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Reduction of entropy uncertainty for qutrit system under non-Markov noisy environment*

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We explore the entropy uncertainty for qutrit system under non-Markov noisy environment and discuss the effects of the quantum memory system and the spontaneously generated interference (SGI) on the entropy uncertainty in detail. The results show that, the entropy uncertainty can be reduced by using the methods of quantum memory system and adjusting of SGI. Particularly, the entropy uncertainty can be decreased obviously when both the quantum memory system and the SGI are simultaneously applied.

Keywords: entropy uncertainty relation, non-Markov noisy, qutrit and spontaneously generated interference

PACS: 03.67.Hk, 03.65.Ud, 73.63.Nm, 85.35.Be **DOI:** 10.1088/1674-1056/ab7ea5

1. Introduction

The uncertainty principle, originally proposed by Heisenberg, clearly illustrates the difference between classical and quantum mechanics.[1] Initially, the expression for the Heisenberg uncertainty principle with respect to the position x and momentum p_x is formulated as $\Delta x \cdot \Delta p_x \geq \hbar/2$, and later was generalized by Kennard and Robertson[2] to two arbitrary observables R and S with a standard deviation

$$\Delta R \cdot \Delta S \geq \frac{|\langle [R,S] \rangle|}{2}, \tag{1}$$

where the variance $\Delta Q = \sqrt{\langle Q^2 \rangle - \langle Q \rangle^2}$ (Q is an arbitrary observable). $\langle \cdot \rangle$ is the expectation of the observable in a quantum system ρ, and $[R,S]$ is the commutator. However, because the lower bound of the above uncertainty relation is state-dependent, the inequality will be trivial if the expectation value of the commutator $[R,S]$ is equal to zero and the variances only contain the second-order statistical moments of the quantum fluctuations of measurements. To remove this pitfall and to obtain a more general form which is independent of the quantum state, Deutsch recast that the uncertainty can be quantified by the Shannon entropy instead of the standard deviation. After a further improvement and proof, the classical entropy uncertainty relation is defined by[3]

$$H(R) + H(S) \geq \log_2 \frac{1}{c}, \tag{2}$$

where $H(R)$ $(H(S))$ is the Shannon entropy of the observable $R(S)$, and $c = \max_{i,j} |\langle r_i | s_j \rangle|^2$ is the maximum overlap of eigenvectors $|r_i\rangle$ and $|s_j\rangle$ corresponding to a pair of incompatible observables R and S, respectively. More recently, a new entropy uncertainty relation has been derived by Renes and Boileau[4,5] and later proved by Berta *et al.*,[6,7] which is named quantum memory assisted entropy uncertainty relation (QMA-EUR)

$$S(R|B) + S(Q|B) \geq \log_2 \frac{1}{c} + S(A|B), \tag{3}$$

where $S(A|B) = S(\rho_{AB}) - S(\rho_B)$ is the conditional entropy and $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy. $S(X|B)$ with $X \in \{R,Q\}$ is the conditional entropy of the post-measurement state $\rho_{XB} = \sum_i (|\psi_i\rangle\langle\psi_i| \otimes I)\rho_{AB}(|\psi_i\rangle\langle\psi_i| \otimes I)$ after particle A is measured by X, where $|\psi_i\rangle$ is the eigenstate of the observable X and I is an identity operator in the Hilbert space of particle B. Generally, one can see the new uncertainty relation in terms of the uncertainty game between two players, Alice and Bob. Firstly Bob sends a particle A to Alice, which may entangle with his quantum memory B in general. Secondly, Alice measures either R or S and notes her outcome. Finally, Alice announces her measurement choice to Bob and Bob's task is to minimize his uncertainty about Alice's measurement outcome.

Recently, the improvement of this new entropic uncertainty principle as well as its dynamics have attracted increasing attention and there are potential applications such as for witnessing entanglement,[6–11] teleportation,[12] quantum phase transitions,[13] and cryptography.[14–16] In this paper, we explore the behavior of the entropic uncertainty for a qutrit system under non-Markov noisy environment. In fact, any systems are essentially open and unavoidably interact with their surrounding environment, which induces decoherence or dissipation phenomena.[17,18] Researchers usually tend to pay attention to the entropy uncertainty of qubit system. Feng *et al.* explored the quantum-memory-assisted entropic uncertainty relation under noises and found that the

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unital noises only increase the entropic uncertainty, whereas the amplitude-damping non-unital noises may reduce the entropic uncertainty in the long-time limit.[19] Fan *et al.* studied the relations between the quantum-memory-assisted entropic uncertainty principle, quantum teleportation, and entanglement witness.[12] Zhang *et al.* studied the quantum-memory-assisted entropic uncertainty principle for the qubit system in non-Markovian environments.[20] However, there are few researches on the entropy uncertainty of high-dimensional quantum systems. Quantum systems with a higher dimension may have advantages over ones with a lower dimension during quantum information processing[21,22] since they provide higher channel capacities, more secure cryptography as well as superior quantum gates.[23,24] Guo *et al.* studied the entropic uncertainty relation for a qutrit system under decoherence.[25] Zhang *et al.* studied entropic uncertainty relation of a two-qutrit Heisenberg spin model in nonuniform magnetic fields and its dynamics under intrinsic decoherence.[20] Recently, the effects of spontaneously generated interference (SGI) stimulate much interests. For instance, researchers studied the effect of SGI on absorption in a V-type system[26] and in three-level systems the nonlinear Kerr absorptions are shown to be greatly enhanced via SGI.[27] The entropy uncertainty of qutirt in non-Markovian environments has not been reported, so our work further enriches the study of the entropy uncertainty.

The paper is organized as follows. In Section 2, we present our physical model which consider a V-type three-level atom coupling to a reservoir of electromagnetic radiation modes. In Section 3, the entropy uncertainty with a memory system is investigated in non-Markovian regimes. In Section 4, we briefly discuss entropy uncertainty with SGI under non-Markovian environment. Finally, we summarize our work in Section 5.

2. Physical model

We will consider a three-level atom which hat is in V-type atomic configuration, with states denoted as $|0\rangle$, $|1\rangle$, and $|2\rangle$, coupling to a reservoir of electromagnetic radiation modes at zero temperature.

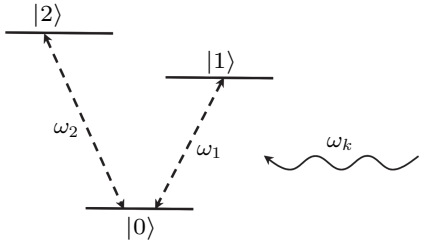


Fig. 1. A three-level atom in V-type configuration.

In this atom, the transitions $|0\rangle \rightleftharpoons |1\rangle$ and $|0\rangle \rightleftharpoons |2\rangle$ are allowed, whereas the transition $|1\rangle \rightleftharpoons |2\rangle$ is forbidden in the electric-dipole approximation, as shown in Fig. 1; the atomic transition frequency is ω_1 between atomic states $|0\rangle$ and $|1\rangle$ and ω_2 between atomic states $|0\rangle$ and $|2\rangle$. The Hamiltonian describing the dynamics of the system in the rotating wave approximation can be written as ($\hbar = 1$)

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm I}, \tag{4}$$

with

$$\hat{H}_0 = \omega_1 |1\rangle\langle 1| + \omega_2 |2\rangle\langle 2| + \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k, \tag{5}$$

$$\hat{H}_{\rm I} = \sum_k (g_{k_1} \hat{a}_k^\dagger |0\rangle\langle 1| + g_{k_2} \hat{a}_k^\dagger |0\rangle\langle 2| + {\rm H.c.}), \tag{6}$$

where $\hat{a}_k$ and $\hat{a}_k^\dagger$ are respectively the annihilation and creation operators of the k-th reservoir, with the k-th field mode frequency ω_k in the reservoir. g_{k_1} represents the coupling constants between the $|0\rangle \rightleftharpoons |1\rangle$ transition and the k-th reservoir. Likewise, the coupling constants between the $|0\rangle \rightleftharpoons |2\rangle$ transition and the k-th reservoir is g_{k_2}.

Considering the atomic transition frequency $\omega_1 = \omega_2$, the evolution of the atom subsystem can be obtained by Kraus operators[28] (see Appendix A for specific derivations)

$$\rho_{\rm s}(t) = \sum_{i=1}^{3} K_i \rho_{\rm s}(0) K_i, \tag{7}$$

where

$$K_i = U^\dagger \mathcal{K}_i U \quad (i \in \{1,2,3\}), \tag{8}$$

with

$$\mathcal{K}_1 = \begin{pmatrix} G_+(t) & 0 & 0 \\ 0 & G_-(t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\mathcal{K}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{1-|G_+(t)|^2} & 0 & 0 \end{pmatrix},$$
$$\mathcal{K}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{1-|G_-(t)|^2} & 0 \end{pmatrix}. \tag{9}$$

In this equation, $G_\pm(t)$ is given by

$$G_\pm(t) = \exp\left(\frac{-\lambda t}{2}\right)\left[\cosh\left(\frac{d_\pm t}{2}\right) + \frac{\lambda}{d_\pm}\sinh\left(\frac{d_\pm t}{2}\right)\right] \tag{10}$$

with $d_\pm = \sqrt{\lambda^2 - 2\lambda\gamma_\pm}$, $\gamma_\pm = (\gamma_1 + \gamma_2 \pm q)/2$ and $q = \sqrt{(\gamma_1-\gamma_2)^2 + 4\gamma_{12}^2}$, where λ is the spectral width of the distribution. When $\lambda \gg \gamma_\pm$, the behavior of the system is Markovian; when $\lambda < 2\gamma_\pm$, it means the strong coupling regime and the behavior of the system is non-Markovian. γ_i is the spontaneous decay constant of the excited sublevel i to the ground level $|0\rangle$ and $\gamma_{ij} = \sqrt{\gamma_i\gamma_j}\theta$ ($i \neq j$ and $|\theta| \leq 1$) represents the effect of quantum interference resulting from the cross coupling between the transitions $|2\rangle \rightarrow |0\rangle$ and $|1\rangle \rightarrow |0\rangle$. θ depends on the relative angle between two dipole moment element related

to the mentioned transitions. $\theta = 0$ means that the dipole moments of two transitions are perpendicular to each other corresponding to the case that there is no spontaneously generated interference (SGI) between two decay channels. On the other hand, $\theta = \pm 1$ indicate that the two dipole moments are parallel or antiparallel corresponding to the interference effect between the two decay channels being maximal. In Eq. (8), the unitary transformation U is given by

$$U = \begin{pmatrix} A & -B & 0 \\ B & A & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

where

$$A = \sqrt{(q+\gamma_1-\gamma_2)/2q}, \quad B = \sqrt{(q-\gamma_1+\gamma_2)/2q}.$$

And then, using the unitary transformation U, we obtain the following expressions for K_i

$$K_1 = \begin{pmatrix} G_+A^2+G_-B^2 & (-G_++G_-)AB & 0 \\ (-G_++G_-)AB & G_+B^2+G_-A^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$K_2 = \sqrt{(1-|G_+|^2)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{A} & -\sqrt{B} & 0 \end{pmatrix},$$

$$K_3 = \sqrt{(1-|G_-|^2)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{B} & \sqrt{A} & 0 \end{pmatrix}, \tag{12}$$

hence, it is concluded that $\sum_{i=1}^{3} K_i^\dagger K_i = I_3$ and $\rho_s(t) = \sum_{i=1}^{3} K_i \rho_s(0) K_i^\dagger$.

In this paper, we consider the relaxation rates of the two decay channels are equal, *i.e.*, $\gamma_1 = \gamma_2 = \gamma$.

3. Reduction of entropy uncertainty for qutrit with a memory system

Now let us consider that Bob initially prepares a pair of qutrits A and B being entangled, as follows:

$$\rho_k(0) = \frac{1-k}{9} I + k|\psi_+\rangle\langle\psi_+|, \tag{13}$$

in this equation, $|\psi_+\rangle = (1/\sqrt{3})(|00\rangle + |11\rangle + |22\rangle)$ is the maximally entangled pure state. Bob sends qutrit A to Alice who measures either R or Q on qutrit A and informs Bob of her choice. Meanwhile Bob serves qutrit B as quantum memory. In this case, each part of the two qutrits locally interacts with an independent reservoir. The evolution of a two-qutrit state based on the techniques of Kraus can be characterized as[29]

$$\rho_{AB}(t) = \sum_{i=1,j=1}^{3} (K_i \otimes K_j)\rho_k(0)(K_i \otimes K_j)^\dagger, \tag{14}$$

where K_i is Kraus operator. In this section, we choose the spin-1 observable $R(Q) = S_x(S_z)$. The conditional von-Neumann entropy after qutrit A was measured by Alice (S_x or S_z) is given as follows:

$$S(S_x|B) = S(\rho_{S_xB}) - S(\rho_B),$$

$$S(S_z|B) = S(\rho_{S_zB}) - S(\rho_B), \tag{15}$$

the post measurement state

$$\rho_{XB} = \sum_i (|\psi_i\rangle\langle\psi_i| \otimes I)\rho_{AB}(|\psi_i\rangle\langle\psi_i| \otimes I),$$

where $|\psi_i\rangle$ is the eigenstate of the observable X and I is an identity operator in the Hilbert space of particle B. $\rho_B = \mathrm{tr}_A(\rho_{AB(t)})$ is the density matrix of subsystem B. So the left-hand side of Eq. (3) can be given by

$$U_L = S(S_x|B) + S(S_z|B), \tag{16}$$

and the right-hand side of Eq. (3) can be represented as

$$U_b = \log_2 \frac{1}{c} + S(\rho_{AB}) - S(\rho_B). \tag{17}$$

$S(\rho) = \sum_i \lambda_i \log_2 \lambda_i$ is the von Neumann entropy where λ_i is the eigenvalue of ρ.

In the traditional entropy uncertainty relation equation (2), if we choose spin-1 observable $R(Q) = S_x(S_z)$, the limit of the traditional lower bound of the entropy uncertainty can be calculated

$$\log_2 \frac{1}{c} = \log_2 \frac{1}{2} = 1. \tag{18}$$

The degree of entanglement for qutrits system can be easily computed by the negativity $\mathcal{N}$ given by

$$\mathcal{N} = \frac{\| \rho^{T_A} \| - 1}{2}. \tag{19}$$

The matrix ρ^{T_A} is the partial transpose with respect to the subsystem A, that is $\rho^{T_A}_{ik,jl} = \rho_{jk,il}$, and the trace norm $\| \cdot \|$ is defined as $\| \rho \| = \mathrm{tr}(\rho\rho^\dagger)^{1/2}$.

In order to study the dynamics of the entropy uncertainty for qutrits system in a non-Markov environment, we draw the change curve of the entropy uncertainty, its lower bound and the negativity $\mathcal{N}$ with time in Fig. 2, both of them considering the strong non-Markov case $\lambda = 0.001\gamma$. In Fig. 2(a), we choose initial state as $\rho_{k=0}(0)$ which means that two qutrits are free of entanglement and the system has no memory effects. From Fig. 2(a), on the one hand, one can see the negativity is always 0 because two qutrits are free of entanglement. On the other hand, it can be seen that the entropy uncertainty U_L, U_b exhibit oscillations with the evolutionary period of $\gamma t \approx 149$. If we select the parameter of initial state as $k = 0.4$, $k = 0.6$, and $k = 1$, as shown in Figs. 2(b)–2(d), there are memory effects. It is found that the entropy uncertainty and its lower limit are reduced with memory effect, by comparing Fig. 2(a) with Figs. 2(b)–2(d). The stronger strength of the initial state entanglement is, the more the entropy uncertainty reduces: from Figs. 2(a)–2(d), the maximum of the entropy uncertainty $3.169 \to 2.890 \to 2.697 \to 2.368$. In addition, the initial entropy uncertainty is 0 when the initial state is max entanglement which breaks the limit of the traditional lower bound of the entropy uncertainty as shown in Fig. 2(d).

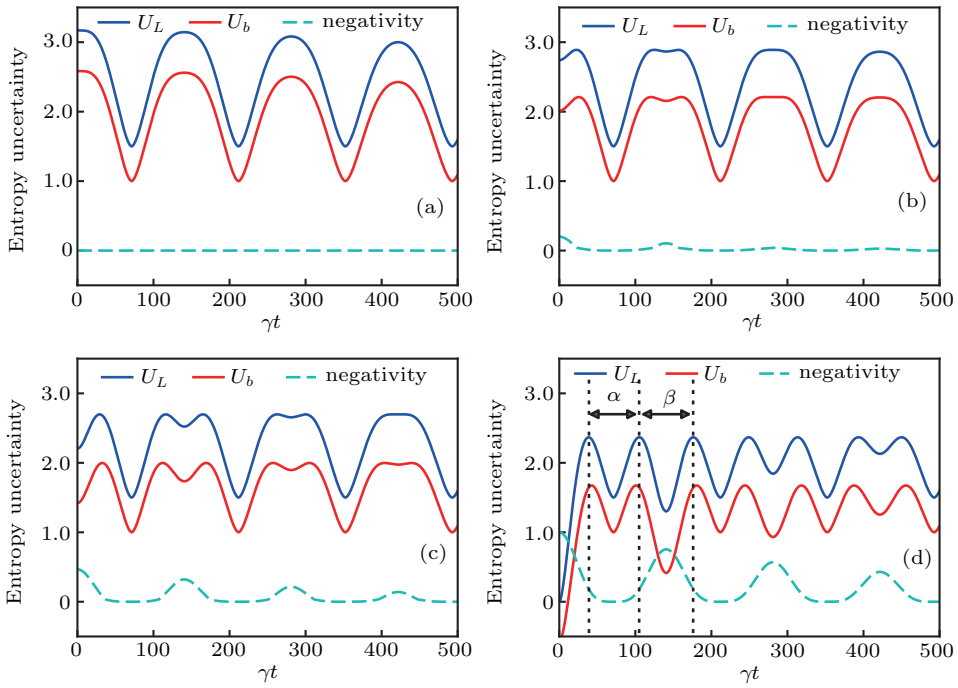


Fig. 2. The evolution of the entropy uncertainty (the blue solid line), its lower bound (the red solid line), and the negativity (the cyan dotted line) in no SGI ($\theta = 0$) and $\lambda = 0.001\gamma$ case considering the following parameter of initial state: (a) $k = 0$; (b) $k = 0.4$; (c) $k = 0.6$; (d) $k = 1$.

Figure 2(d) also reveals that, the evolutionary period of the entropy uncertainty can be divided into two regions: α and β regions. Our result is different from the conventional wisdom that the entropy uncertainty is negatively correlated with entanglement. In the α region the entropy uncertainty decreases and then slowly increases meanwhile the negativity decays asymptotically to 0 and after a freezed time, the negativity began to increase. In the β region, as the time goes the entropy uncertainty decreases with the negativity increasing and then the entropy uncertainty increases with the negativity decreasing. In other word, the entropy uncertainty is negatively correlated with entanglement in the β region, not in the α region. Beside in the β region, the minimum value of the entropy uncertainty can be reduced more. These results show that the memory systems formed by two qutrits entangled reduce effectively the entropy uncertainty and break the traditional lower bound of the entropy uncertainty when degree of the entanglement is large. In the non-Markov environment, entanglement only affects the entropy uncertainty locally and cannot completely control the evolution of the entropy uncertainty. In the β region, entanglement is beneficial to reduce entropy uncertainty.

4. Reduction of entropy uncertainty for qutrit with SGI

So far, we have only discussed the case without SGI. Next we will study the case of the strongest SGI, which means the two dipole moment elements of qutrit are parallel (or antiparallel). Figure 3 presents the numerical results of the entropy uncertainty, its lower bounds, and negativity in $\lambda = 0.001\gamma$, when the two dipole moment elements of qutrit are parallel. In Fig. 3(a), we have assumed the initial state parameters $k = 0$. The dynamics of the entropy uncertainty is similar to that found in the no-SGI case in Fig. 2(a), except for the evolutionary period. One can see that in the case of no memory effects, the SGI effect cannot reduce the entropy uncertainty. But if we consider the memory effects, the entropy uncertainty can be further reduced by the SGI effect. The maximum of the entropy uncertainty from 2.890 to 2.853 in the $k = 0.4$; from 2.697 to 2.555 in the $k = 0.6$, and from 2.368 to 1.596 in the $k = 1$ are revealed in Figs. 3(b)–3(d). It is clearly seen that the stronger strength of the initial state entanglement is, the more entropy uncertainty can be reduced by the SGI effect. In Fig. 3(d), it should be noted that in the first β region, the minimum value of the entropy uncertainty is $0.575 < 1$. It is lower than the limit of the traditional lower bound of the entropy uncertainty which is not found in Fig. 2(d). These results indicate that the strongest SGI can effectively reduce the entropy uncertainty in a two-qutrit entangled system.

Finally, the influence of non-Markov strengths on the entropy uncertainty is discussed. Figure 4 shows our findings considering the initial state in maximum entanglement in the case of no SGI for different non-Markov strengths. From Fig. 4 one can see that with the non-Markov strengths increasing, the maximum value of the entropy uncertainty is not reduced, but the oscillation frequency of the entropy uncertainty increases. It is noted that with the non-Markov strengths increasing, the evolutionary period of the entropy uncertainty can be divided into two regions: α and β regions, as discussed in Section 3. In other word, the non-Markov effect can reduce the local minimum of the entropy uncertainty but does not affect the maximum of the entropy uncertainty.

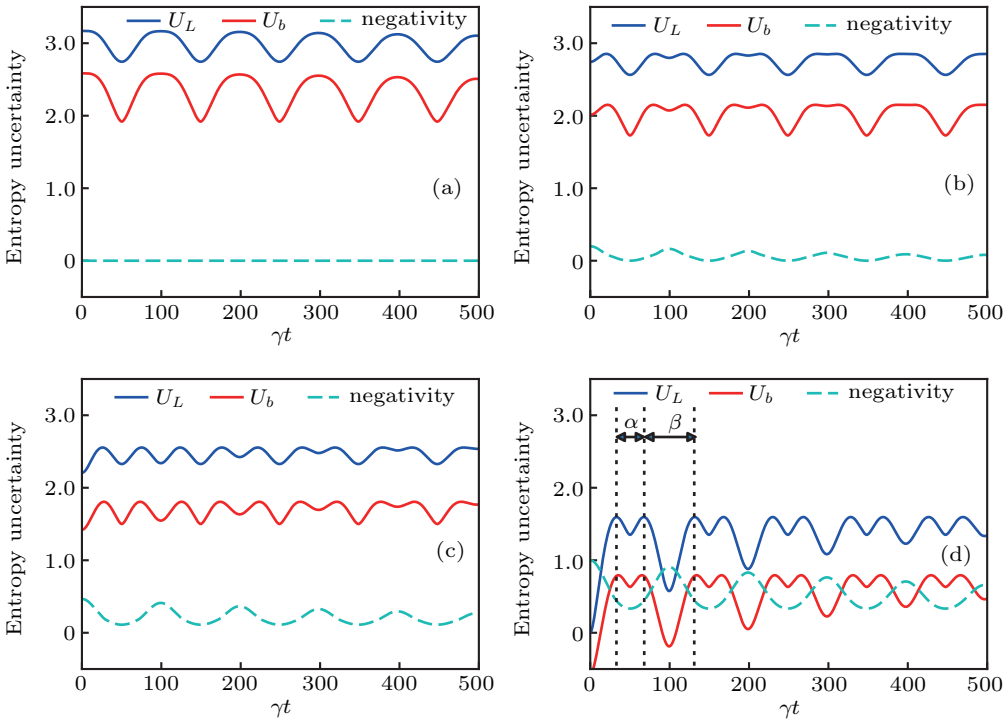


Fig. 3. The evolution of the entropy uncertainty (the blue solid line), its lower bound (the red solid line), and the negativity (the cyan dotted line) in the strongest SGI ($\theta = 1$) and $\lambda = 0.001\gamma$ case considering the following parameter of initial state: (a) $k = 0$; (b) $k = 0.4$;(c) $k = 0.6$; (d) $k = 1$.

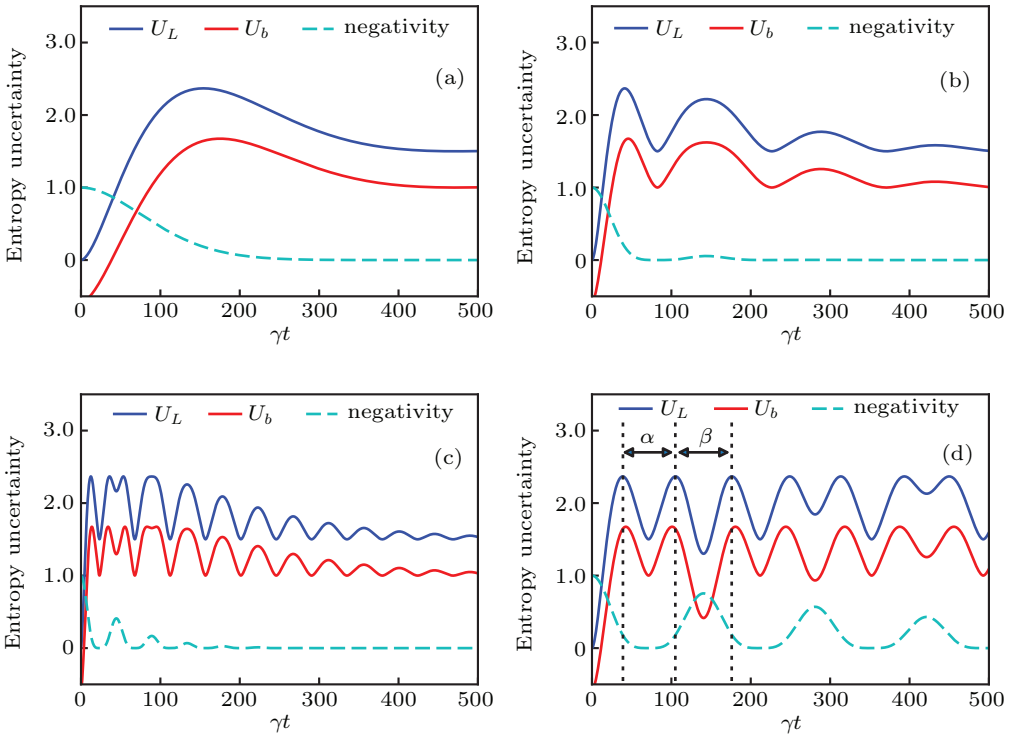


Fig. 4. The evolution of the entropy uncertainty (the blue solid line), its lower bound (the red solid line), and the negativity (the cyan dotted line) in no SGI ($\theta = 0$) and considering the initial state in maximum entanglement for different non-Markov strengths: (a) $\lambda = \gamma$; (b) $\lambda = 0.1\gamma$; (c) $\lambda = 0.01\gamma$; (d) $\lambda = 0.001\gamma$.

5. Conclusion

In conclusion, we have investigated the dynamics of the entropy uncertainty for a qutrit with and without a memory system under non-Markov noisy environment in two cases: no-SGI and the strongest SGI. We have found that the entropy uncertainty is reduced effectively with a memory system, the stronger strength of the initial state entanglement is, the more the entropy uncertainty reduces. The traditional lower bound of uncertainty is broken when degree of the entanglement is large. Furthermore, in the strongest SGI case the entropy un-

certainty can be further reduced for two-qutrit entangled system. The stronger strength of the initial state entanglement is, the more entropy uncertainty can be reduced by the SGI effect. To sum up, our investigation might offer an insight into the dynamics of the entropy uncertainty in a realistic system, and it is nontrivial to quantum precision measurement in prospective quantum information processing.

Appendix A

In the interaction picture, the time evolution of state obeys the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = H_{\mathrm{int}}|\Psi(t)\rangle, \tag{A1}$$

where

$$\begin{aligned} H_{\mathrm{int}} = &\sum_k (g_{k_1}\hat{a}_k^{\dagger}|0\rangle\langle 1|\,\mathrm{e}^{-\mathrm{i}(\omega_1-\omega_k)t} \\ &+ g_{k_2}\hat{a}_k^{\dagger}|0\rangle\langle 2|\,\mathrm{e}^{-\mathrm{i}(\omega_2-\omega_k)t} + \mathrm{H.c.}), \end{aligned} \tag{A2}$$

$$|\Psi(t)\rangle = \sum_0^l c_l(t)|l\rangle_S \otimes |0\rangle_{\mathrm{E}} + \sum_k c^k(t)|0\rangle_S \otimes |1_k\rangle_{\mathrm{E}}. \tag{A3}$$

The evolutions for probability amplitudes take the form

$$\begin{aligned} \dot{c}_0(t) &= 0, \\ \dot{c}_1(t) &= -\mathrm{i}\sum_k g_{k_1} c^k(t)\,\mathrm{e}^{\mathrm{i}(\omega_1-\omega_k)t}, \\ \dot{c}_2(t) &= -\mathrm{i}\sum_k g_{k_2} c^k(t)\,\mathrm{e}^{\mathrm{i}(\omega_2-\omega_k)t}, \\ \dot{c}^k(t) &= -\mathrm{i}g_{k_1}^* c^1(t)\,\mathrm{e}^{-\mathrm{i}(\omega_1-\omega_k)t} \\ &\quad -\mathrm{i}g_{k_2}^* c^2(t)\,\mathrm{e}^{-\mathrm{i}(\omega_2-\omega_k)t}. \end{aligned} \tag{A4}$$

Assuming that $(\dot{c})^k(0) = 0$, there is no photon in the initial state, so we obtain a closed equation for $\dot{c}_l(t)$

$$\dot{c}_l(t) = -\sum_{m=1}^{2}\int_0^t f_{lm}(t-\tau)c_m(\tau)\,\mathrm{d}\tau, \quad l = 1,2, \tag{A5}$$

where the correlation function $f_{lm}(t-\tau)$ is given by

$$f_{lm}(t-\tau) = \int_0^{\infty}\mathrm{d}\omega J_{lm}(\omega)\,\mathrm{e}^{\mathrm{i}(\omega_l-\omega)t-\mathrm{i}(\omega_m-\omega)\tau} \tag{A6}$$

and $J_{lm}(\omega)$ is chosen as Lorentzian distribution

$$J_{lm}(\omega) = \frac{1}{2\pi}\frac{\gamma_{lm}\lambda^2}{(\omega_0-\Delta-\omega)^2+\lambda^2}. \tag{A7}$$

If we take Laplace transform from Eq. (A5), it becomes

$$\begin{aligned} &\begin{pmatrix} s\tilde{c}_2(s)-c_2(0) \\ s\tilde{c}_1(s)-c_1(0) \end{pmatrix} \\ &= -\frac{\lambda}{2(s+\lambda)}\begin{pmatrix} \gamma_2 & -\sqrt{\gamma_1\gamma_2}\theta \\ \sqrt{\gamma_1\gamma_2}\theta & \gamma_1 \end{pmatrix}\begin{pmatrix} \tilde{c}_2(s) \\ \tilde{c}_1(s) \end{pmatrix}, \end{aligned} \tag{A8}$$

where $\tilde{c}_l(s)$ is the Laplace transform of $c_l(s)$. By considering the subspace spanned by $\{|1\rangle,|2\rangle\}$, and doing the following unitary transformation

$$U_1 = \begin{pmatrix} \sqrt{(q+\gamma_1+\gamma_2)/2q} & -\sqrt{(q-\gamma_1+\gamma_2)/2q} \\ \sqrt{(q-\gamma_1+\gamma_2)/2q} & \sqrt{(q+\gamma_1+\gamma_2)/2q} \end{pmatrix} \tag{A9}$$

to Eq. (A8) and taking inverse Laplace transform we get

$$c_{\pm}(t) = G_{\pm}c_{\pm}(0), \tag{A10}$$

where

$$\begin{pmatrix} c_+(0) \\ c_-(0) \end{pmatrix} = U_1\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix}, \tag{A11}$$

$$G_{\pm}(t) = \exp\left(\frac{-\lambda t}{2}\right)\left[\cosh\left(\frac{d_{\pm}t}{2}\right) + \frac{\lambda}{d_{\pm}}\sinh\left(\frac{d_{\pm}t}{2}\right)\right]. \tag{A12}$$

It should be noted that the extended unitary transformation on the whole space of the system spanned by $|0\rangle,|1\rangle,|2\rangle$, becomes

$$U_2 = \begin{pmatrix} \sqrt{(q+\gamma_1+\gamma_2)/2q} & -\sqrt{(q-\gamma_1+\gamma_2)/2q} & 0 \\ \sqrt{(q-\gamma_1+\gamma_2)/2q} & \sqrt{(q+\gamma_1+\gamma_2)/2q} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A13}$$

By these considerations, the density matrix of a V-type system at time t becomes

$$\begin{pmatrix} |G_+(t)|^2|c_+(0)|^2 & G_+(t)c_+(0)G_-^*(t)c_-^*(0) & G_+(t)c_+(0)c_0^* \\ G_+^*(t)c_+^*(0)G_-(t)c_-(0) & |G_-(t)|^2|c_-(0)|^2 & G_-(t)|c_-(0)c_0^* \\ G_+^*(t)c_+^*(0)c_0 & G_-^*(t)c_-^*(0)c_0 & 1-|G_+(t)|^2|c_+(0)|^2-|G_-(t)|^2|c_-(0)|^2 \end{pmatrix}, \tag{A14}$$

so we can easily obtain the Krauss representation such as Eqs. (7)–(9).

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