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Hidden attractors in a new fractional–order discrete system: Chaos, complexity, entropy, and control*

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This paper studies the dynamics of a new fractional-order discrete system based on the Caputo-like difference operator. This is the first study to explore a three-dimensional fractional-order discrete chaotic system without equilibrium. Through phase portrait, bifurcation diagrams, and largest Lyapunov exponents, it is shown that the proposed fractional-order discrete system exhibits a range of different dynamical behaviors. Also, different tests are used to confirm the existence of chaos, such as 0–1 test and C_0 complexity. In addition, the quantification of the level of chaos in the new fractional-order discrete system is measured by the approximate entropy technique. Furthermore, based on the fractional linearization method, a one-dimensional controller to stabilize the new system is proposed. Numerical results are presented to validate the findings of the paper.

Keywords: discrete chaos, discrete fractional calculus, hidden attractor

PACS: 05.45.–a, 05.45.Ac, 05.45.Gg, 05.30.Pr **DOI:** 10.1088/1674-1056/ab820d

1. Introduction

Throughout the years, interest grew about chaos in discrete dynamical systems and a large number of discrete chaotic systems were proposed.[1–6] Discrete chaos has been shown to provide accurate modeling of numerous natural physical phenomena in the fields of biology, chemistry, and physics.[7] The control and synchronization of such systems also known as maps have been widely investigated.[8–11]

In the chaos literature, there is an active interest in the study of chaotic systems without equilibrium points due to its importance in engineering application. The attractors associated with these systems are called hidden attractors. Such attractors are namely observed in the nonlinear system with no equilibrium point, infinite equilibrium, or with only one stable equilibrium, where their basin of attraction does not intersect with the neighborhoods of the equilibrium.[12,13] Recently, many research works with respect to continuous chaotic systems with hidden attractors have been published.[14,15] Those works have demonstrated that such systems can be applied in different engineering fields, such as aircraft control systems. Compared with the continues-time system with hidden attractors, the fractional-order discrete counterpart is relatively new. There are only a few investigations of hidden attractors in discrete chaotic systems.[16–18]

In recent years, discrete fractional calculus has received considerable attention. A few researchers have attempted to develop a theoretical framework for fractional calculus on time scale.[19–23] Recently, researchers have diverted their attention to discrete chaotic systems with fractional order. Some conventional discrete chaotic systems have been extended to the fractional case using discrete fractional calculus.[24–30] From what has been reported, the fractional-order discrete systems are sensitive to variation in the fractional order in addition to their natural sensitivity to variation in the initial condition and parameter.

In the current paper, we consider a new fractional-order discrete system without equilibrium and focus on the study of the effect of the fractional order on the dynamic behaviors of this new system. Bifurcations and chaos properties are obtained through the largest Lyapunov exponents, 0–1 test, C_0 algorithm, and entropy. The remainder of this paper is organized as follows. In Section 2, the basic notions related to discrete fractional calculus are introduced. Section 3 presents the new fractional-order discrete system. In Section 4, the complex dynamics of the proposed fractional-order discrete system are investigated numerically by varying the fractional order, initial conditions, and bifurcation parameters. In Section 5, chaos and complexity of the new system are confirmed by using 0–1 test, C_0 complexity, and entropy. Section 6 proposes an active control scheme to stabilize the states of the fractional-order

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discrete system. Section 7 contains the conclusion.

2. Theoretical background

In this section, we recall some basic concepts of discrete fractional calculus and stability result that will aid the reader in understanding the analysis in the manuscript. We shall use the time scale $N_a = \{a, a+1, a+2, \ldots\}$, where $a \in \mathbb{N}$.

Definition 1 The ν-th fractional sum of $X(t) : \mathbb{N}_a \to \mathbb{R}$ is defined as[22]

$$\Delta_a^{-\nu} X(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} X(s), \tag{1}$$

where $t \in \mathbb{N}_{a+\nu}$ and $\nu > 0$. The term $t^{(\nu)}$ denotes the falling function defined in terms of the Gamma function Γ as

$$t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}. \tag{2}$$

Definition 2 The ν-th Caputo like difference operator of a function $X(t)$[32] is defined as

$$\begin{aligned} {}^{\mathrm{C}}\Delta_a^{\nu} X(t) &= \Delta_a^{-(n-\nu)} \Delta^n X(t) \\ &= \frac{1}{\Gamma(n-\nu)} \sum_{s=a}^{t-(n-\nu)} (t-s-1)^{(n-\nu-1)} \Delta_s^n X(s), \end{aligned} \tag{3}$$

for $\nu \notin \mathbb{N}$, $t \in \mathbb{N}_{a+n-\upsilon}$, and $n = \lceil \nu \rceil + 1$.

Theorem 1[23] For the fractional difference equation

$$\begin{cases} {}^{\mathrm{C}}\Delta_a^{\nu} u(t) = f(t+\nu-1, u(t+\nu-1)), \\ \Delta^k u(a) = u_k, \ n = \lceil \nu \rceil + 1, \ k = 0, 1, \ldots, n-1, \end{cases} \tag{4}$$

the equivalent discrete integral equation can be obtained as

$$\begin{aligned} u(t) \ = \ & u_0(t) + \frac{1}{\Gamma(\nu)} \sum_{s=a+n-\nu}^{t-\nu} (t-\sigma(s))^{(\nu-1)} \\ & \times f(s+\nu-1, u(s+\nu-1)), \ t \in \mathbb{N}_{\alpha+n}, \end{aligned} \tag{5}$$

where

$$u_0(t) = \sum_{k=0}^{n-1} \frac{(t-a)^{(k)}}{\Gamma(k+1)} \Delta^k u(a). \tag{6}$$

Theorem 2[31] The zero equilibrium of the linear fractional-order discrete-time system

$${}^{\mathrm{C}}\Delta_a^{\nu} \boldsymbol{e}(t) = \boldsymbol{M} \boldsymbol{e}(t+\nu-1), \tag{7}$$

where $\boldsymbol{e}(t) = (e_1(t), \ldots, e_n(t))^{\mathrm{T}}$, $0 < \nu \leq 1$, $\boldsymbol{M} \in \mathbb{R}^{n \times n}$, and $\forall t \in \mathbb{N}_{a+1-\nu}$, is asymptotically stable if

$$\lambda \in \left\{ z \in \mathbb{C} : \ |z| < \left(2\cos \frac{|\arg z| - \pi}{2-\nu} \right)^{\nu} \text{ and } |\arg z| > \frac{\nu\pi}{2} \right\} \tag{8}$$

for all the eigenvalues λ of $\boldsymbol{M}$.

3. Description of the new fractional-order discrete system

A new fractional-order discrete system is reported in this paper, and is described by a set of three-dimensional fractional-order difference equation as follows:

$$\begin{cases} {}^{\mathrm{C}}\Delta_a^{\nu} x(t) = y(t-1+\nu) - x(t-1+\nu), \\ {}^{\mathrm{C}}\Delta_a^{\nu} y(t) = z(t-1+\nu) - y(t-1+\nu), \\ {}^{\mathrm{C}}\Delta_a^{\nu} z(t) = -a_1 z^2(t-1+\nu) + a_2 x(t-1+\nu) y(t-1+\nu) \\ \qquad + a_3 x(t-1+\nu) z(t-1+\nu) + a_4 - z(t-1+\nu), \end{cases} \tag{9}$$

where a is the starting point, $0 < \nu < 1$ is the fractional order, x, y, z are the state variables, and a_1, a_2, a_3, a_4 are the bifurcation parameters. The equilibrium points of the fractional-order discrete system (9) are found by solving the following system of equation:

$$\begin{cases} y - x = 0, \\ z - y = 0, \\ -a_1 z^2 + a_2 xy + a_3 xz - z + a_4 = 0. \end{cases} \tag{10}$$

From Eq. (10), we have the simplified form

$$(-a_1 + a_2 + a_3) z^2 + a_4 - z = 0. \tag{11}$$

The aim is to demonstrate that the fractional-order discrete system (9) can generate hidden chaotic attractors, for that we only need to determine the cases in which equation (11) has no solution. In particular, when $-a_1 + a_2 + a_3 \neq 0$ and $1 < 4a_4(-a_1 + a_2 + a_3)$, equation (11) has no solution, which means that the system (9) has no equilibrium. We must therefore impose the condition (11) on the system parameters a_i in order to obtain the hidden attractors.

To continue with our analysis, we need to define the numerical formula of the system (9). Using Theorem 1, the state solution of the fractional-order discrete system (9) can be obtained as follows:

$$\begin{cases} x(t) = x(a) + \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t-\nu} (t-\sigma(s))^{(\nu-1)} \\ \qquad \times (y(s+\nu-1) - x(s+\nu-1)), \\ y(t) = y(a) + \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t-\nu} (t-\sigma(s))^{(\nu-1)} \\ \qquad \times (z(s+\nu-1) - y(s+\nu-1)), \\ z(t) = z(a) + \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t-\nu} (t-\sigma(s))^{(\nu-1)} \\ \qquad \times (-a_1 z^2(s+\nu-1) + a_2 x(s+\nu-1) y(s+\nu-1) \\ \qquad + a_3 x(s+\nu-1) z(s+\nu-1) + a_4 - z(s+\nu-1)). \end{cases} \tag{12}$$

By replacing the discrete kernel function $\frac{(t-\sigma(s))^{(\nu-1)}}{\Gamma(\nu)}$ by $\frac{\Gamma(t-s)}{\Gamma(\nu)\Gamma(t-s-\nu+1)}$ and assuming that $a = 0$, the above equation is converted to

$$\begin{cases} x(n) = x(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (y(j-1) - x(n-1)), \\ y(n) = y(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (z(j-1) - y(j-1)), \\ z(n) = z(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \big(-a_1 z^2(j-1) \\ \qquad + a_2 x(j-1) y(j-1) + a_3 x(j-1) z(j-1) + a_4 - z(j-1) \big), \end{cases} \tag{13}$$

where $x(0), y(0)$, and $z(0)$ are the initial conditions. For the system parameters choice $a_1 = 1$, $a_2 = 0.7$, $a_3 = 1$, $a_4 = 0.7$, initial conditions $x(0) = 1.5$, $y(0) = 2.84$, $z(0) = -1.2$, and fractional order $\nu = 0.998$, system (9) can display a hidden attractor as exhibited in Fig. 1. More basic characteristics of this fractional-order discrete system will be given in the following section.

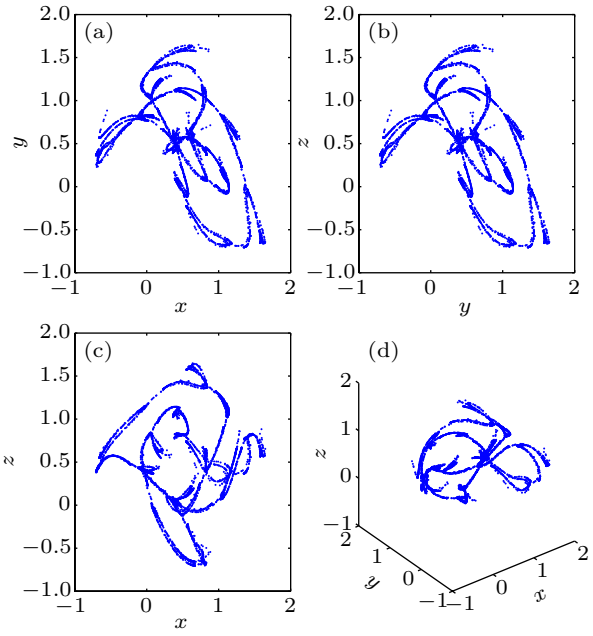


Fig. 1. The hidden chaotic attractor of the fractional-order discrete system for $\nu = 0.998$.

4. Bifurcation diagrams and largest Lyapunov exponents

In this section, the bifurcation diagrams, largest Lyapunov exponent, and phase portraits are used to analyze the dynamic behavior of the fractional-order discrete system (9). Firstly, we investigate the bifurcation diagrams of the fractional-order discrete system (9) as the system parameter a_1 is varied when the other parameters are taken as $a_2 = 0.7$, $a_3 = 1$, $a_4 = 0.7$. The bifurcation diagrams are obtained by plotting the local maxima of the state $x(n)$ in term of system parameter a_1 for three different values of ν as shown in Fig. 2. By comparing Fig. 2(a) with Figs. 2(b) and 2(c), we find that the interval in which chaos exists shrinks as ν decreases. In particular, when ν decreases from 0.9935 to 0.9434, the area of chaotic motion shrinks from $[1, 1.008] \cup [1.66, 1.267] \cup [1.045, 1.196]$ to $[1.022, 1.116] \cup [1.164, 1.216]$. When the parameters are assigned as $a_1 = 1, a_2 = 0.7, a_4 = 0.7$, the bifurcation diagrams with a_3 varying are shown in Fig. 3. From this diagram, it can be seen that the bifurcation structure is apparently the same. When ν reduces to 0.9434, the fractional-order discrete system (9) becomes totally periodic in the interval $[1, 1.141]$. Next, the dynamic behavior of the fractional-order discrete system (9) with parameter a_4 varied is studied in Figs. 4(a) and 5(a). It can be seen that the states of system (9) change qualitatively with the variation of order ν and parameter a_4. The bifurcation diagram of the fractional-order discrete system (10) with $\nu = 0.9935$ is plotted in Fig. 4(a). As can be seen, when $0.87 \leq \nu \leq 0.8$, system (9) exhibits chaotic behavior. A series of period windows occur in the region $(0.7, 0.8)$ as ν reduces to 0.9434 (see Fig. 5(a)). Although bifurcation plots are a useful tool in determining the existence of chaos and quantifying it, the most agreed upon tool is Lyapunov exponents. The largest Lyapunov exponents of this new fractional-order discrete system are calculated by the Jacobian matrix algorithm.[33] Figures 4(b) and 5(b) show the estimated largest Lyapunov exponents for $\nu = 0.9935$, $a_4 = 0.7$ and for $\nu = 0.9434$, $a_4 = 0.7$, respectively. Obviously, when we select $\nu = 0.9935$, the corresponding largest Lyapunov exponent is positive, which implies that system (9) is in chaotic state. And when the fractional order is set to $\nu = 0.9434$, the largest Lyapunov exponent is zero, which implies that system (9) is in periodic state.

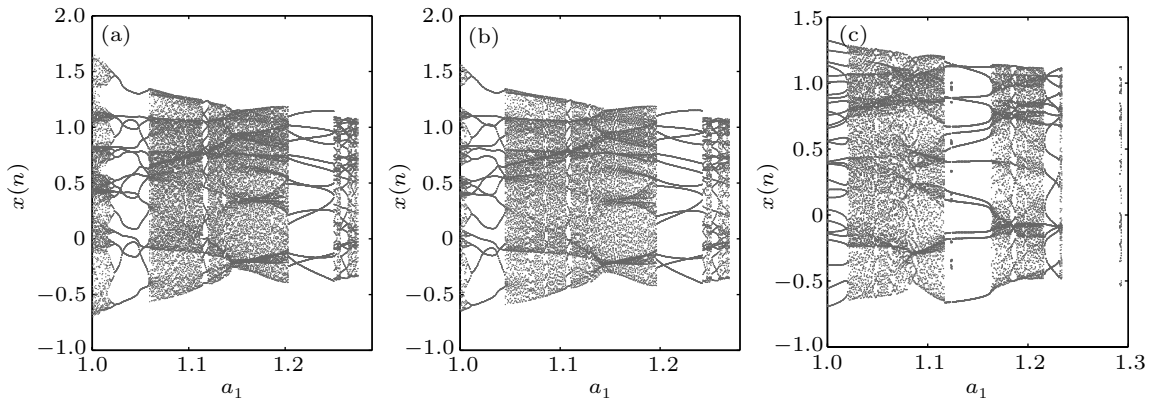


Fig. 2. Bifurcation diagrams of the fractional-order discrete system (9) versus a_1 for (a) $\nu = 1$, (b) $\nu = 0.9935$, and (c) $\nu = 0.9434$.

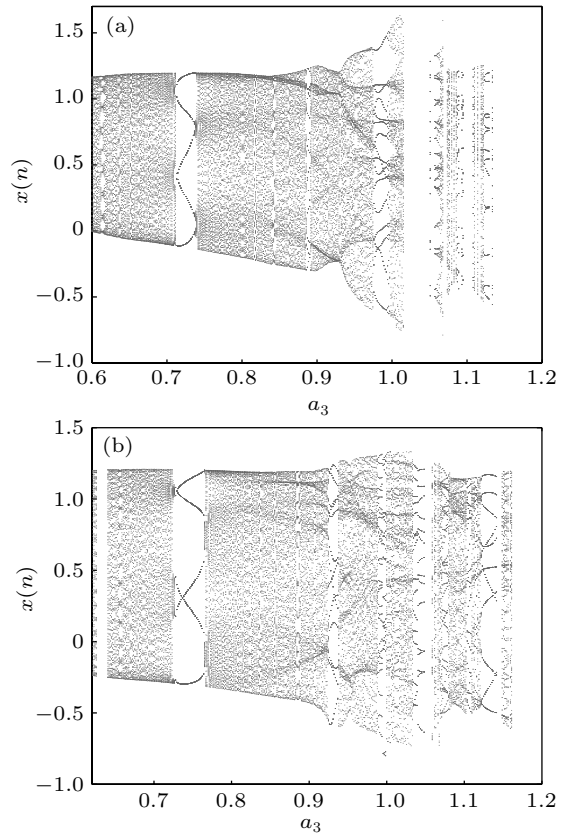


Fig. 3. Bifurcation diagrams of the fractional-order discrete system (9) versus a_3 for (a) $\nu = 0.9935$, and (b) $\nu = 0.9434$.

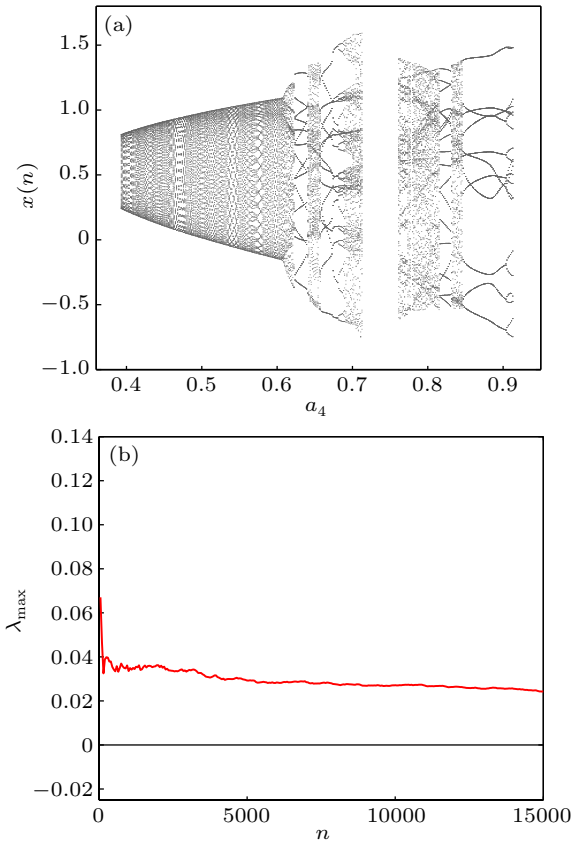


Fig. 4. (a) Bifurcation diagrams of the fractional-order discrete system (9) versus a_4 for $\nu = 0.9935$. (b) The estimated largest Lyapunov exponent of the fractional-order discrete system for $\nu = 0.9935$ and $a_4 = 0.7$.

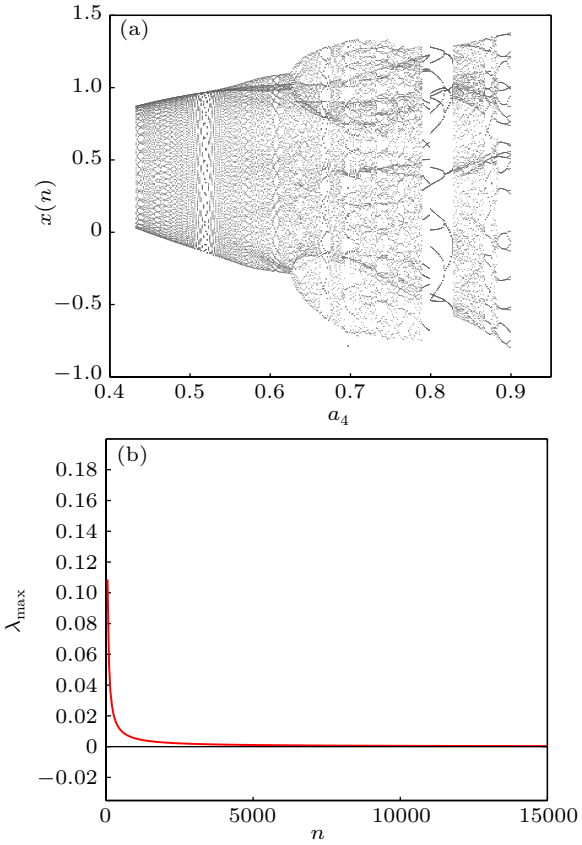


Fig. 5. (a) Bifurcation diagrams of the fractional-order discrete system (9) versus a_4 for $\nu = 0.9434$. (b) The estimated largest Lyapunov exponent of the fractional-order discrete system for $\nu = 0.9434$ and $a_4 = 0.7$.

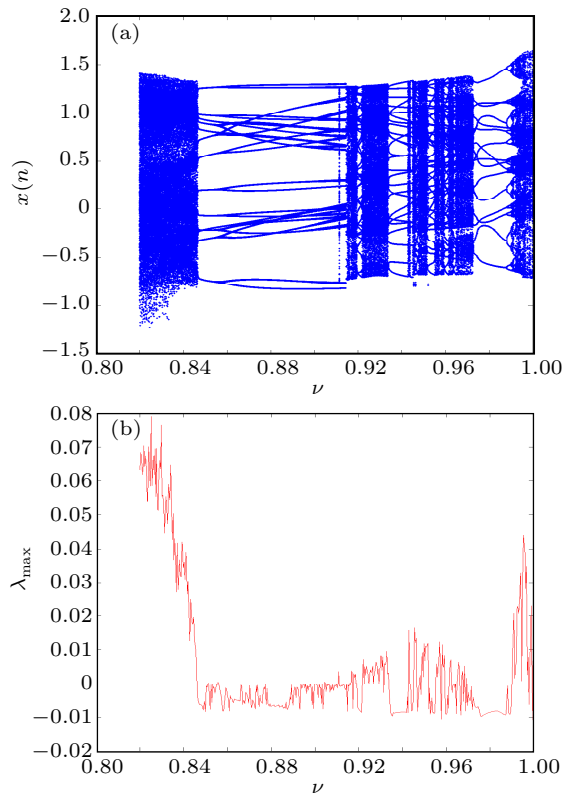


Fig. 6. (a) Bifurcation diagram versus ν for $a_1 = 1, a_2 = 0.7, a_3 = 1$, and $a_4 = 0.7$. (b) Largest Lyapunov exponent diagram corresponding to diagram (a).

In order to observe the influence of the fractional order ν on the dynamic of the fractional-order discrete system, we fix the system parameters $a_1 = 1$, $a_2 = 0.7$, $a_3 = 1$, $a_4 = 0.7$, and vary ν in the interval $[0.8, 1]$. Figure 6 shows that the fractional-order discrete system displays limit cycles, periodic and chaotic motions with the variation of order ν. In particular, we find that when $0 \leq \nu \leq 0.81$, the system diverges to infinity. When $\nu \in [0.9153, 0.9339] \cup [0.9429, 0.9726] \cup [0.9883, 1]$, the largest Lyapunov exponent changes its value quickly between negative and positive, which implies that there are periodic windows in the chaotic region. When ν is decreased from 0.915, the fractional-order discrete system converges to a periodic orbit. When $\nu \in [0.8211, 0.8467]$, the system has more complex hidden chaotic attractors as its largest Lyapunov exponent takes the highest values. The phase portraits in the x–y plane for different values of order ν are shown in Fig. 7.

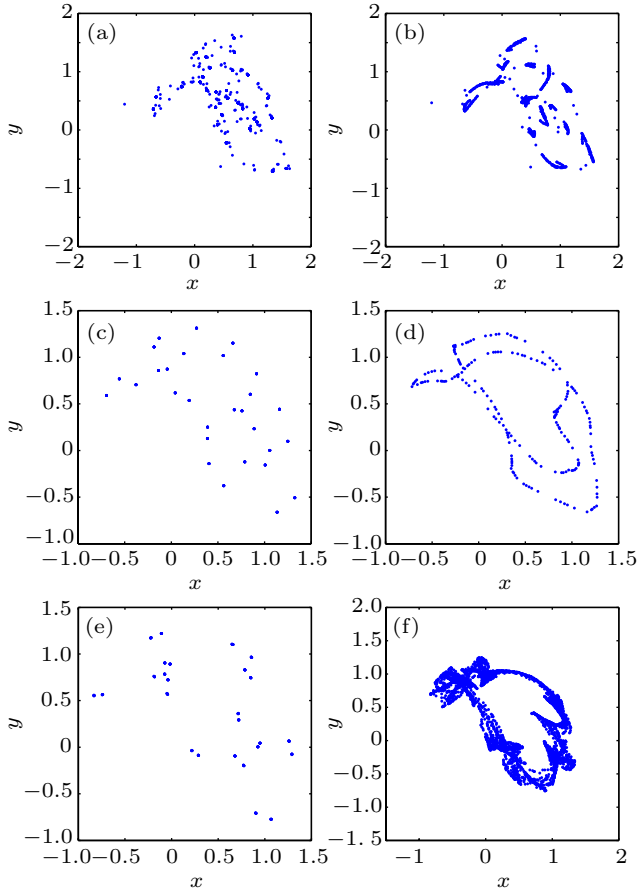


Fig. 7. Phase portraits of the fractional-order discrete system (9): (a) periodic orbit for $\nu = 0.9996$, (b) chaotic hidden attractor for $\nu = 0.9935$, (c) periodic attractor for $\nu = 0.9434$, (d) two limit cycles for $\nu = 0.9177$, (e) periodic orbit for $\nu = 0.895$, (f) hidden chaotic attractor for $\nu = 0.84$.

5. The 0–1 test and complexity

5.1. The 0–1 test

Another method to validate the existence of chaos in the fractional-order discrete system is the 0–1 test method.[34] The 0–1 test method for chaos is a binary test that reveals chaotic behavior in nonlinear systems, where the input is a series of data and the output is 0 or 1 depending on whether the dynamic is chaotic or non chaotic. We briefly review the steps of the test algorithm. First, we consider the state $\{x(j)\}$ of the system (9). Then, we define the translation components p and q as follows:

$$p(n) = \sum_{j=1}^{n} x(j)\cos(jc), \quad q(n) = \sum_{j=1}^{n} x(j)\sin(jc), \tag{14}$$

where c in an arbitrary constant in the rang $(0, 2\pi)$. Here, the dynamics of the components p–q provides a visual test. Basically, if the dynamic of the system is regular, then the behavior of trajectories in the p–q plane is bounded, whereas if the dynamic is chaotic, then the p–q trajectories show Brownian like behavior. Next, we define the mean square displacement $M_c(n)$ as

$$M_c(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} [p(j+n) - p(j)]^2 + [q(j+n) - q(j)]^2, \quad \text{for } n \ll N. \tag{15}$$

Finally, we define the asymptotic growth rate K as

$$K = \text{median}(K_c) = \lim_{n\to\infty} \frac{\log M_c(n)}{\log(n)}. \tag{16}$$

Now, we apply the 0–1 test to the discrete series data $x(n)$ for different values of ν where the system parameters are assigned as $a_1 = 1$, $a_2 = 0.7$, $a_3 = 1$, $a_4 = 0.7$. The translation components p–q of the fractional-order discrete system (9) are calculated and illustrated in Fig. 8. In particular, when $\nu = 0.9996$, $\nu = 0.9434$, $\nu = 0.895$ and $\nu = 0.9177$, the trajectories in the p–q plane show bounded behavior, system (9) is in periodic state. When ν is set to $\nu = 0.9935$ and $\nu = 0.84$, the trajectories in the p–q plane show unbounded trajectories, system (9) is chaotic, which confirms very well the results in Fig. 8.

5.2. The C_0 complexity

In order to observe the effect of fractional order ν on the dynamic properties of the fractional-order discrete system (9), we measure the complexity of the new system using C_0 algorithm.[35] Suppose that $\{x(j); j = 0, 1, \ldots, N-1\}$ is a series of data and

$$\psi_N(j) = \frac{1}{N} \sum_{k=0}^{N-1} x(k) \exp^{-2\pi \mathrm{i}\left(\frac{kj}{N}\right)}, \; j = 0, 1, 2, \ldots, N-1, \tag{17}$$

is the corresponding Fourier transformation. The mean square value of ψ_N can be written as

$$G_N = \frac{1}{N} \sum_{j=0}^{N-1} |\psi_N(j)|^2. \tag{18}$$

Next, import a parameter r and keep all the amplitude

spectra components unchanged if their square is larger than rG_N. Otherwise, replace all the parts that are equal to or less than rG_N with zero, as follows:

$$\bar{\psi}_N(j) = \begin{cases} \psi_N(j), & \text{if } |\psi_N(j)|^2 > rG_N, \\ 0, & \text{if } |\psi_N(j)|^2 \le rG_N. \end{cases} \tag{19}$$

Then, define the inverse Fourier transformation of $\bar{\psi}_N$ as

$$\bar{x}(k) = \frac{1}{N}\sum_{j=0}^{N-1} \bar{\psi}_N(j) \exp^{2\pi \mathrm{i}\left(\frac{kj}{N}\right)}, \ k = 0,1,2,\ldots,N-1. \tag{20}$$

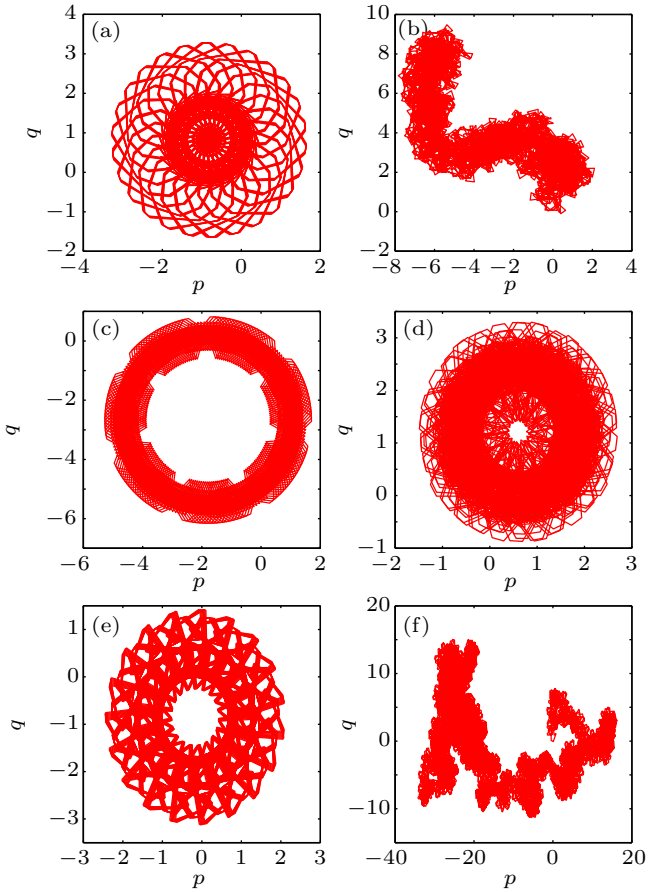


Fig. 8. The 0–1 test of the fractional-order discrete system (9): (a) $\nu = 0.9996$, (b) $\nu = 0.9935$, (c) $\nu = 0.9434$, (d) $\nu = 0.9177$, (e) $\nu = 0.895$, (f) $\nu = 0.84$.

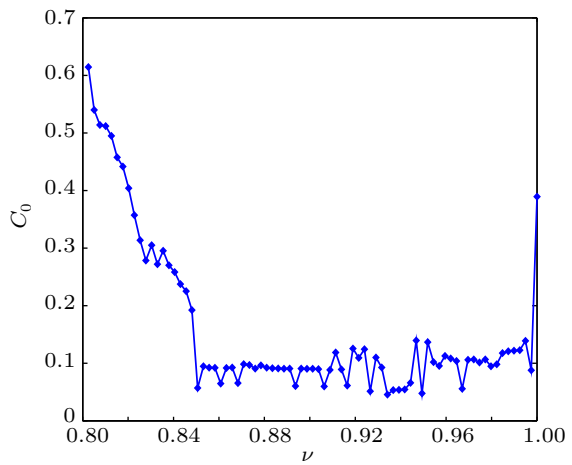


Fig. 9. The C_0 complexity of the fractional-order discrete system (9) versus ν with $a_1 = 1$, $a_2 = 0.7$, $a_3 = 1$, and $a_4 = 0.7$.

Finally, the C_0 complexity is given by

$$C_0 = \frac{\sum_{k=0}^{N-1} |x(k) - \bar{x}(k)|^2}{\sum_{k=0}^{N-1} |x(k)|^2}. \tag{21}$$

Fixing system parameters to $a_1 = 1$, $a_2 = 0.7$, $a_3 = 1$, $a_4 = 0.7$ and $\nu \in [0.8, 1]$, the result of C_0 complexity is plotted in Fig. 9. We notice that the complexity of the fractional-order discrete system increases when ν passes to the range $[0.8211, 0.8467]$, which agrees well with the result in Fig. 7.

5.3. Approximate entropy

The complexity of the fractional-order discrete system (9) can be described by employing approximate entropy (ApEn),[36,37] which is briefly described as follow. Consider a set of discrete data $x(1), x(2), \ldots, x(n)$, define $n-m+1$ vectors as follows:

$$X(i) = [x(i), \ldots, x(i+m-1)]. \tag{22}$$

These vectors represent m consecutive x values starting with the i-th data. With tolerance r and for each $i \in [1, n-m+1]$, define the following equation:

$$C_i^m(r) = \frac{K}{n-m+1}, \tag{23}$$

where K is the number of $X(i)$ having $d(X(i), X(j)) \le r$. In this case, $d(X(i), X(j))$ denote the maximum absolute difference between $X(i)$ and $X(j)$. We define the approximate entropy by

$$\mathrm{ApEn} = \phi^m(r) - \phi^{m+1}(r), \tag{24}$$

where $\phi^m(r)$ is considered as

$$\phi^m(r) = \frac{1}{n-m-1} \sum_{i=1}^{n-m+1} \log C_i^m(r). \tag{25}$$

Figure 10 shows the ApEn of the proposed fractional-order discrete system (9) for different fractional order values when $a_1 = 1$, $a_2 = 0.7$, $a_3 = 1$, $a_4 = 0.7$. As one can see, the complexity of the fractional- order discrete system (9) varies as we vary ν. When $\nu \in [0.8211, 0.8467]$, the system (9) is more complex. Therefore, we must be aware of the selected fractional order in order to have a relatively high structural complexity.

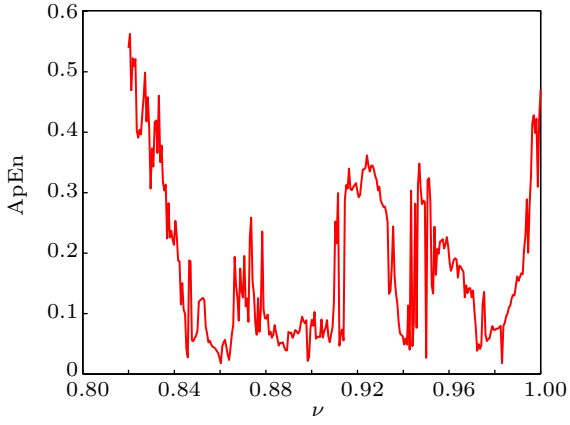


Fig. 10. Approximate entropy of the fractional-order discrete system (9) versus ν with $a_1 = 1$, $a_2 = 0.7$, $a_3 = 1$, and $a_4 = 0.7$.

6. Chaos control

In this section, we present a control law to stabilize the states of the fractional-order discrete chaotic system (9). Moreover, the control law will force the states of the map (9) to zero in sufficient time.

Theorem 3 The three-dimensional fractional-order discrete system (9) can be stabilized under the following one-dimensional control law:

$$u(t) = z(t)^2 - 0.7x(t)y(t) - x(t)z(t) - 0.7, \quad t \in \mathbb{N}_{a+n-\nu}. \tag{26}$$

Proof The controlled fractional-order discrete system that is obtained by adding the time varying control parameter $u(t)$ to the system (9) is given by

$$\begin{cases} {}^{\mathrm{C}}\Delta_a^{\nu} x(t) = y(t-1+\nu) - x(t-1+\nu), \\ {}^{\mathrm{C}}\Delta_a^{\nu} y(t) = z(t-1+\nu) - y(t-1+\nu), \\ {}^{\mathrm{C}}\Delta_a^{\nu} z(t) = -a_1 z(t-1+\nu)^2 + a_2 x(t-1+\nu)y(t-1+\nu) \\ \qquad\quad + a_3 x(t-1+\nu)z(t-1+\nu) + a_4 - z(t-1+\nu) \\ \qquad\quad + u(t-1+\nu). \end{cases} \tag{27}$$

If we substitute the proposed control law (26) into Eq. (27), we obtain the simplified dynamics

$$\begin{cases} {}^{\mathrm{C}}\Delta_a^{\nu} x(t) = -x(t-1+\nu) + y(t-1+\nu), \\ {}^{\mathrm{C}}\Delta_a^{\nu} y(t) = -y(t-1+\nu) + z(t-1+\nu), \\ {}^{\mathrm{C}}\Delta_a^{\nu} z(t) = -z(t-1+\nu). \end{cases} \tag{28}$$

The system (28) can be written in the compact form

$$\begin{aligned} &{}^{\mathrm{C}}\Delta_a^{\nu} \left(x(t), y(t), z(t)\right)^{\mathrm{T}} \\ &= \boldsymbol{M} \left(x(t-1+\nu), y(t-1+\nu),\ z(t-1+\nu)\right)^{\mathrm{T}}, \end{aligned} \tag{29}$$

where

$$\boldsymbol{M} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}. \tag{30}$$

The eigenvalues of matrix $\boldsymbol{M}$ are $\lambda_1 = \lambda_2 = \lambda_3 = -1$, and satisfy the conditions

$$|\lambda_i| < \left(2\cos\frac{|\arg\arg\lambda_i| - \pi}{2-\nu}\right)^{\nu}, \quad |\arg\lambda_i| > \frac{\nu\pi}{2}. \tag{31}$$

Therefore, using Theorem 2, the zero equilibrium of Eq. (29) is asymptotically stable, and so the three-dimensional fractional-order discrete chaotic system (9) is stabilized.

To illustrate the result given in Theorem 3, we display the evolution after control of the chaotic states of system (9) when $a = 0$, $(a_1, a_2, a_3, a_4) = (1, 0.7, 1, 0.7)$, initial conditions $x_0 = 1.5$, $y_0 = 2.84$, $z_0 = -1.2$, and $\nu = 0.84$ in Fig. 11. From Fig. 11, it is easily observed that the evolution of the system states converges, and so the control law given in Eq. (26) stabilizes the new fractional-order discrete chaotic system.

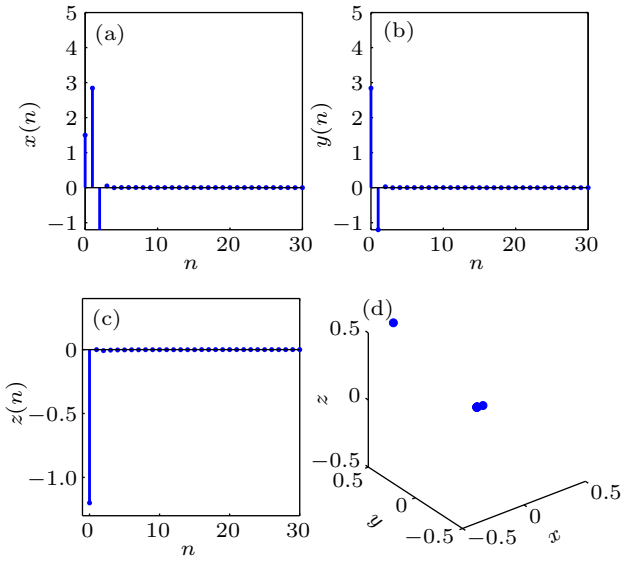


Fig. 11. The states and phase space of the controlled system (9).

7. Conclusion

A fractional-order discrete system without equilibrium, which exhibits rich dynamics and hidden chaotic attractors, was examined in this work. The proposed fractional-order discrete system has constructed based on the Caputo-like difference operator with no equilibrium. There are some important concluding remarks to indicate here.

First, based on the fractional difference order, we have shown that varying the fractional order impacts the parameter interval over which chaos is observed, as well as the shape of the resulting attractors. The 0–1 test and C_0 complexity have confirmed the results found by the phase-space portraits and bifurcation diagrams. We have also quantified the level of chaos present in the proposed map by means of the approximate entropy measure. Second, the suggested stabilization controller can impose the fractional-order discrete system states to approach to zero asymptotically. Third, the executed numerical simulations have supported the dynamics, complexity, and control results. Furthermore, many applications of such fractional maps are expected to appear in the near future.

In future works, we will aim to define new chaotic discrete systems using fractal derivative with fractional kernel functions. This new derivative will be more suitable for modeling real world phenomena. Preliminary idea concerning this issue can be found in Refs. [38–41].

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