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Approximate solution to the time-dependent Kratzer plus screened Coulomb potential in the Feinberg–Horodecki equation

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We obtain the quantized momentum eigenvalues P_n together with space-like coherent eigenstates for the space-like counterpart of the Schrödinger equation, the Feinberg–Horodecki equation, with a combined Kratzer potential plus screened coulomb potential which is constructed by temporal counterpart of the spatial form of these potentials. The present work is illustrated with two special cases of the general form: the time-dependent modified Kratzer potential and the time-dependent screened Coulomb potential.

Keywords: quantized momentum states, Feinberg–Horodecki equation, the time-dependent screened Coulomb potential, and time-dependent modified Kratzer potential

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1. Introduction

Any physical phenomenon in nature is usually characterized by solving differential equations. A good example is the time-dependent Schrödinger equation that describes quantum-mechanical phenomena, in which it dictates the dynamics of a quantum system. Solving this differential equation by means of any method results in the eigenvalues and eigenfunctions of that Schrödinger quantum system. However, solving the time-dependent Schrödinger equation analytically is not straightforward except in some cases where the time-dependent potentials are constant, linear and quadratic functions of the coordinates.[1–4]

The Feinberg–Horodecki (FH) equation is a space-like counterpart of the Schrödinger equation derived by Horodecki[5] from the relativistic Feinberg equation.[6] This equation has been demonstrated in a possibility of describing biological systems[7,8] in terms of the time-like supersymmetric quantum mechanics.[9] The space-like solutions to the Feinberg–Horodecki (FH) equation can be employed to test its relevance in different areas of science including physics, biology and medicine.[7,8]

The space-like quantum systems with the Feinberg–Horodecki equation have attracted a great deal of attention from many scientists especially in some branches of physics, such as extended special relativity and extended quantum mechanics.[7,10–12] For example, they are used to explain the force between electric charges, the electric charge source, and the mass in a stable particle.[7,13] Among these studies, Molski has also constructed the space-like coherent states of a time-dependent Morse potential with the Feinberg–Horodecki equation and showed that the obtained results for space-like coherent states can be used for Gompertzian systems.[7]

Molski constructed the space-like coherent states of a time-dependent Morse oscillator on the basis of the FH quantal equation for minimizing the time-energy uncertainty relation and showed that the results are useful for interpreting the formation of the specific growth patterns during crystallization process and biological growth.[7] In addition, Molski obtained the FH equation to demonstrate a possibility of describing the biological systems in terms of the space-like quantum supersymmetry for an-harmonic oscillators.[8] Hamzavi *et al.*[14] obtained the exact bound state solutions to the FH equation with a rotating time-dependent Deng–Fan oscillator potential by means of the parametric NU method. Eshghi *et al.*[15] solved the FH equation for a time-dependent mass distribution (TDM) harmonic oscillator quantum system with a certain interaction applied to a mass distribution $m(t)$ to provide a particular spectrum of stationary energies. In addition, the spectrum of harmonic oscillator potential $V(t)$ acting on a TDM $m(t)$ oscillator was obtained.

In the non-relativistic level, the Nikiforov–Uvarov (NU) method was used to obtain the bound state solutions to the arbitrary angular momentum Schrödinger equation with a modified Kratzer potential.[16] The factorization method was also used to obtain the solution of the non-central modified Kratzer potential for the diatomic molecules.[17] The exact solutions to the Schrödinger equation with modified Kratzer and corrected Morse potentials for position-dependent mass were also found.[18] The coherent states for a particle in Kratzer-type potentials are constructed by solving Feynman's path integral.[19] Furthermore, the exact solution to the Schrödinger equation

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for the modified Kratzer potential plus a ring-shaped potential was solved.[20]

On the other hand, in the relativistic level, approximate solutions to the D-dimensional Klein–Gordon equation are obtained for the scalar and vector general Kratzer potential for any l by using the ansatz method and the solutions of the Dirac equation with equal scalar and vector ring-shaped modified Kratzer potential were found by means of the Nikiforov–Uvarov method.[21,22]

At the level of applications, some authors have studied the modified Morse–Kratzer potential for alkali hydrides,[23] and the effect of the modified Kratzer potential on the confinement of an exciton in a quantum dot.[24] Moreover, there have been analyses of applications of the modified Kratzer potential, the bound states of two limiting cases of interest of the interactions and hence this approximation to obtain the solution to the Schrödinger equation for the Morse potential.[25]

Very recently, a superposition of modified Kratzer potential plus screened Coulomb potential was suggested to study diatomic molecules.[26] Edet *et al.* have obtained an approximate solution to the Schrödinger equation for the modified Kratzer potential plus the screened Coulomb potential model, within the framework of the Nikiforov–Uvarov method. They obtained bound state energy eigenvalues for N_2, CO, NO, and CH diatomic molecules for various vibrational and rotational quantum numbers. Special cases were considered when the potential parameters were altered, resulting in modified Kratzer potential, screened Coulomb potential, and standard Coulomb potential.[26] Further, Okorie *et al.* have solved the Schrödinger equation with the modified Kratzer plus a screened Coulomb potential using the modified factorization method. They have also employed both the Greene–Aldrich approximation scheme and a suitable transformation scheme to obtain the energy eigenvalue equation and its corresponding energy eigenfunctions for CO, NO, and N_2 diatomic molecules. They have used the energy eigenvalues of the modified Kratzer plus the screened Coulomb potential to obtain the vibrational partition functions and other thermodynamic functions for the selected diatomic molecules.[27]

However, recently some authors used the improved Rosen–Morse potential and improved Tietz potential to represent the internal vibrations of diatomic molecules, including NO,[28–30] N_2,[31] and CO,[32–34] and some triatomic molecules,[35–37] and successfully predicted the vibrational partition functions and important thermodynamic properties for some pure substances.

In the present work, we study the solutions to the Feinberg–Horodecki equation and extend the subject of coherent states to the space-like coherent states for the temporal counterpart of the Kratzer plus the screened Coulomb potential. The motivation of the present work is to obtain the eigen-solution to the Feinberg–Horodecki equation with a time-dependent modified Kratzer plus a screened Coulomb potential by means of the NU method. The rest of this work is organized as follows: the NU method is briefly introduced in Section 2. The approximate solution to the FH equation for the time-dependent general form of Kratzer potential plus the screened Coulomb potential is solved to obtain its quantized momentum states and eigenfunctions in Section 3. We generate the solutions of a few special potentials mainly found from our general form solution in Section 4. Finally we present our discussions and conclusions.

2. Feinberg–Horodecki equation with time-dependent combined Kratzer plus screened Coulomb potential

It is necessary to note that as the potential $V = V(t)$ depends on t,

$$E = T + V(r,t) \tag{1}$$

is not a constant of motion, and the mechanical energy reads

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}(\frac{1}{2}mv^2)}{\mathrm{d}t} + \frac{\mathrm{d}V(r,t)}{\mathrm{d}t} = \frac{\partial V}{\partial t}. \tag{2}$$

It is known that conservation of energy is a universal principle of physics. If V depends on time, then energy will still be conserved and, therefore, the energy must be changing in another part of the system in order to be conserved. On the other hand, if the potential is independent of time, then the energy will be a constant of the motion, i.e. energy will be conserved,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}(\frac{1}{2}mv^2)}{\mathrm{d}t} + \frac{\mathrm{d}V(r)}{\mathrm{d}t} = 0. \tag{3}$$

A very general principle in modern theoretical physics states that for every symmetry there is a conserved quantity. For example, translation invariance in time symmetry implies a conservation energy and spatial translation symmetry implies a conservation in momentum.

The NU method[38] is being applied to find the approximate solutions to the FH equation for the Kratzer plus a screened Coulomb potential, then the eigenvalues and eigenfunctions of two special cases are obtained from the results.

The time-dependent Kratzer potential is given by[39–41]

$$V(t) = D - \frac{B}{t} + \frac{qC}{t^2}, \tag{4}$$

where D, B, and C are adjustable real potential parameters, and q is dimensionless parameter. Further, the screened Coulomb potential is defined as[26]

$$V(t) = \frac{A\,\mathrm{e}^{-\alpha t}}{t}. \tag{5}$$

If equations (4) and (5) are substituted in the FH equation, one can obtain

$$\left[-\frac{\hbar^2}{2mc^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \left(D - \frac{B}{t} + \frac{qC}{t^2} + \frac{A\,\mathrm{e}^{-\alpha t}}{t}\right)\right]\psi_n(t)$$

$$= cP_n\psi_n(t), \tag{6}$$

where c is the speed of light and P_n $(n = 0, 1, 2, \ldots)$ is the momentum eigenvalues.

Using the Greene–Aldrich approximation[26] defined as

$$\frac{1}{t} \approx \frac{\alpha}{(1-\mathrm{e}^{-\alpha t})}, \tag{7}$$

then letting $s = \mathrm{e}^{-\alpha(t)}$, where $s \in (0,1)$, we obtain

$$\frac{\mathrm{d}^2\psi_n(s)}{\mathrm{d}s^2} + \frac{(1-s)}{s(1-s)}\frac{\mathrm{d}\psi_n(s)}{\mathrm{d}s} + \frac{-\varepsilon_1^2 - \varepsilon_3 s + \varepsilon_2 s^2}{s^2(1-s)^2}\psi_n(s) = 0, \tag{8}$$

where the equation satisfies the asymptotic behavior where $\psi_n(s=0) = 0$ and $\psi_n(s=1) = 0$, and

$$\varepsilon_1^2 = \frac{2mc^2}{\hbar^2\alpha^2}\left(D - B\alpha + qC\alpha^2 - cP_n\right), \tag{9}$$

$$\varepsilon_2 = \frac{2mc^2}{\hbar^2\alpha^2}\left(A\alpha - D + cP_n\right), \tag{10}$$

$$\varepsilon_3 = \frac{2mc^2}{\hbar^2\alpha^2}\left(B\alpha + A\alpha - 2D + 2cP_n\right). \tag{11}$$

After comparing Eq. (8) with Eq. (A1), one obtains $\tilde{\tau}(s) = 1 - s$, $\sigma(s) = s(1-s)$, and $\tilde{\sigma}(s) = -\varepsilon_1^2 - \varepsilon_3 s + \varepsilon_2 s^2$. When these values are substituted in Eq. (A10), we obtain

$$\pi(s) = -\frac{s}{2} \pm \sqrt{\left(\frac{1}{4} - \varepsilon_2 - k\right)s^2 + (k + \varepsilon_3)s + \varepsilon_1^2}. \tag{12}$$

As mentioned in the NU method, the discriminant under the square root, in Eq. (12), has to be zero, so that the expression of $\pi(s)$ becomes the square root of a polynomial of the first degree. This condition can be written as

$$\left(\frac{1}{4} - \varepsilon_2 - k\right)s^2 + (k + \varepsilon_3)s + \varepsilon_1^2 = 0. \tag{13}$$

After solving this equation, we have

$$s = \frac{-(k+\varepsilon_3) \pm \sqrt{(k+\varepsilon_3)^2 - 4\varepsilon_1^2(\frac{1}{4} - \varepsilon_2 - k)}}{2(\frac{1}{4} - \varepsilon_2 - k)}. \tag{14}$$

Then, for our purpose we assume

$$(k + \varepsilon_3)^2 - 4\varepsilon_1^2\left(\frac{1}{4} - \varepsilon_2 - k\right) = 0. \tag{15}$$

Arranging this equation and solving it to obtain an expression for k which is given by

$$k_\pm = -\varepsilon_3 - 2\varepsilon_1^2 \pm 2\varepsilon_1\left(\frac{1}{Q} - \frac{1}{2}\right), \tag{16}$$

where the expression between the parentheses is given by

$$\frac{1}{Q} = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2mc^2}{\hbar^2\alpha^2}(qC\alpha^2)}. \tag{17}$$

If we substitute k_- into Eq. (12) we obtain a possible expression for $\pi(s)$, which is given by

$$\pi_-(s) = \varepsilon_1 - s\left(\varepsilon_1 + \frac{1}{Q}\right). \tag{18}$$

This solution satisfies the condition that the derivative of $\tau(s)$ is negative. Therefore, the expression of $\tau(s)$ which satisfies these conditions can be written as

$$\tau(s) = 1 - s + 2\varepsilon_1 - 2s\left(\varepsilon_1 + \frac{1}{Q}\right). \tag{19}$$

Now, substituting the values of $\tau_-'(s)$, $\sigma''(s)$, $\pi_-(s)$ and k_- into Eqs. (A6) and (A11), we obtain

$$\lambda_n = \frac{2mc^2}{\hbar^2\alpha}(-A + B - 2qC\alpha) + \frac{2\varepsilon_1}{Q} - \frac{1}{Q}, \tag{20}$$

$$\lambda = \lambda_n = n\left(n + \frac{2}{Q}\right) + 2n\varepsilon_1. \tag{21}$$

Now, from Eqs. (20) and (21), we obtain the eigenvalues of the quantized momentum as

$$P_n = \frac{1}{c}\left[(D - B\alpha + qC\alpha^2) - \frac{\alpha^2\hbar^2}{2mc^2} \times \left(\frac{\frac{2mc^2}{\alpha\hbar^2}(A - B + 2qC\alpha) + n(n + \frac{2}{Q}) + \frac{1}{Q}}{2(n + \frac{1}{Q})}\right)^2\right]. \tag{22}$$

Obviously, the quantized momentum eigenvalues of the time-dependent FH equation are dependent on the values of the potential parameters. The sign of the momentum eigenvalues is dependent on the strength of these parameters and their signs.

Due to the NU method used to obtain the eigenvalues, the polynomial solutions of the hypergeometric function $y_n(s)$ depend on the weight function $\rho(s)$, which can be determined by solving Eq. (A9) to achieve

$$\rho(s) = s^{2\varepsilon_1}(1-s)^{(\frac{2}{Q})-1}. \tag{23}$$

Substituting $\rho(s)$ into Eq. (A8), we have an expression for the wave functions as follows:

$$y_n(s) = A_n s^{-2\varepsilon_1}(1-s)^{-(\frac{2}{Q}-1)} \times \frac{\mathrm{d}^n}{\mathrm{d}s^n}\left[s^{n+2\varepsilon_1}(1-s)^{n+\frac{2}{Q}-1}\right], \tag{24}$$

where A_n is the normalization constant. Solving Eq. (24) gives the final form of the wave function in terms of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ as follows:

$$y_n(s) = A_n n! P_n^{(2\varepsilon_1, \frac{2}{Q}-1)}(1-2s). \tag{25}$$

Now, substituting $\pi_-(s)$ and $\sigma(s)$ into Eq. (A4) and then solving it we have

$$\phi_n(s) = s^{\varepsilon_1}(1-s)^{\frac{1}{Q}}. \tag{26}$$

Substituting Eqs. (25) and (26) into Eq. (A2), and using $s = \mathrm{e}^{-\alpha(t)}$ we can obtain

$$\psi_n(t) = B_n \mathrm{e}^{-\alpha\varepsilon_1 t}(1-\mathrm{e}^{-\alpha t})^{\frac{1}{Q}} P_n^{(2\varepsilon_1,\frac{2}{Q}-1)}(1-2\mathrm{e}^{-\alpha t}), \quad (27)$$

where B_n is the normalization constant with $\frac{1}{Q}$ given in Eq. (17). Obviously, the above wave function is finite at both $t = 0$ and $t \longrightarrow \infty$.

3. Special cases

3.1. The time-dependent modified Kratzer potential

To obtain the modified Kratzer potential from the general form, D, A and α are set to zero, $q = 1$, $B = 2t_\mathrm{e}D_\mathrm{e}$ and $C = {t_\mathrm{e}}^2 D_\mathrm{e}$ and then substituted in Eq. (4) to reduce the general form to the special case,[25]

$$V(t) = \frac{-2t_\mathrm{e}D_\mathrm{e}}{t} + D_\mathrm{e}\left(\frac{t_\mathrm{e}}{t}\right)^2, \quad (28)$$

where t_e and D_e represent the equilibrium time point and the dissociation energy of the system, respectively. In addition, by substituting the same constants in Eq. (22) we achieve the eigenvalues of the time-dependent HF equation with the modified Kratzer potential. Our result becomes

$$P_n = \frac{1}{c}\left[-\frac{2mc^2}{\hbar^2}\left(\frac{2t_\mathrm{e}D_\mathrm{e}}{2(n+\frac{1}{Q})}\right)^2\right], \quad (29)$$

where

$$\frac{1}{Q} = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2mc^2 t_\mathrm{e}^2 D_\mathrm{e}}{\hbar^2}}. \quad (30)$$

On the other hand, to determine the eigenfunctions associated with the modified Kratzer potential, substituting the same parameters in Eq. (9) yields

$$\psi_n(s) = B_n \mathrm{e}^{-\alpha\varepsilon_1 t}(1-\mathrm{e}^{-\alpha t})^{\frac{1}{Q}} P_n^{(2\varepsilon_1,\frac{2}{Q}-1)}(1-2\mathrm{e}^{-\alpha t}), \quad (31)$$

where

$$\varepsilon_1 = \sqrt{\frac{2mc^2}{\hbar^2\alpha^2}(-2t_\mathrm{e}D_\mathrm{e}\alpha + t_\mathrm{e}^2 D_\mathrm{e}\alpha^2 - cP_n)}. \quad (32)$$

3.2. The time-dependent screened Coulomb potential

Further by setting the values of B, C, and D to zero and A to $-Ze^2$ we have the screened coulomb potential. By substituting these values in Eq. (22), it gives the eigenvalues of the FH time-dependent equation. These eigenvalues are given by the relation

$$P_n = \frac{1}{c}\left[-\frac{\alpha^2\hbar^2}{2mc^2}\left(\frac{\frac{-2mc^2Ze^2}{\alpha\hbar^2} + n(n+2)+1}{2(n+1)}\right)^2\right]. \quad (33)$$

To determine the eigenfunctions associated with the screened Coulomb potential, substituting the same parameters into Eq. (9) yields

$$\psi_n(s) = B_n \mathrm{e}^{-\alpha\varepsilon_1 t}(1-\mathrm{e}^{-\alpha t}) P_n^{(2\varepsilon_1,1)}(1-2\mathrm{e}^{-\alpha t}), \quad (34)$$

where

$$\varepsilon_1 = \frac{c}{\hbar\alpha}\sqrt{2mcP_n}. \quad (35)$$

Setting $\alpha = 0$ in the screened Coulomb potential, we can obtain the quantized eigenvalues of the FH equation with the Coulomb potential as follows:

$$P_n = \frac{-mc^2z^2e^4}{2\hbar^2(n+1)^2}. \quad (36)$$

3.3. Numerical results and discussion

We compute the momentum eigenvalues of the time-dependent Kratzer plus screened Coulomb potential for some diatomic molecules like CO, NO, O_2 and I_2. This was done using the spectroscopic parameters displayed in Table 1.

Table 1. Spectroscopic parameters of the various diatomic molecules.

Molecule	D_e/eV	t_e /time unit	μ/a.u.
CO	10.84514471	1.1282	6.860586000
NO	8.043782568	1.1508	7.468441000
O_2	5.156658828	1.2080	7.997457504
I_2	1.581791863	2.6620	63.45223502

Its worth noting that when we solve the FH equation in the absence of the interaction potential, i.e., $V(t) = 0$, the quantized momentum eigenvalues are negative. Further, in the presence of Coulomb interaction potential $V(t) = -ze^2/t$, the quantized momentum still stands negative as shown by Eq. (36). On the other hand, if we let the diatomic molecule interact via the Kratzer plus the screened Coulomb potential this interaction shifts the quantized spectrum to the positive region at small values of the screening parameter α. The momentum spectrum for states tends to be continuous as state n is increasing for the studied diatomic molecules. The momentum spacing increases with increasing the screening parameter α for all molecules. The momentum spacing between states decreases with increasing n. This momentum spacing difference between states is largest for CO, NO, and O_2, respectively, while smallest for I_2 as in Tables 2–4. This spacing is almost same for each molecule with different values of the screening parameter. For example, for CO, NO, O_2 and I_2, $P_1 - P_0 = 1.20$ eV/c, 1.00 eV/c, 0.60 eV/c, and 0.07 eV/c, respectively. However, $P_9 - P_8 = 0.40$ eV/c, 0.20 eV/c, 0.13 eV/c, and 0.04 eV/c.

Table 2. The FH quantized momentum eigenvalues (in units of eV/c) with the Kratzer plus the screened Coulomb potential for the diatomic molecules ($\alpha = 0.001$, D_e, t_e and μ as defined in Table 1).

	$\alpha = 0.001$			
n	CO	NO	O_2	I_2
0	1.540356974	1.395600000	1.176000000	0.386600000
1	2.700454092	2.305000000	1.789300000	0.447000000
2	3.656651169	3.040100000	2.271200000	0.503000000
3	4.454075791	3.642700000	2.656900000	0.554900000
4	5.126028125	4.142800000	2.970200000	0.603200000
5	5.697514722	4.562400000	3.228200000	0.648200000
6	6.187609162	4.918000000	3.443300000	0.690100000
7	6.611064425	5.221900000	3.624400000	0.729300000
8	6.979436269	5.483600000	3.778300000	0.766000000
9	7.301880048	5.710700000	3.910300000	0.800400000

Table 3. The FH quantized momentum eigenvalues (in units of eV/c) with the Kratzer plus the screened Coulomb potential for the diatomic molecules ($\alpha = 0.05$, D_e, t_e and μ as defined in Table 1).

	$\alpha = 0.05$			
n	CO	NO	O_2	I_2
0	1.629666341	1.477300000	1.249400000	0.440300000
1	2.857499074	2.440500000	1.901800000	0.509600000
2	3.869194581	3.218800000	2.414300000	0.573800000
3	4.712568654	3.856500000	2.824000000	0.633300000
4	5.422905120	4.385500000	3.156600000	0.688600000
5	6.026700142	4.829000000	3.430300000	0.740000000
6	6.544163497	5.204500000	3.658100000	0.788000000
7	6.990926983	5.525100000	3.849600000	0.832700000
8	7.379234725	5.801000000	4.012100000	0.874600000
9	7.718787450	6.040000000	4.151100000	0.913800000

In Fig. 1, we plot the time-dependent Kratzer plus the screened Coulomb potential for four different diatomic molecules. The behaviors of these diatomic molecules are relatively similar with little difference for I_2. In Fig. 2, the Kratzer plus the screened Coulomb potential is plotted versus both time and screening parameter α for the diatomic molecules. Here colors represent the value of the potential as illustrated by the color bar.

Table 4. The FH quantized momentum eigenvalues (in units of eV/c) for the Kratzer plus the screened Coulomb potential for the diatomic molecules ($\alpha = 0.1$, D_e, t_e and μ as defined in Table 1).

	$\alpha = 0.1$			
n	CO	NO	O_2	I_2
0	1.720708160	1.560500000	1.324200000	0.495200000
1	3.017005473	2.578100000	2.016100000	0.573500000
2	4.084136054	3.399400000	2.558600000	0.645800000
3	4.972738926	4.071500000	2.991600000	0.712800000
4	5.720181209	4.62800000	3.342300000	0.774900000
5	6.354523505	5.093800000	3.630000000	0.832700000
6	6.897168854	5.487200000	3.868600000	0.886400000
7	7.364672017	5.822200000	4.068300000	0.936400000
8	7.770000059	6.109500000	4.237000000	0.983100000
9	8.123426476	6.357500000	4.380400000	1.026600000

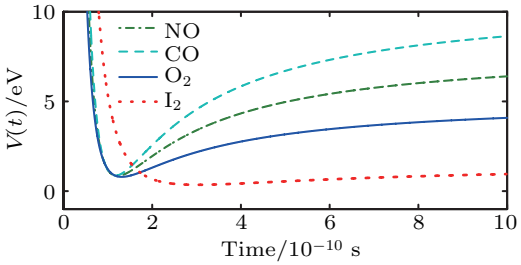


Fig. 1. Kratzer plus the screened Coulomb potential for the diatomic molecules.

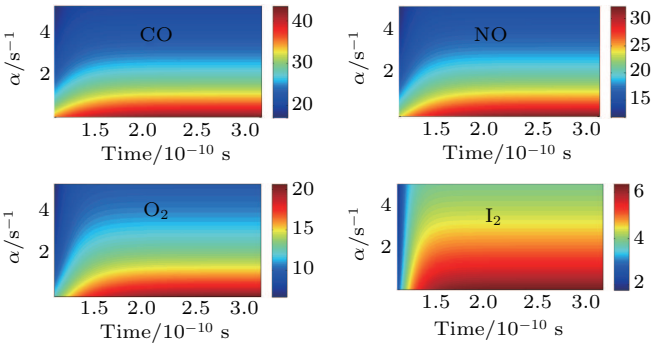


Fig. 2. Kratzer plus the screened Coulomb potential plotted vs both time and screening parameter α for the diatomic molecules. Colors represent the value of the potential as illustrated by the color bar.

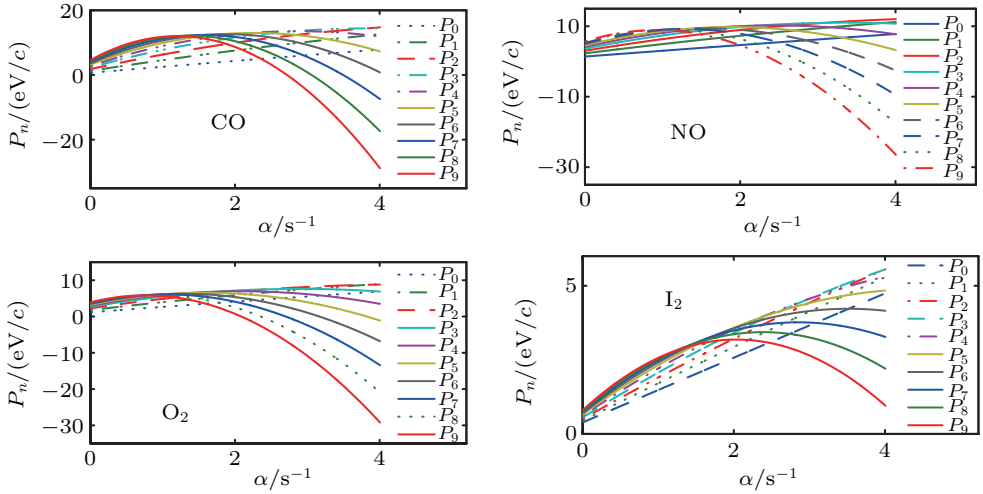


Fig. 3. FH quantized momentum eigenvalues for the Kratzer plus the screened Coulomb potential for the diatomic molecules.

In Fig. 3, the FH quantized momentum states for the Kratzer plus the screened Coulomb potential are plotted versus the screening parameter α for the diatomic molecules (Table 1). We can see that the momentum reaches its continuous value when n increases. We sharpen our analysis by taking $n = 9$. It is found from Fig. 3 that momentum reaches the continuous spectrum when $2 < \alpha < 3$ for CO, $\alpha \approx 2$ for NO, $1 < \alpha < 2$ for O_2 and $\alpha > 4$ for I_2.

In Fig. 4, the FH quantized momentum eigenvalues for the Kratzer plus the screened Coulomb potential are plotted versus the potential strength D_e for $\alpha = 0.005$ 1/s, the momentum increases above 10 eV/c for CO, NO, and O_2 whereas for I_2 it reaches 4 eV/c.

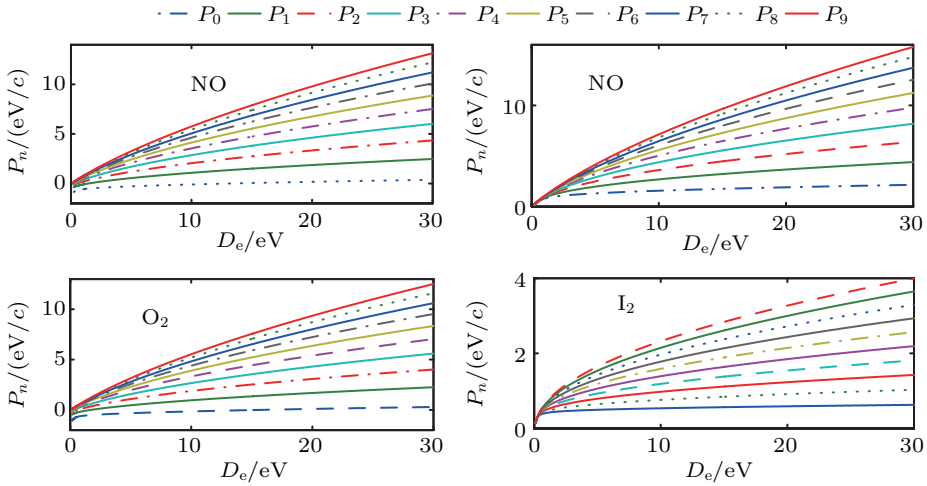


Fig. 4. FH quantized momentum eigenvalues for the Kratzer plus the screened Coulomb potential for the diatomic molecules. The graphs are plotted vs D_e for $\alpha = 0.005$.

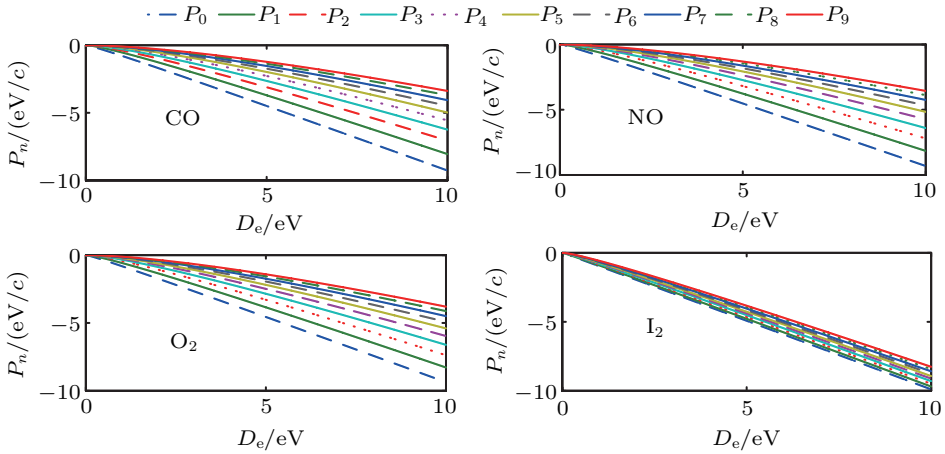


Fig. 5. FH quantized momentum eigenvalues with the modified Kratzer potential for the diatomic molecules.

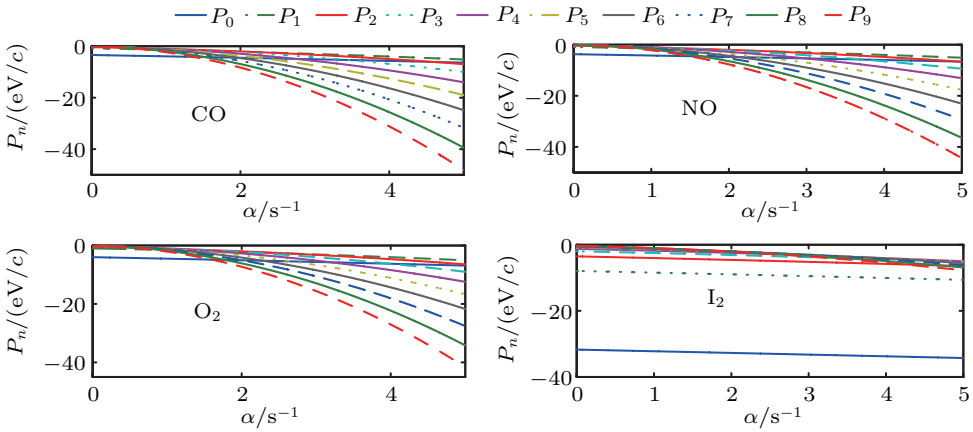


Fig. 6. FH quantized momentum eigenvalues with the screened Coulomb potential for the diatomic molecules.

Figure 5 shows the FH quantized momentum spectrum versus the potential parameter D_e in the negative region with the modified Kratzer potential for the diatomic molecules. Finally, in Fig. 6, the FH quantized momentum eigenvalues of the screened Coulomb potential versus the screening parameter for the diatomic molecules reach the continuous region for small values of $\alpha < 1$.

4. Conclusions

We have solved the Feinberg–Horodecki equation for the time-dependent general form of the Kratzer potential via the Nikiforov–Uvarov method, and obtain the approximate quantized momentum eigenvalues solution to the FH equation. It is worth mentioning that the method is elegant and powerful. Our results can be applied in biophysics and other branches of physics. In this paper, we have applied our result for the modified Kratzer and screened coulomb potentials, as special cases of the used potential, for quantized momentum eigenvalues.

Appendix: Review to the Nikiforov–Uvarov method

The Nikiforov–Uvarov method[38] is usually used to reduce the second-order differential equation into a general form of a hypergeometric type. In that sense, any second-order differential equation can be transformed, using a suitable coordinate transformation $s = s(r)$, into the form

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{A1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To solve Eq. (A1), the following wave function can be proposed:

$$\psi_n(s) = \phi_n(s)y_n(s), \tag{A2}$$

which transforms Eq. (A1) into the following hypergeometric form:

$$\sigma(s)y_n''(s) + \tau(s)y_n'(s) + \lambda y_n(s) = 0, \tag{A3}$$

where

$$\sigma(s) = \pi(s)\frac{\phi_n(s)}{\phi_n'(s)}, \tag{A4}$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \tag{A5}$$

λ in Eq. (A3) is a parameter defined as

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \tag{A6}$$

where $n = 0, 1, 2, \ldots$, and $\tau(s)$ is a polynomial with a negative first derivative to produce an appropriate solution for the hypergeometric equation. The first part of the proposed wave function is a solution to the equation

$$\sigma(s)\phi'(s) - \pi(s)\phi(s) = 0. \tag{A7}$$

However, the second part of the wave function (A2) represents a hypergeometric form,[24] which can be obtained using the Rodrigues formula

$$y_n(s) = \frac{A_n}{\rho(s)}\frac{\mathrm{d}^n}{\mathrm{d}s^n}[\sigma^n(s)\rho(s)], \tag{A8}$$

where B_n is a normalization constant and $\rho(s)$ is a weight function that can be calculated from the relation,

$$\sigma(s)\rho'(s) + [\sigma(s) - \tau(s)]\rho(s) = 0. \tag{A9}$$

$\pi(s)$ in Eq. (A4) is defined as

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}, \tag{A10}$$

and λ in Eq. (A3) is defined as

$$\lambda = \lambda_n = k + \pi'(s), \tag{A11}$$

where $\pi(s)$ is a polynomial depending on the transformation function $s(r)$, and k should be determined by calculating $\pi(s)$, for which the discriminant under the square root in Eq. (A10) is set to zero, in order to let $\pi(s)$ be a first order polynomial. Finally the eigenvalues can be found by solving Eqs. (A6) and (A11).

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