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Bose–Einstein condensates in an eightfold symmetric optical lattice*

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We investigate the properties of Bose–Einstein condensates (BECs) in a two-dimensional quasi-periodic optical lattice (OL) with eightfold rotational symmetry by numerically solving the Gross–Pitaevskii equation. In a stationary external harmonic trapping potential, we first analyze the evolution of matter-wave interference pattern from periodic to quasi-periodic as the OL is changed continuously from four-fold periodic to eight-fold quasi-periodic. We also investigate the transport properties during this evolution for different interatomic interaction and lattice depth, and find that the BEC crosses over from ballistic diffusion to localization. Finally, we focus on the case of eightfold symmetric lattice and consider a global rotation imposed by the external trapping potential. The BEC shows vortex pattern with eightfold symmetry for slow rotation, becomes unstable for intermediate rotation, and exhibits annular solitons with approximate axial symmetry for fast rotation. These results can be readily demonstrated in experiments using the same configuration as in *Phys. Rev. Lett.* **122** 110404 (2019).

Keywords: quasicrystals, Bose–Einstein condensates, quantum transport, vortex and soliton

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1. Introduction

Quasicrystals, which can exhibit long-range order without translational symmetry, have been used to study quantum states between the limits of periodic order and disorder.[1–4] According to the theorems of crystallography, the rotational symmetries of periodic lattices are highly restricted to a few possibilities, namely, two-, three-, four-, and six-fold symmetries. However, quasicrystals can show all the rotational symmetries forbidden to crystals, including five-, seven-, eight-, and higher-fold symmetries,[5–7] as can be retrieved from the experimental data of electron microscopy, spectroscopy, surface imaging methods, and diffraction pattern.[6]

The remarkable tunability and dynamical control of optical lattices (OLs) offer an opportunity to investigate quantum many-body states in periodic systems.[8,9] Ultracold atomic gases trapped in OLs can work as promising candidates to simulate versatile quantum phenomena in various fields of physics, such as condensed matter physics, high energy physics, and astrophysics. For example, the novel quantum transition from extended to localized states has been investigated in numerical simulation and observed in experimental exploration in the presence of OLs.[10–13] In addition, a carefully arranged configuration of laser beams can generate a quasi-periodic optical lattice with fascinating spatial patterns which are neither periodic as crystals (i.e., lack of translational symmetry) nor totally disordered (i.e., possession of long-range order).[14–17] Recently, a Bose–Einstein condensate (BEC) has been realized experimentally in a two-dimensional (2D) quasi-periodic OL with eightfold rotational symmetry.[18] This achievement paves the route to study the quasi-periodic systems which are expected to present hybrid features of crystals and amorphous matters.

Stimulated by the experimental achievement of BECs in eightfold symmetric OL,[18] we study in this work the effects of quasi-periodicity on various static and dynamic properties of the BECs. By numerically solving the Gross–Pitaevskii (GP) equation, we first discuss the matter-wave interference pattern of the BECs obtained from time-of-flight images, which reveals the eightfold symmetry and the self-similarity of the quasicrystal lattice. Next, we investigate the diffusion of particles upon released from the harmonic trap while the background lattice potential is present. A crossover from the ballistic diffusion to spatial localization is observed by continuously tuning the lattice configuration from a periodic square lattice to an eightfold symmetric quasi-periodic lattice. The effects of the interaction and lattice depth on the particle diffusion are also discussed. In addition, by applying a tilt of the lattice potential, we study the quasi-periodic Bloch oscillation and compare to the conventional Bloch oscillation induced by static force in a regular OL. Finally, we impose an external rotation on the system and numerically solve for the ground state of the BECs in the combined potential of OL and harmonic

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trap. When the rotation is slow in comparison to the harmonic trapping frequency, vortices can be generated to form a lattice structure with eightfold symmetry. When the rotating frequency is much greater than the trapping frequency, annular solitons can be formed along the radial direction with an approximate axial symmetry. For intermediate rotating speed, the system is dynamically unstable with no steady solution. These results extend our understanding on BECs through the crossover from ordered to disordered lattices, and can be readily implemented in experiments.

The remainder of this manuscript is organized as follows. In Section 2 we introduce the configuration of the system under consideration and present the GP equation. We then solve the GP equation with no external rotation and discuss the matter-wave interference pattern, particle diffusion, and Bloch oscillation in the presence of quasi-periodic lattices in Section 3. By imposing an overall rotation with different frequencies, we study the emergence and structure of vortices and solitons in Section 4. Finally, we summarize the main results in Section 5.

2. Formalism

The structure of a two-dimensional eightfold rotationally symmetric lattice can be created by four mutually incoherent one-dimensional (1D) optical lattices, which are yielded by retro-reflection of four single frequency laser beams, lying in the x–y plane and separated by an angle of $\pi/4$. The four lasers are all linearly polarized to be perpendicular to the plane as illustrated in Fig. 1(a).[18] In general, the resulting potential of such a laser configuration can be expressed as

$$V_{\rm latt}(\boldsymbol{r}) = V_0 \sum_{i=1}^{4} \varepsilon_i \cos^2\left(\frac{\hat{\boldsymbol{G}}_i}{2}\cdot\boldsymbol{r}\right), \tag{1}$$

where $\boldsymbol{r}=(x,y)$ labels the 2D spatial coordinate in the lattice plane, $0\le\varepsilon_i\le 1$ stands for the relative dimensionless intensity of the i-th laser beam, and $V_0<0$ is the overall lattice depth. The reciprocal lattice vectors $\hat{\boldsymbol{G}}_i$ of the four 1D lattices are arranged as $\hat{\boldsymbol{G}}_1\propto(1,0)$, $\hat{\boldsymbol{G}}_2\propto(1,1)/\sqrt{2}$, $\hat{\boldsymbol{G}}_3\propto(0,1)$, and $\hat{\boldsymbol{G}}_4\propto(-1,1)/\sqrt{2}$, respectively. The magnitudes of the lattice vectors are normalized as $|\hat{\boldsymbol{G}}_i|=2k_{\rm L}$, where $k_{\rm L}$ is the lattice wave vector. In the following discussion, we set the lattice wavelength $\lambda=2\pi/k_{\rm L}$ as the length unit, the recoil energy $E_{\rm R}=\hbar^2k_{\rm L}^2/2m$ (with m the atomic mass) as the energy unit, and the recoil frequency $\omega_{\rm R}=E_{\rm R}/\hbar$ as the unit of frequency. With this configuration, one can tune the lattice potential continuously from a 2D cubic lattice with $\varepsilon_1=\varepsilon_3=1$ and $\varepsilon_2=\varepsilon_4=0$ to an eightfold symmetric quasi-periodic lattice with all $\varepsilon_i=1$. In Fig. 1(b), we show a false color plot of the lattice potential for the latter scenario with eightfold symmetry for $V_0=-2E_{\rm R}$. The quasi-periodic lattice shows no translational invariance but some apparent long-range order.

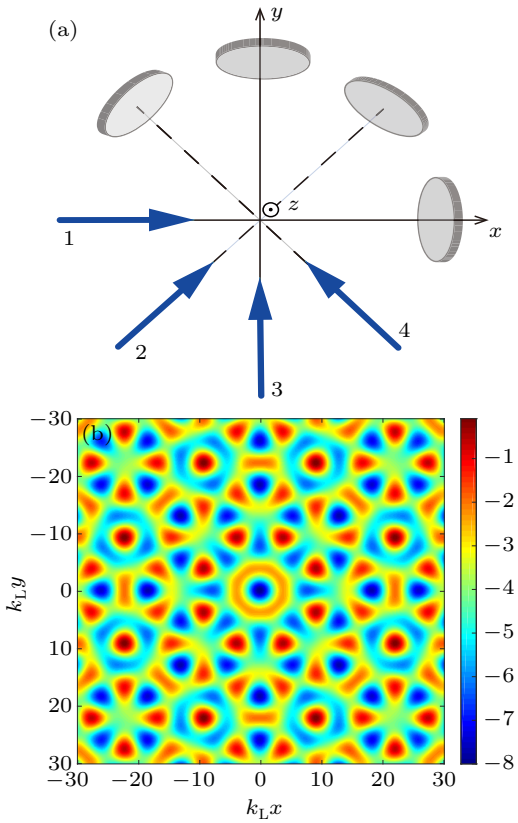


Fig. 1. (a) Schematic illustration of laser arrangement to create an eightfold symmetric optical lattice. (b) A false-color plot of the quasi-periodic lattice potential with $V_0=-2E_{\rm R}$.

We consider interacting BECs trapped in a combined potential of the lattice potential (1) and a cylindrically symmetric external harmonic trap $V_{\rm trap}=m\omega_r^2r^2/2+m\omega_z^2z^2/2$. In the quasi-2D regime where the trapping frequency along the z-axis ω_z is much greater than that in the radial plane ω_r, the atoms are tightly confined along the z-direction such that only the ground harmonic oscillator is occupied. Thus, one can integrate out the degrees of freedom along the z-axis, and consider a 2D system trapped in a combined potential

$$V(\boldsymbol{r}) = V_{\rm latt}(\boldsymbol{r}) + \frac{1}{2}\omega_r'^2r^2. \tag{2}$$

Here, we have employed the proper units to obtain a dimensionless form. The coordinate r and lattice potential $V_{\rm latt}(\boldsymbol{r})$ are replaced by their dimensionless counterparts, while other parameters are defined as $V_0'=V_0/E_{\rm R}$ and $\omega_r'=\omega_r/\omega_{\rm R}$.

The dynamical evolution of the BEC wave function $\psi(\boldsymbol{r},t)$ in such a potential can be described by the time-dependent GP equation,[19–22] which takes the following dimensionless form:

$$\mathrm{i}\frac{\partial\psi(\boldsymbol{r},t)}{\partial t} = \left[-\frac{1}{2}\nabla^2+V(\boldsymbol{r})+g|\psi(\boldsymbol{r},t)|^2-\Omega L_z\right]\psi(\boldsymbol{r},t). \tag{3}$$

Here, we allow the possibility to impose a global rotation with frequency Ω about the z-axis, which couples to the z-component of the angular momentum $L_z=-\mathrm{i}(x\partial_y-y\partial_x)$. The interaction strength g is the two-body interaction in the 2D

plane, which can be varied to a large extent by either tuning the three-dimensional scattering length through a magnetic Feshbach resonance[23] or by changing the z-axis confinement through a confinement-induced resonance.[24–29]

3. Matter-wave interference and transport properties

In this section, we first analyze the case without global rotation, i.e., $\Omega = 0$. One typical experiment to reveal the quasi-periodic lattice structure is the matter-wave interference pattern in the momentum space extracted from time-of-flight images. Figure 2 depicts the simulated results of the BECs released from a combined potential of harmonic trap and OL, where the initial state has the wave function ψ_0 of ground state obtained by applying the imaginary time-splitting sine-spectral (TSSP) method to solve GP Eq. (3) with $V_0' = -0.2$ and $g = 10$.[30] Coherence peaks can be clearly observed in the time-of-flight images, indicating the existence of a long-range order. The peak positions correspond to the linear combinations of elementary basis vectors of the reciprocal lattice. For the case of a 2D cubic lattice ($\varepsilon_1 = \varepsilon_3 = 1, \varepsilon_2 = \varepsilon_4 = 0$) as shown in Fig. 2(a), the distribution of the peaks displays a clear fourfold symmetry associated with the periodic long-rang order. For the case of quasicrystal ($\varepsilon_i = 1$) as shown Fig. 2(b), the distribution of the peaks exhibits a clear eightfold rotational symmetry. Contrasting to the periodic case, some interference peaks also present at smaller momenta inside the first Brillouin zone, revealing the distinctive self-similar fractal structure of the quasicrystals.[18]

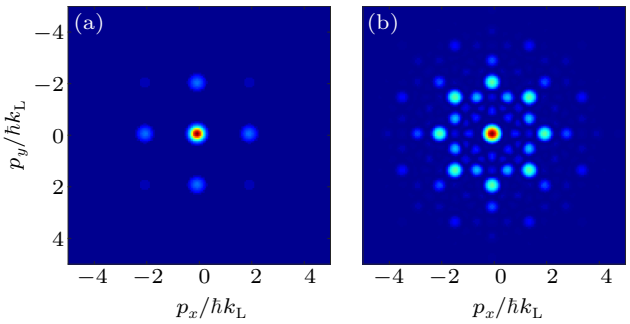


Fig. 2. Matter-wave interference patterns of BECs released from a combined potential of OL and harmonic trap for (a) a 2D cubic OL with $\varepsilon_1 = \varepsilon_3 = 1$ and $\varepsilon_2 = \varepsilon_4 = 0$, and (b) a quasi-periodic OL with $\varepsilon_i = 1$. Other parameters used in this figure are $V_0' = -0.2$, $g = 10$, $\omega_r' = 0.08$, and $\Omega = 0$.

Another characteristic feature of quasicrystal is the transport properties subjected to an external potential gradient. In a periodic lattice, the coherent diffusion of BEC is frictionless as a result of the off-diagonal long-range order. On the other hand, a strong enough disorder would hinder the transport of the particles and lead to an insulating phase. Thus, one would expect an evolution from diffusive to localized states by continuously evolving the OL from periodic to quasi-periodic, although a quantitative description of the process could be very complicated in the presence of interaction.[31–34] To understand the transport properties, we assume that the BECs are initially prepared in a ground state ψ_0 in the combined potential of OL and harmonic trap. At time $t = 0$, the harmonic trap is suddenly switched off while the OL is maintained as the background. The diffusion of the BECs is then described by applying the real TSSP method to numerically solve the GP Eq. (3), and the effect of interaction g and lattice depth V_0' can be investigated as in Fig. 3. Since the quasicrystal structure is a superposition of two square lattices tilted by $\pi/4$, the two lattices can be regarded as their mutual quasi-disorder. Thus, by varying the intensities $\varepsilon_2 = \varepsilon_4 = \Gamma$ of one square lattice while keeping $\varepsilon_1 = \varepsilon_3 = 1$, we can quantitatively analyze the effect of the quasi-disorder on the coherent diffusion of the BECs.

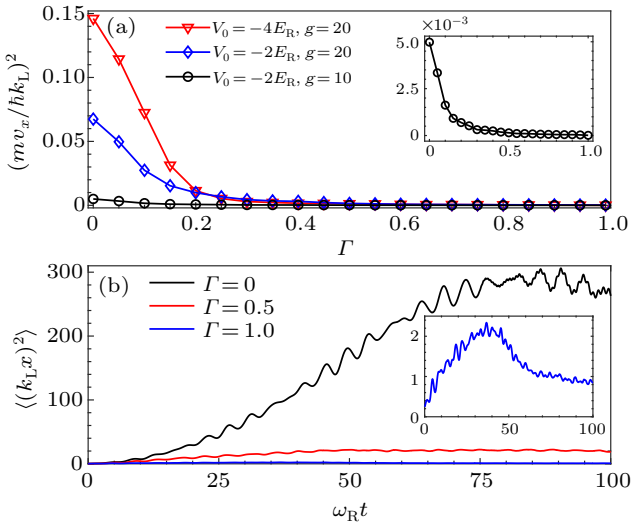


Fig. 3. (a) The mean diffusion velocity along the x-axis by varying the quasi-disorder Γ ($\Gamma = \varepsilon_2 = \varepsilon_4$, $\varepsilon_1 = \varepsilon_3 = 1$) for different interaction strength g and lattice depth V_0'. (b) Time evolution of the mean-square position along the x-axis upon released from the harmonic trap in periodic ($\Gamma = 0$) and quasi-periodic ($\Gamma = 0.5, 1$) lattices with $V_0' = -2$ and $g = 10$. Other parameters used in this figure are $\omega_r' = 0.08$ and $\Omega = 0$.

Upon released from the harmonic trap, the BEC tends to expand in space as the potential and interaction energies are transformed into kinetic energy. In short period of time, the mean-square position of the particles along the x-direction can be approximated by a diffusive behavior $\langle x^2 \rangle \sim v_x^2 t^2$ with v_x the diffusion velocity, as shown in Fig. 3(b) for typical choices of parameters. The expansion gradually ceases and the mean-square position starts to saturate at long-enough time, as the kinetic energy of the particles becomes less than the energy difference between adjacent sites of the underlying lattice. The expansion of the BECs is significantly suppressed by the presence of quasi-disorder, while the overall time-dependence remains the same. This behavior is qualitatively consistent with the results observed in a fivefold symmetric OL.[15] Thus, we can extract the diffusion velocity by fitting the mean-square displacement at short evolving time, and study the change of $\langle v_x \rangle$ versus quasi-disorder Γ as shown

in Fig. 3(a). A crossover from ballistic diffusion to spatial localization is clearly identified with increasing Γ from 0 to 1, during which the lattice configuration evolves from a periodic cubic lattice to an eightfold symmetric quasi-periodic lattice. Although the diffusion–localization crossover is very broad since the system is only mesoscopic with a finite lattice size, we can still observe from all curves in Fig. 3(a) that the particles cease to tunnel when $\Gamma \gtrsim 0.5$.

In the diffusive regime $\Gamma \lesssim 0.5$, the interaction strength g and overall lattice depth V_0' can obviously affect the diffusion velocity $\langle v_x \rangle$. When the lattice depth V_0' of the quasi-periodic OL is fixed [see the black-circle and the blue-diamond lines in Fig. 3(a)], a stronger interaction strength induces more potential energy within the system, which leads to higher kinetic energy and faster diffusion upon releasing from the harmonic trap. On the other hand, a deeper potential well V_0' with fixed interaction g also enhances the diffusion of the BECs [see the blue-diamond and red-triangle lines in Fig. 3(a)]. In fact, since the band width becomes narrower with increasing lattice depth, the interaction and the harmonic trap become relatively stronger in deeper lattices and both lead to faster diffusion.

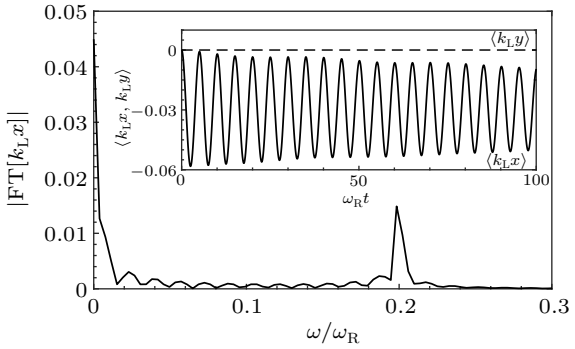


Fig. 4. Fourier transform of the mean position of the BECs in the x-direction in the presence of a tilted optical lattice. Inset: Time evolution of the mean position of the particles. Parameters used in this figure are $V_0' = -2$, $V_{\mathrm{tilt}} = 0.3 k_L x$, $g = 0$, $\omega_r' = 0.08$, and $\Omega = 0$.

Before concluding this section, we investigate the transport of the BECs under a static and homogeneous force. In periodic OLs, a constant driving force induces oscillatory motion of the particles known as Bloch oscillation. In cold atoms, Bloch oscillation can be observed experimentally by applying a constant force via tilt of the optical lattice or gravity.[35–38] In the context of quasicrystals, numerical stimulations of 1D tilted Fibonacci lattice and 2D tilted Penrose tiling predict a quasi-periodic Bloch oscillation.[15,39] Here, we consider a tilting along the x-axis of the eightfold symmetric quasi-periodic OL with $\Gamma = 1$, and analyze the displacement of the particles upon time evolution. In order to eliminate the effect of the interaction which can induce damping and suppression of the Bloch oscillation, we assume that the BEC is non-interacting with $g = 0$, and the initial state is the wave function ψ_0 of ground state for BECs in the combined potential of harmonic trap and OL. At $t = 0$, the harmonic trap is switched off and the tilting potential is applied, so that the system is subjected to a constant force along the x-axis. As shown in the inset of Fig. 4, the particles display quasi-periodic oscillation along the x-direction, while the motion along the y-direction is completely frozen. The appearance of discrete peaks in the Fourier transform of mean position $\langle x(t) \rangle$ also verifies an ordered structure in the quasi-periodic Bloch oscillation, as depicted in Fig. 4.

4. Vortices and solitons

Similar method can be implemented to create a rotating quasi-periodic OL to investigate the dynamical steady states of BECs therein. Here we obtain the stationary solutions of BECs in the combined potential of rotating OL and static harmonic trap by applying the backward Euler pseudospectral (BESP) method to compute the GP Eq. (3),[44] where the lattice depth V_0' and the interaction g are fixed. When the quasi-periodic OL is stationary with $\Omega = 0$ as shown in Fig. 5(a), the particles mainly reside at the center of the harmonic trap, as well as the lattice sites of lower potential energy. As a consequence, the spatial distribution of the particles preserves the eightfold rotational symmetry of the underlying lattice, as one would expect. The corresponding phase is uniform throughout the entire x–y plane without any singularity.

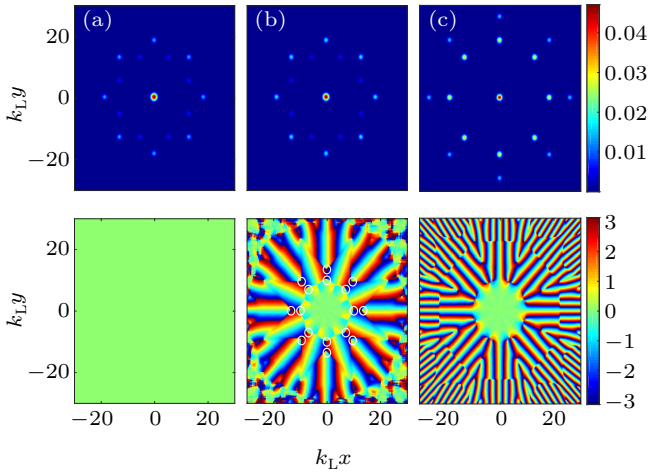


Fig. 5. Spatial distributions of particle density $|\psi_0|^2$ (top row) and phase θ (bottom row) of the ground state of a rotating BEC in the combined potential of harmonic trap and quasi-periodic OL. The rotating frequency is (a) $\Omega = 0$, (b) $\Omega = 0.05$, and (c) $\Omega = 0.1$. Other parameters used are $\omega_r' = 0.1$, $V_0' = -4$, and $g = 50$.

By imposing a relatively slow rotation with $\Omega = 0.05$ and 0.1 as in Figs. 5(b) and 5(c), respectively, the density distribution of the particles still preserves the eightfold symmetry with dominated probability at the trap center and lower potential wells. However, the particles are pushed away from the center as a result of the centrifugal force. Meanwhile, vortices circulating along the same direction of Ω are generated and form an eightfold symmetric lattice structure, as can be identified from the phase plots. In the phase plots, the phase changes

continuously from $-\pi$ (blue) to π (red). The vortices can be identified at the ends of the branch cuts between the phases $-\pi$ and π, as highlighted by the white circles.

Next, we impose an external rotation along the z-axis and analyze the steady state of BECs in the presence of harmonic trap and quasi-periodic OL. This configuration attracts great attention as the underlying lattice potential can have nontrivial effect on the structure and stability of the BEC.[40,41] For example, a structural crossover from a triangular to a square lattice of vortices with increased potential of square OL has been observed in a rotating frame.[42] It was also predicted that fractional quantum Hall effect can present in a dilute gas trapped in a rotating lattice.[43]

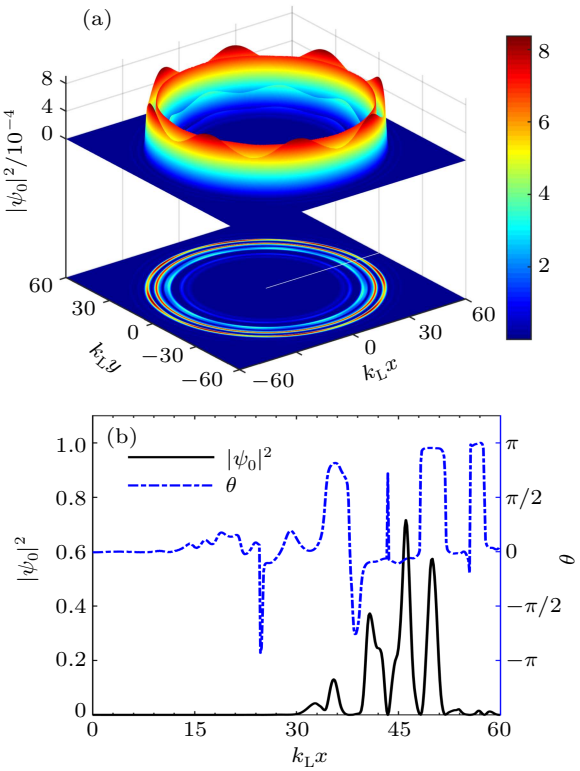


Fig. 6. (a) Spatial distribution of particle density $|\psi_0|^2$ in the x–y plane. Annular solitons along the radial direction can be clearly observed with an approximate axial symmetry. From the surface plot at the top, we find that the density distribution acquires a slight axial modulation, reflecting the presence of background lattice potential with eightfold rotational symmetry. (b) Density and phase profiles along the x-direction. Notice that the phase (blue dashed-dotted) changes abruptly at the positions of zeros or minima of the particle density (black solid). In this plot, the rotating frequency $\Omega = \pi$, while other parameters are the same as those in Fig. 5.

When the external rotation becomes much faster ($\Omega \simeq 10\omega_r'$), the vortices decouple from the background lattice in the azimuthal direction, and an annular soliton mode appears. As depicted in Fig. 6(a), the axial symmetry of the soliton is slightly broken owing to the eightfold symmetry of the underlying quasi-periodic OL. Moreover, vortical solitons have been found in the static fivefold symmetrical OL.[45] By moving along the x-direction from the trap center to the edge, we observe multiple sharp phase jumps as shown in Fig. 6(b). The positions of these jumps coincide well with the zeros or minima of the density distribution. For intermediate rotation frequency $\omega_r' \lesssim \Omega \lesssim 10\omega_r'$, we find that the system is dynamically unstable with no steady solution. Similar instability has been discussed in a simulation of rotating BECs in regular periodic lattices.[46]

5. Conclusion

We discuss the properties of Bose–Einstein condensates in an eightfold symmetric quasicrystal optical lattice by numerically solving the Gross–Pitaevskii equation. Starting from a ground state within a combined potential of optical lattice and harmonic trap, we investigate the matter-wave interference pattern, coherent diffusion of particles, and Bloch oscillation subjected to a constant force, focusing on the characteristic features induced by the quasi-periodic lattice potential. In particular, we find that the matter-wave interference pattern can reveal the underlying eightfold symmetry of the system, a crossover from diffusive to localized behavior can be experienced by continuously tuning the lattice configuration from periodic to quasi-periodic, and the Bloch oscillation also acquires a quasi-periodic nature. In addition, we also study the emergence and structure of vortices and solitons by imposing an overall rotation of the system. For slow rotation, vortices can be generated and form a lattice structure with eightfold symmetry. For fast rotation, however, annular solitons would emerge along the radial direction with approximate axial symmetry. In the intermediate regime, we find that the system is unstable with no steady state solution. Our results provide useful information to the understanding of the crossover regime between ordered and disordered geometries, and can be readily implemented in experiments.

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