3}(\pi^2)^3. \tag{85}$$

So the Hamilton–Jacobi equation can be written as

$$\frac{\partial S}{\partial t}+\frac{1}{2}\left(\frac{\partial S}{\partial \pi^1}\right)^2+\frac{1}{2}\left(\frac{\partial S}{\partial \pi^2}\right)^2+\frac{1}{3}(\pi^2)^3=0. \tag{86}$$

Equation (86) can be solved by the separation of variables. The complete solution is

$$S = a^1 t + a^2 \pi^1 - \int \sqrt{-2a^1 - (a^2)^2 - \frac{2}{3}(\pi^2)^3}\, \mathrm{d}\pi^2 + a^3, \quad (87)$$

where a^3 is a trivial arbitrary constant. According to proposition 5, we can obtain the following four first integrals of the particle.

$$\begin{aligned} b_1 &= \int \left(-2a^1 - (a^2)^2 - \frac{2}{3}(\pi^2)^3\right)^{-1/2} \mathrm{d}\pi^2 + t, \\ b_2 &= \int a^2 \left(-2a^1 - (a^2)^2 - \frac{2}{3}(\pi^2)^3\right)^{-1/2} \mathrm{d}\pi^2 + \pi^1, \\ \xi_1 &= a^2, \\ \xi_2 &= -\sqrt{-2a^1 - (a^2)^2 - \frac{2}{3}(\pi^2)^3}. \end{aligned} \quad (88)$$

In order to obtain the first integral expressed by x^i and $\dot{x}^i$, we calculate the derivative of Eqs. (88) with respect to time, and then obtain

$$\begin{aligned} \dot{\pi}^1 &= a^2, \\ \dot{\pi}^2 &= -\sqrt{-2a^1 - (a^2)^2 - \frac{2}{3}(\pi^2)^3}. \end{aligned} \quad (89)$$

Substituting Eqs. (89) into the mapping (81) yields

$$\begin{aligned} \dot{x}^1 &= \frac{a^2(x^1 + x^2)}{\sqrt{(x^1 + x^2)^2 + 4(x^3)^2}}, \\ \dot{x}^2 &= -\sqrt{-2a^1 - (a^2)^2 - \frac{2}{3}(x^2)^3}, \\ \dot{x}^3 &= -\frac{2a^2 x^3}{\sqrt{(x^1 + x^2)^2 + 4(x^3)^2}}. \end{aligned} \quad (90)$$

Equations (90) are three first integrals expressed by x^i and $\dot{x}^i$. Because of the constraint equation (80), there are only two independent first integrals in Eqs. (90).

The second equation of Eqs. (90) can be further integrated into

$$t + c_1 = \int -\left(-2a_1 - (a_2)^2 - \frac{2}{3}(x^2)^3\right)^{-1/2} \mathrm{d}x^2. \quad (91)$$

Combining Eqs. (90) and (91), we obtain three independent first integrals of the nonholonomic system by the Hamilton–Jacobi method.

6. Discussions

The quasi-canonicalization of constrained systems depends on finding a set of suitable quasi-velocities. It is natural. The partial differential equations (21) is the key to find a set of suitable quasi-velocities. That is, if we get a particular solution of the partial differential equations (21), a set of suitable quasi-velocities can be inversely solved from the first-order linear mapping (29). But it is usually difficult to find the particular solution of the partial differential equations (21) directly. So how to find a set of suitable quasi-velocities according to some properties of constrained systems is worth further study.

From the above two examples, we can see that $\dot{\pi}^\mu$ and $\dot{\xi}_\mu$ in the quasi-Hamilton's equations (48) are actually the derivatives of quasi-coordinates π^μ and quasi-momenta ξ_μ with respect to time. From the physical point of view, it means that the quasi-coordinates π^μ satisfying the partial differential equations (21) have some properties of generalized coordinates. From the geometric point of view, it means that the space Π defined by the quasi-coordinates π^μ satisfying the partial differential equations (21) is an immersed submanifold of the high-dimensional manifold M.

How to extend the quasi-canonicalization method in this paper to other differential constrained systems (such as linear nonhomogeneous differential constrained systems) is also worthy of further study. For linear nonhomogeneous differential constrained systems, the difficulty of the generalization lies in that the image space Π of the first-order linear mapping implying constraints has a time dimension. When the time dimension and the space dimension cannot be separated, the constraint force cannot be geometrized.[43]

7. Conclusions

We extend the canonicalization and the Hamilton–Jacobi method from holonimic systems to linear homogeneous nonholonomic systems in this paper. It means that the motion equations of conservative linear homogeneous nonholonomic systems can be written into the canonical form with quasi-coordinates π^μ and quasi-momenta ξ_μ. The canonicalization realized by quasi-coordinates π^μ is called the quasi-canonicalization in this paper. In order to achieve the quasi-canonicalization, a set of suitable quasi-coordinates π^μ need to be defined by a first-order linear mapping, which implies the constraints, so that the reduced configuration space Π of linear homogeneous nonholonomic system is a Riemann space with quasi-coordinates π^μ. Since there is naturally a cotangent bundle $T^*\Pi$ with the symplectic form $\mathrm{d}\pi^\mu \wedge \mathrm{d}\xi_\mu$ on the Riemann space Π, the motion equations can be written as the canonical form by the set of suitable quasi-coordinates π^μ and quasi-momenta ξ_μ. And the Hamilton–Jacobi method for linear homogeneous nonholonomic systems is a natural result of the quasi-canonicalization. Because the Hamilton–Jacobi method based on the quasi-canonicalization does not depend on the symmetry of linear homogeneous nonholonomic systems, it is suitable for non-Chaplygin nonholonomic systems. How to extend the quasi-canonicalization and the Hamilton–Jacobi

method to more general nonholonomic systems is worthy of further study.

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