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Two integrable generalizations of WKI and FL equations: Positive and negative flows, and conservation laws*

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With the aid of Lenard recursion equations, an integrable hierarchy of nonlinear evolution equations associated with a 2×2 matrix spectral problem is proposed, in which the first nontrivial member in the positive flows can be reduced to a new generalization of the Wadati–Konno–Ichikawa (WKI) equation. Further, a new generalization of the Fokas–Lenells (FL) equation is derived from the negative flows. Resorting to these two Lax pairs and Riccati-type equations, the infinite conservation laws of these two corresponding equations are obtained.

Keywords: integrable generalizations, positive flow and negative flow, infinite conservation laws

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1. Introduction

Integrable nonlinear evolution equations have attracted wide attention in mathematics and theoretical physics because they successfully describe and explain nonlinear phenomena in natural science. A key feature of integrable nonlinear evolution equations is the fact that they can be expressed as the compatibility condition of two linear spectral problems,[1] i.e., a Lax pair, which plays a crucial role in the inverse scattering transformation,[2–4] Darboux and Bäcklund transformation,[5–8] algebraic-geometrical solutions,[9,10] Riemann–Hilbert problem,[11,12] rogue waves,[13,14] Hamiltonian structure,[15,16] etc. Among many integrable systems, the WKI model, proposed by Wadati, Konno, and Ichikawa, could describe the nonlinear oscillation of elastic beams under tension.[17] Under the sense of Lax integrability, it is discovered that the AKNS and WKI models are connected through a generalized gauge transformation.[18] Some related studies on the WKI system have been carried out, such as integrable extensions[19–21] and explicit solutions via different approaches.[22–28]

In this article, we propose an integrable hierarchy of nonlinear evolution equations, where the first nontrivial member in the positive flows is a generalization of the WKI equation and that in the negative flows is a generalized Fokas–Lenells (FL) equation. Then we derive the infinite conservation laws of these two equations respectively. In Section 2, we first introduce a 2×2 matrix spectral problem, from which we derive a novel hierarchy of integrable nonlinear evolution equations with the aid of the zero-curvature equation. The first nontrivial member in the hierarchy is

$$\begin{aligned} u_{t_0} &= \frac{1}{2}\left(\frac{\alpha_0 u}{\sqrt{1+uv}}\right)_{xx} + \left(\frac{\alpha_0 u}{\sqrt{1+uv}}\right)_x, \\ v_{t_0} &= -\frac{1}{2}\left(\frac{\alpha_0 v}{\sqrt{1+uv}}\right)_{xx} + \left(\frac{\alpha_0 v}{\sqrt{1+uv}}\right)_x, \end{aligned} \tag{1}$$

where u and v are two potentials, and α_0 is a constant. By choosing $\alpha_0=-2\mathrm{i}$, $v=u^*$, $t_0=t$, then equation (1) is reduced to a new integrable generalization of the WKI equation,

$$\mathrm{i}u_t = \left(\frac{u}{\sqrt{1+|u|^2}}\right)_{xx} + \left(\frac{2u}{\sqrt{1+|u|^2}}\right)_x. \tag{2}$$

Based on the Lax pair and the Riccati-type equation, we obtain the infinite conservation laws of Eq. (1). In Section 3, different from the positive flow in Section 2, we propose a hierarchy of negative flow equations associated with the 2×2 matrix spectral problem. The first nontrivial member can be reduced to an evolution equation with cubic nonlinearity

$$u_{xt} = -\left(|u|^2 u_x\right)_x - 2\mathrm{i}|u|^2 u_x - \mathrm{i}u_x + 2u, \tag{3}$$

which is a new generalization of the FL equation.[29,30] Accordingly, we can derive the infinite many conservation laws for the generalization of the FL equation.

2. A hierarchy of generalized WKI equations

In this section, we first introduce a 2×2 matrix spectral problem

$$\psi_x = U\psi, \quad U = \begin{pmatrix} \lambda & (1+\lambda)u \\ \lambda v & -\lambda \end{pmatrix}, \tag{4}$$

where u and v are two potentials, and λ is a constant spectral parameter. In order to derive a hierarchy of nonlinear evolution equations associated with spectral problem (4), we solve

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the stationary zero curvature equation

$$\boldsymbol{V}_x - [\boldsymbol{U}, \boldsymbol{V}] = 0, \tag{5}$$

$$\boldsymbol{V} = \begin{pmatrix} \lambda(1+\lambda)a & (1+\lambda)(c+\lambda ua) \\ \lambda(b+\lambda va) & -\lambda(1+\lambda)a \end{pmatrix}, \tag{6}$$

where

$$a = \sum_{j\geq 0} a_j \lambda^{-j}, \quad b = \sum_{j\geq 0} b_j \lambda^{-j}, \quad c = \sum_{j\geq 0} c_j \lambda^{-j}, \tag{7}$$

and coefficients a_j, b_j, c_j are functions to be determined. Equation (5) is equivalent to

$$\begin{aligned} &c_x + \lambda(ua)_x - 2\lambda c + 2\lambda ua = 0, \\ &b_x + \lambda(va)_x + 2\lambda b - 2\lambda va = 0, \\ &a_x - ub + vc = 0, \end{aligned}$$

which can be rewrite into the following Lenard recursion equation:

$$\boldsymbol{K}\boldsymbol{G}_j = \boldsymbol{J}\boldsymbol{G}_{j+1}, \quad j \geq 0, \quad \boldsymbol{J}\boldsymbol{G}_0 = 0, \tag{8}$$

with $\boldsymbol{G}_j = (a_j, b_j, c_j)^{\mathrm{T}}$, $j \geq 0$, and the two operators $\boldsymbol{K}, \boldsymbol{J}$ defined as

$$\boldsymbol{K} = \begin{pmatrix} 0 & 0 & \partial \\ 0 & \partial & 0 \\ \partial & -u & v \end{pmatrix}, \quad \boldsymbol{J} = \begin{pmatrix} -2u - \partial u & 0 & 2 \\ 2v - \partial v & -2 & 0 \\ \partial & -u & v \end{pmatrix}. \tag{9}$$

From $\boldsymbol{J}\boldsymbol{G}_0 = 0$, it is easy to calculate that

$$a_0 = \frac{\alpha_0}{\sqrt{1+uv}}, \quad b_0 = va_0 - \frac{1}{2}\partial va_0, \quad c_0 = ua_0 + \frac{1}{2}\partial ua_0, \tag{10}$$

where α_0 is an arbitrary constant. Using the recursion relation (8), we can derive a sequence $\boldsymbol{G}_j, j \geq 1$, with a series of integration constants $\alpha_l, l \geq 0$. For example, set $j = 0$ in Eq. (8), we can obtain

$$\begin{aligned} a_1 = &-\frac{\alpha_0}{4}\left[\frac{2u}{1+uv}\left(\frac{v}{\sqrt{1+uv}}\right) - \frac{u}{1+uv}\left(\frac{v}{\sqrt{1+uv}}\right)_x \right. \\ &\left. + \frac{v}{1+uv}\left(\frac{u}{\sqrt{1+uv}}\right)_x\right] + \frac{\alpha_1}{\sqrt{1+uv}}, \\ b_1 = va_1 - &\frac{1}{2}(va_1)_x - \frac{1}{2}b_{0,x}, \quad c_1 = ua_1 + \frac{1}{2}(ua_1)_x + \frac{1}{2}c_{0,x}. \end{aligned} \tag{11}$$

Let ψ satisfy the spectral problem (4) and the following auxiliary problem:

$$\psi_{t_m} = \boldsymbol{V}^{(m)}\psi, \quad \boldsymbol{V}^{(m)} = (V_{ij}^{(m)})_{2\times 2}, \tag{12}$$

with each entry $V_{ij}^{(m)} = V_{ij}(a^{(m)}, b^{(m)}, c^{(m)})$,

$$a^{(m)} = \sum_{j=0}^{m} a_j \lambda^{m-j}, \quad b^{(m)} = \sum_{j=0}^{m} b_j \lambda^{m-j}, \quad c^{(m)} = \sum_{j=0}^{m} c_j \lambda^{m-j}. \tag{13}$$

Then the compatibility condition of Eqs. (4) and (12) yields the zero-curvature equation

$$\boldsymbol{U}_{t_m} - \boldsymbol{V}_x^{(m)} + \left[\boldsymbol{U}, \boldsymbol{V}^{(m)}\right] = 0, \tag{14}$$

which is equivalent to a hierarchy of nonlinear evolution equations

$$(u_{t_m}, v_{t_m})^{\mathrm{T}} = \boldsymbol{X}_m, \quad m \geq 0, \tag{15}$$

where the vector fields $\boldsymbol{X}_m = \boldsymbol{P}(\boldsymbol{K}\boldsymbol{G}_m) = \boldsymbol{P}(\boldsymbol{J}\boldsymbol{G}_{m+1})$, $\boldsymbol{P}$ is the projective map $\boldsymbol{P} : (\gamma^1, \gamma^2, \gamma^3)^{\mathrm{T}} \longrightarrow (\gamma^1, \gamma^2)^{\mathrm{T}}$. The first nontrivial member in the hierarchy for $m = 0$ is just the nonlinear equation (1) that can be reduced to a generalization of the WKI equation (2).[17]

3. Infinite many conversation laws

In what follows, we shall derive the infinite sequences of conversation laws for the generalization of the WKI equation (2). Define $\rho = (\ln \psi_1)_x$. From the spectral problem (4), it is easy to see that ρ satisfies the following Riccati-type equation:

$$u\rho_x + u\rho^2 - u_x\rho + \lambda(u_x - u^2 v) - \lambda^2 u(1+uv) = 0. \tag{16}$$

Let $\lambda = \zeta^{-1}$. By inserting the ansatz

$$\rho = \sum_{j=1}^{\infty} \rho_j \zeta^{-j} \tag{17}$$

into Eq. (16) and comparing the coefficients of ζ with the same power, we obtain ρ_1, ρ_2, and the recursion formulas for $\rho_j, j \geq 3$ as follows:

$$\begin{aligned} &\rho_1 = 1 + u\partial^{-1}v, \\ &\rho_2 = u\partial^{-1}[v - u(\partial^{-1}v)^2 - 2\partial^{-1}v], \\ &u\rho_{j,x} - u_x\rho_j + u \sum_{\substack{m+n=j \\ 1\leq m,n\leq j-1}} \rho_m\rho_n = 0, \quad j \geq 3. \end{aligned} \tag{18}$$

On the other hand, denote $\theta = (\ln \psi_1)_{t_m}$ and from the auxiliary spectral problem (12), we have

$$\theta = \lambda a^{(m)} + \lambda a^{(m)}\rho - \lambda\frac{c^{(m)}}{u} + \frac{c^{(m)}\rho}{u}. \tag{19}$$

For $m = 0$, expanding θ into the following series of ζ ($\zeta = \lambda^{-1}$):

$$\theta = \sum_{j=1}^{\infty} \theta_j \zeta^{-j}, \tag{20}$$

we can gain the expressions of θ_j immediately from Eqs. (18)–(20)

$$\begin{aligned} &\theta_1 = \frac{\alpha_0}{\sqrt{1+uv}} + \left[\frac{1}{2}\left(\frac{\alpha_0 u}{\sqrt{1+uv}}\right)_x + \left(\frac{\alpha_0 u}{\sqrt{1+uv}}\right)\right]\partial^{-1}v, \\ &\theta_j = \frac{c_0}{u}\rho_j + a_0\rho_{j-1}, \quad j \geq 2. \end{aligned} \tag{21}$$

Since $(\ln \psi_1)_{t_0,x} = (\ln \psi_1)_{x,t_0}$, we obtain

$$\rho_{t_0} = \theta_x, \tag{22}$$

in which the coefficients ρ_j and θ_j in the expansions of ρ and θ are called conservation densities and currents, respectively. Specially, when $j = 1$, the first conservation law is as follows:

$$\begin{aligned} \left(1 + u\partial^{-1}v\right)_{t_0} = &\left\{\frac{\alpha_0}{\sqrt{1+uv}} + \left[\frac{1}{2}\left(\frac{\alpha_0 u}{\sqrt{1+uv}}\right)_x \right.\right. \\ &\left.\left. + \left(\frac{\alpha_0 u}{\sqrt{1+uv}}\right)\right]\partial^{-1}v\right\}_x. \end{aligned} \tag{23}$$

4. Negative flows

In this section, we shall derive the nonlinear evolution equations of the negative flows associated with the matrix spectral problem (4). In the field of integrable systems, the nonlinear evolution equations of negative flows have important significance in physics and mathematics, such as the Camassa–Holm equation, the Degasperis–Procesi equation, and the Vakhnenko equation. In order to obtain the negative flows, we consider the spectral problem

$$\psi_x = \boldsymbol{U}\psi, \quad \boldsymbol{U} = \begin{pmatrix} \lambda & (1+\lambda)u_x \\ \lambda v_x & -\lambda \end{pmatrix}, \tag{24}$$

in which we transform the potentials u and v in the matrix spectral problem (4) into the potentials u_x and v_x. To solve the stationary zero curvature equation $\boldsymbol{V}_x - [\boldsymbol{U},\boldsymbol{V}] = 0$, we set

$$\boldsymbol{V} = \begin{pmatrix} (1+\lambda)a & \lambda^{-1}(1+\lambda)[(1+\lambda)u_x a - c] \\ b+(1+\lambda)v_x a & -(1+\lambda)a \end{pmatrix}, \tag{25}$$

where

$$a = \sum_{j\le 0} a_j\lambda^{-j}, \quad b = \sum_{j\le 0} b_j\lambda^{-j}, \quad c = \sum_{j\le 0} c_j\lambda^{-j}. \tag{26}$$

Through direct calculations, we can obtain the Lenard recursion equation

$$\boldsymbol{K}_1\boldsymbol{S}_j = \boldsymbol{J}_1\boldsymbol{S}_{j+1}, \quad j\le -1, \quad \boldsymbol{K}_1\boldsymbol{S}_0 = 0, \tag{27}$$

where $\boldsymbol{S}_j = (a_j,b_j,c_j)^{\mathrm{T}}, j\le 0$, and the two operators $\boldsymbol{K}_1,\boldsymbol{J}_1$ are defined as

$$\boldsymbol{K}_1 = \begin{pmatrix} -\partial u_x & 0 & \partial \\ -\partial v_x & -\partial & 0 \\ \partial & -u_x & -v_x \end{pmatrix}, \quad \boldsymbol{J}_1 = \begin{pmatrix} \partial u_x & 0 & 2 \\ \partial v_x & 2 & 0 \\ \partial & -u_x & -v_x \end{pmatrix}. \tag{28}$$

From recursion relation (27), we obtain a sequence $\boldsymbol{S}_j$, in which the first three members are as follows:

$$a_0 = \beta_0, \quad b_0 = -\beta_0 v_x, \quad c_0 = \beta_0 u_x, \tag{29}$$

$$\begin{aligned} a_{-1} &= 2\beta_0 uv + \beta_1, \\ b_{-1} &= \beta_0[-v_x + 2(v - uvv_x)] - \beta_1 v_x, \\ c_{-1} &= \beta_0[u_x + 2(u + uvu_x)] + \beta_1 u_x, \end{aligned} \tag{30}$$

$$\begin{aligned} a_{-2} &= 2\beta_0\big[uv + 2\partial^{-1}v_x\partial^{-1}(u+uvu_x) \\ &\quad -2\partial^{-1}u_x\partial^{-1}(v-uvv_x)\big] + 2\beta_1 uv + \beta_2, \\ b_{-2} &= -v_x a_{-2} - v_x a_{-1} - 2\partial^{-1}b_{-1}, \\ c_{-2} &= u_x a_{-2} + u_x a_{-1} + 2\partial^{-1}c_{-1}, \end{aligned} \tag{31}$$

where $\partial\partial^{-1} = \partial^{-1}\partial = 1$, and β_0,β_1,β_2 are arbitrary constants.

Let ψ satisfy the spectral problem (24) and the auxiliary problem

$$\psi_{t_{-n}} = \boldsymbol{V}^{(-n)}\psi, \quad \boldsymbol{V}^{(-n)} = (V_{ij}^{(-n)})_{2\times 2}, \tag{32}$$

with each entry $V_{ij}^{(-n)} = V_{ij}(a^{(-n)},b^{(-n)},c^{(-n)})$,

$$a^{(-n)} = \sum_{j=-n}^{0} a_j\lambda^{-n-j}, \quad b^{(-n)} = \sum_{j=-n}^{0} b_j\lambda^{-n-j},$$

$$c^{(-n)} = \sum_{j=-n}^{0} c_j\lambda^{-n-j}. \tag{33}$$

Then the compatibility condition of Eqs. (24) and (32) yields a hierarchy of negative flow nonlinear evolution equations

$$(u_{xt_{-n}}, v_{xt_{-n}})^{\mathrm{T}} = \boldsymbol{P}\boldsymbol{J}_1\boldsymbol{S}_{-n} = \boldsymbol{P}\boldsymbol{K}_1\boldsymbol{S}_{-n-1}, \quad n\ge 0. \tag{34}$$

When $n=1$, from hierarchy (34), we obtain a new interesting nonlinear evolution equation

$$\begin{aligned} u_{xt_{-1}} &= 2\beta_0[(u_x uv)_x + 2uvu_x + u_x + 2u] + \beta_1(u_{xx} + 2u_x), \\ v_{xt_{-1}} &= 2\beta_0[(v_x uv)_x - 2uvv_x - v_x + 2v] + \beta_1(v_{xx} - 2v_x). \end{aligned} \tag{35}$$

Let $x\to -\mathrm{i}x, \beta_0 = -\mathrm{i}/2, \beta_1 = 0, v = u^*, t_{-1} = t$, equation (35) can reduce to the novel integrable generalization of the FL equation (3).

Next, we shall give the infinite sequences of conversation laws for equation (35). Similar to the above calculation method, defining $\rho = (\ln\psi_1)_x$ and $\theta = (\ln\psi_1)_{t_{-1}}$, we find that ρ and θ satisfy the following two expressions:

$$\rho_x u_x - u_{xx}\rho + \rho^2 u_x + \lambda(-u_x^2 v_x + u_{xx}) - \lambda^2(u_x^2 v_x + u_x) = 0, \tag{36}$$

$$\begin{aligned} \theta &= \left(a_0 - \frac{c_0}{u_x}\right)\rho\lambda^{-2} + \left(a_{-1} + a_0 - \frac{c_{-1}}{u_x}\right)\rho\lambda^{-1} \\ &\quad + a_{-1}\rho + \frac{c_0}{u_x}\lambda^{-1} + \frac{c_{-1}}{u_x}. \end{aligned} \tag{37}$$

Let $\lambda = \zeta^{-1}$. By inserting the ansatz

$$\rho = \sum_{j=-1}^{\infty}\rho_j\zeta^j, \quad \theta = \sum_{j=-1}^{\infty}\theta_j\zeta^j \tag{38}$$

into the above two expressions and comparing the coefficients of ζ with the same power, we obtain two sequences of conserved quantities as follows:

$$\begin{aligned} &\rho_{-1} = \sqrt{1+u_x v_x}, \quad \rho_0 = \frac{u_{xx}}{2u_x} - \frac{(u_x v_x)_x}{4(1+u_x v_x)} - \frac{-u_x^2 v_x + u_{xx}}{2u_x\sqrt{1+u_x v_x}}, \\ &u_x\rho_{j,x} - u_{xx}\rho_j + u_x \sum_{\substack{m+n=j \\ -1\le m,n\le j+1}} \rho_m\rho_n = 0, \quad j\ge 0, \end{aligned} \tag{39}$$

with

$$\begin{aligned} \theta_{-1} &= (2\beta_0 uv + \beta_1)\sqrt{1+u_x v_x}, \\ \theta_0 &= \beta_0\left(-\frac{2u}{u_x}\rho_{-1} + 2uv\rho_0 + \frac{2u}{u_x} + 2uv + 1\right) + \beta_1\rho_0 + \beta_1, \\ \theta_1 &= \beta_0\left(-\frac{2u}{u_x}\rho_0 + 2uv\rho_1 + 1\right) + \beta_1\rho_1, \\ \theta_j &= \beta_0\left(-\frac{2u}{u_x}\rho_{j-1} + 2uv\rho_j\right) + \beta_1\rho_j, \quad j\ge 2. \end{aligned} \tag{40}$$

5. Conclusion

In this paper, we propose a hierarchy of nonlinear evolution equations associated with a 2×2 matrix spectral problem, the integrability of which is guaranteed by the zero-curvature equation. The first nontrivial member in the positive flows can be reduced to a new generalization of the WKI equation, and a new generalization of the FL equation is derived from

the negative flows. These two new nonlinear evolution equations closely related to the WKI and FL models have the potential applications in physics. On the one hand, these nonlinear evolution equations can enrich the existing integrable models and possibly describe new nonlinear phenomena. On the other hand, researchers paid more attention to the integrable nonlinear evolution equations of the positive flows than to the negative flows. Recently, however, there has been an interest in studying the integrable nonlinear evolution equations of negative flows, such as the Camassa–Holm equation, the Degasperis–Procesi equation, the Vakhnenko equation, etc. Moreover, the infinite conservation laws of these two new nonlinear evolution equations are obtained by using the Riccati-type equations. It remains an open question whether these two new equations have solitons, breathers, rogue waves, rogue periodic waves, and other properties, which will be left to a future discussion.

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